Boundedness of two-point correlators
covariant under the meta-conformal algebra

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Abstract. Covariant two-point functions are derived from Ward identities. For
several extensions of dynamical scaling, notably Schrödinger-invariance, con-
formal Galilei invariance or meta-conformal invariance, the results become un-
bounded for large time- or space-separations. Standard ortho-conformal invari-
ance does not have this problem. An algebraic procedure is presented which
corrects this difficulty for meta-conformal invariance in \((1 + 1)\) dimensions. A
canonical interpretation of meta-conformally covariant two-point functions as
correlators follows. Galilei-conformal correlators can be obtained from meta-
conformal invariance through a simple contraction. All these two-point func-
tions are bounded at large separations, for sufficiently positive values of the
scaling exponents.

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1 Introduction

Dynamical symmetries are powerful tools in investigations of many complex
systems. The best-known examples are conformal invariance in equilibrium
phase transitions \([4, 6]\) and Schrödinger-invariance in time-dependent phenom-
ena \([8, 12]\). One of the most elementary predictions of dynamical symmetries
concerns the form of the co-variant two-point functions, to be derived form the
(e.g. conformal or Schrödinger) Ward identities, \([6, 12]\). These are built from
quasi-primary scaling operators \(\phi_i(t_i, r_i)\), depending locally on a ‘time’ coor-
dinate \(t_i \in \mathbb{R}\) and a ‘space’ coordinate \(r_i \in \mathbb{R}^d\). Since both conformal and
Schrödinger groups contain time- and space-translations, and also spatial ro-
tations, we can restrict to the difference \(t := t_1 - t_2\) and the absolute value
\(r := |r| = |r_1 - r_2|\). For conformal \([18]\) and Schrödinger-invariance \([8]\), re-
spectively, the covariant two-point functions of scalar quasi-primary operators
The properties of the conformally invariant two-point function are described by the scaling dimensions \( x_i \). It is a correlator and is symmetric under permutation of the two scaling operators, viz. \( C_{12}(t; r) = C_{21}(-t; -r) \). The result (1) is a physically reasonable correlator which decays to zero, if \( x_i > 0 \), for large time- or space separations, viz. \( |t| \to \infty \) or \( r \to \infty \).

The Schrödinger-invariant two-point function is a (linear) response function – recast here formally as a correlator by appealing to Janssen-de Dominicis theory [21], where \( \phi_i \) is the response operator conjugate to the scaling operator \( \phi_i \). The two-point function is now characterised by the pair \( (x_i, M_i) \) of a scaling dimension and a mass \( M_i \) associated to each scaling operator \( \phi_i \). For ‘usual’ scaling operators, masses are positive by convention, whereas response operators \( e^{\phi_i} \) have formally negative masses, viz. \( M_i = -M_i < 0 \). Because of causality, a response function must vanish for \( t < 0 \), viz. \( R_{12} = 0 \) but one has \( R_{12} \neq 0 \) for \( t > 0 \); hence it is maximally asymmetric under permutation of the scaling operators.

The result (2) of Schrödinger-invariance does not contain the causality requirement \( t > 0 \). In addition, it is not obvious why the response should vanish for large separations, even if \( x_i > 0 \) is admitted. Although one might insert these features by hand, it is preferable to derive such conditions formally. One may do so following the procedure [11]:

(i) consider the mass \( M \) as an additional coordinate and dualise by Fourier-transforming with respect to \( M_i \), which introduces dual coordinates \( \zeta_i \). The terminology is borrowed from non-relativistic versions of the AdS/CFT correspondence.

(ii) construct an extension of the Schrödinger Lie algebra \( \text{sch}(d) \) := \( \text{sch}(d) \oplus \mathbb{C}N \), where the new generator \( N \) is in the Cartan sub-algebra of \( \text{sch}(d) \).

(iii) use the extended Schrödinger Ward identities, in the dual coordinates, to find the co-variant two-point function \( R(\zeta_1 - \zeta_2, t, r) \).

(iv) finally, transform back to the fixed masses \( M_i \). The result is [11, 13]

\[
R_{12,\text{Schr}}(t; r) = \delta_{x_1, x_2} \delta(M_1 + \overline{M_2}) \Theta(M_1) t^{-x_1} \exp \left[ -\frac{M_1 r^2}{2t} \right] \tag{3}
\]

With the convention \( M_1 > 0 \), the Heaviside function \( \Theta \) expresses the causality condition \( t > 0 \). In addition, if \( x_1 > 0 \), the response function decays to zero for large time- or space separations, as physically expected.
Meta-conformal invariance

A similar problem with boundedness of two-point function arise also for conformal galilean algebra \([2, 9, 10, 16, 17]\) \(cga(d)\)\(^1\). However, for the \(cga(d)\) algebra, it has been shown recently that a procedure analogous to the one of the Schrödinger algebra, as outlined above, can be applied to assure the boundedness of two-point function which in this case does obey the symmetry relations of a correlator \([14]\).

In this paper\(^2\) we wish to demonstrate that an algebraically sound procedure, as outlined above, to the formulation of Ward identities which physically reasonable results, can be applied to meta-conformal algebra. This may not appear obvious, since it is semi-simple, in contrast to Schrödinger and conformal galilean algebra which are not. Our results are stated in Theorems 1 and 2 in section 4.

2 Meta-conformal algebra and two-point function

We shall call meta-conformal algebra \(mconf(d)\) a non-standard representation of conformal algebra which leads to a two-point correlation function distinct from (1). To be precise, we shall distinguish between ortho- and meta-conformal transformations.\(^3\)

Definition 1. (i) Meta-conformal transformations are maps \((t, r) \mapsto (t', r') = M(t, r)\), depending analytically on several parameters, such that they form a Lie group. The associated Lie algebra is isomorphic to the conformal Lie algebra \(conf(d)\).

(ii) Ortho-conformal transformations (called ‘conformal transformations’ for brevity) are those meta-conformal transformations \((t, r) \mapsto (t', r') = O(t, r)\) which keep the angles in the time-space of points \((t, r) \in \mathbb{R}^{1+d}\) invariant.

In this paper, we study the meta-conformal transformations, in \((1 + 1)\) time and space dimensions, with the following infinitesimal generators \([10, 12]\):

\[
X_n = -t^{n+1} \partial_t - \frac{1}{\mu - 1}[(t + \mu r)^{n+1} - t^{n+1}]\partial_r \\
\quad - (n + 1)\frac{\gamma}{\mu}[(t + \mu r)^n - t^n] - (n + 1)x t^n
\]

\[
Y_n = -(t + \mu r)^{n+1} \partial_r - (n + 1)\gamma(t + \mu r)^n
\]

such that \(\mu^{-1}\) can be interpreted as a velocity (‘speed of light or sound’) and where \(x, \gamma\) are constants (‘scaling dimension’ and ‘rapidity’). The generators obey the Lie algebra, for \(n, m \in \mathbb{Z}\):

\[
[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, Y_m] = (n - m)Y_{n+m}
\]

\[
[Y_n, Y_m] = \mu(n - m)Y_{n+m}
\]

\(^1\)\(cga(d)\) is non-isomorphic to either the standard Galilei algebra or else the Schrödinger algebra \(sch(d)\). It is a maximal finite-dimensional sub-algebra of non-semi-simple ‘altern-Virasoro algebra’ \(altv(1)\) (but without central charges) \([3, 7, 9, 11]\).

\(^2\)Following mainly our original work \([20]\).

\(^3\)From the greek prefixes \(o\varrho\theta\): right, standard; and \(\mu\varphi\tau\alpha\): of secondary rank.
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The isomorphism of (5) with the conformal Lie algebra $\text{conf}(2)$ is seen for example in [10, 14, 20].

The meta-conformal Lie algebra (5) acts as a dynamical symmetry on the linear advection equation [15]

$$S\phi(t, r) = (-\mu \partial_t + \partial_r)\phi(t, r) = 0$$  \hspace{1cm} (6)

in the sense that a solution $\phi$ of $S\phi = 0$, with scaling dimension $x_\phi = \gamma / \mu$, is mapped to another solution of the same equation. This follows from ($n \in \mathbb{Z}$)

$$[S, Y_n] = 0, \quad [S, X_n] = -(n + 1)t^n \dot{S} + n(n + 1)(\mu x - \gamma)t^{n-1}$$  \hspace{1cm} (7)

Hence the space of solutions of $S\phi = 0$ is meta-conformal invariant [10] (extended to Jeans-Poisson systems in [19]).

Now, quasi-primary scaling operators [4] are characterised by the parameters $(x_i, \gamma_i)$ ($\mu$ is simply a global dimensionful scale) and by co-variance under the maximal finite-dimensional sub-algebra $(X_{\pm 1, 0}, Y_{\pm 1, 0}) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ for $\mu \neq 0$. Explicitly

$$X_{-1} = -\partial_t, \quad X_0 = -t\partial_t - r\partial_r - x$$
$$X_1 = -t^2\partial_t - 2tr\partial_r - \mu^2\partial_r - 2xt - 2\gamma r$$
$$Y_{-1} = -\partial_r, \quad Y_0 = -t\partial_r - \mu r\partial_r - \gamma$$
$$Y_1 = -t^2\partial_r - 2\mu t\partial_r - \mu^2 r^2\partial_r - 2\gamma t - 2\gamma \mu r$$  \hspace{1cm} (8)

Here, the generators $X_{-1}, Y_{-1}$ describe time- and space-translations, $Y_0$ is a (conformal) Galilei transformation, $X_0$ gives the dynamical scaling $t \rightarrow \lambda t$ of $r \rightarrow \lambda r$ (with $\lambda \in \mathbb{R}$) such that the so-called ‘dynamical exponent’ $z = 1$ since both time and space are re-scaled in the same way and finally $X_{+1}, Y_{+1}$ give ‘special’ meta-conformal transformations.

Using the generators (8) (in their two-body forms $X_n^{[2]}, Y_n^{[2]}$) we construct the meta-conformal Ward identities $X_n^{[2]} \langle \phi_1 \phi_2 \rangle = Y_n^{[2]} \langle \phi_1 \phi_2 \rangle = 0$. One obtains the co-variant two-point function, up to normalisation [10, 12]

$$\langle \phi_1(t_1, r_1)\phi_2(t_2, r_2) \rangle = \delta_{x_1, x_2}\delta_{\gamma_1, \gamma_2}(t_1 - t_2)^{-2x_1} \left(1 + \frac{r_1 - r_2}{t_1 - t_2}\right)^{-2\gamma_1 / \mu}$$  \hspace{1cm} (9)

clearly distinct from the result (1) of ortho-conformal invariance. However, the result (9) raises immediately the following questions:

1. is $\langle \phi_1 \phi_2 \rangle$ a correlator or rather a response, since neither of the symmetry or causality conditions are obeyed?
2. even if $x_i > 0$ and $\gamma_i / \mu > 0$, why does $\langle \phi_1 \phi_2 \rangle$ not always decay to zero for large separations $|t_1 - t_2| \rightarrow \infty$ or $|r_1 - r_2| \rightarrow \infty$?
3. why is there a singularity at $\mu(r_1 - r_2) = -(t_1 - t_2)$?
Meta-conformal invariance

3 Two-point function in dual space

Our construction follows the same steps as outlined above which have already been used to recast the co-variant two-point functions of Schrödinger- and conformal Galilean invariance into a physically reasonable form, see [11, 13, 14].

First, we consider the ‘rapidity’ $\gamma$ as a new variable and dualise it through a Fourier transformation, which gives the quasi-primary scaling operator

$$\hat{\phi}(\zeta, t, r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\gamma e^{i\gamma\zeta} \phi_\gamma(t, r)$$  \hspace{1cm} (10)

The representation (4) of the meta-conformal algebra becomes

$$X_n = \frac{i(n+1)}{\mu} [(t+\mu r)^n - t^n] \partial_\zeta$$
$$-t^{n+1} \partial_t - \frac{1}{\mu} \left[ (t+\mu r)^{n+1} - t^{n+1} \right] \partial_r - (n+1) xt^n$$
$$Y_n = i(n+1) (t+\mu r)^n \partial_\zeta - (t+\mu r)^{n+1} \partial_r$$  \hspace{1cm} (11)

Second, we seek an extension of the Cartan sub-algebra $\mathfrak{h}$ by looking for a new generator $N$ such that $[X_n, N] = \alpha_n X_n$ and $[Y_m, N] = \beta_m Y_m$ where $\alpha_n, \beta_m$ are constants to be determined. It turned out [20] that $N$ must have the form

$$N := -r \partial_r - (\zeta + c) \partial_\zeta + \mu \partial_\mu - \nu$$
$$[X_n, N] = 0, \quad [Y_n, N] = -Y_n \quad n \in \mathbb{Z}.$$  \hspace{1cm} (12)

and satisfy (13). $N$ is a dynamical symmetry of (6), since $[S, N] = -S$. This achieves the construction of the extended meta-conformal algebra $mconf(2) := mconf(2) \oplus \mathbb{C}N$, with commutators (5, 13).

Third, co-variant two-point functions of quasi-primary scaling operators are found from the Ward identities $X_n^{[2]}(\phi_1 \phi_2) = Y_n^{[2]}(\phi_1 \phi_2) = N^{[2]}(\phi_1 \phi_2) = 0$, with $X_n, Y_n, N \in mconf(2)$ and $n = \pm 1, 0$ [12]. Given the form of $N$, we also consider $\mu$ to be a further variable and set

$$\langle \hat{\phi}(\zeta_1, t_1, r_1; \mu_1) \hat{\phi}(\zeta_2, t_2, r_2; \mu_2) \rangle = \hat{F}(\zeta_1, \zeta_2, t_1, t_2, r_1, r_2; \mu_1, \mu_2)$$  \hspace{1cm} (14)

Clearly, co-variance under $X_{-1}$ and $Y_{-1}$ implements time- and space-translation-invariance, such that $\hat{F} = \hat{F}(\zeta_1, \zeta_2, t, r; \mu_1, \mu_2)$, with $t = t_1 - t_2$ and $r = r_1 - r_2$. Next, co-variance under other generators of $mconf(2)$ provides
there exist the following integral representation

A well-known mathematical result on Fourier analysis on Hardy spaces states

Forth, to un-dualise, we write

Since the constant \( \mu \) is merely a parameter, \( \hat{G}_1(\mu) \) is just a normalisation constant.

**Proposition.** The dual two-point function, covariant under the generators \( X_{\pm 1,0}, Y_{\pm 1,0} \), \( N \) of the dual representation (11,12) of the meta-conformal algebra mconf(2), is up to normalisation

\[
\hat{F}(\zeta_1, \zeta_2, t, r) = (\hat{\phi}_1(t, r, \zeta_1) \hat{\phi}_2(0, 0, \zeta_2)) = \delta_{x_1, x_2} |t|^{-2x_1} \left( \frac{\zeta_1 + \zeta_2}{2} + i \frac{\ln(1 + \mu r/t)}{\mu} \right)^{-\nu_1 - \nu_2}.
\]

### 4 Inverse dual transformation

Forth, to un-dualise, we write

\[
\hat{F} = \delta_{x_1, x_2} |t|^{-2x_1} \hat{f}_\lambda(\zeta_+),
\]

such that

\[
\hat{f}_\lambda(\zeta_+) := \hat{f}(\zeta_+ + i\lambda) = (\zeta_+ + i\lambda)^{-\nu_1 - \nu_2}, \quad \lambda := \frac{\ln(1 + \mu r/t)}{\mu}.
\]

A well-known mathematical result on Fourier analysis on Hardy spaces states there exist the following integral representation \( \hat{F}_\pm(\gamma_+) \) are square-integrable.

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4for more details see [20]
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functions) \[1, 14, 20\]

\[
\sqrt{2\pi} \hat{f}(\zeta) = \Theta(\lambda) \int_0^{\infty} d\gamma_+ e^{i(\gamma_+ + i\lambda)\gamma} \hat{\mathcal{F}}_+(\gamma) \\
+ \Theta(-\lambda) \int_0^{\infty} d\gamma_+ e^{-i(\gamma_+ + i\lambda)\gamma} \hat{\mathcal{F}}_-(\gamma).
\]

The inverse Fourier transformation is found by distinguishing the cases \(\lambda > 0\) and \(\lambda < 0\). In the case \(\lambda > 0\), we have from (23)

\[
F = \frac{|t|^{-2x}}{\pi^{1/2}} G_1(\mu) \int_{\mathbb{R}^2} d\zeta_+ d\zeta_- e^{-i(\gamma_1 + \gamma_2)\zeta_+} e^{-i(\gamma_1 - \gamma_2)\zeta_-} \\
\times \int d\gamma_+ \Theta(\gamma_+) \hat{\mathcal{F}}_+(\gamma) e^{-\gamma_+\lambda} e^{i\gamma_+ \zeta_+} \\
= \frac{|t|^{-2x}}{\pi^{1/2}} G_1(\mu) \int_{\mathbb{R}^2} d\zeta_- e^{-(\gamma_1 - \gamma_2)\zeta_-} \int d\zeta_+ e^{i(\gamma_1 + \gamma_2 - \gamma_2)\zeta_+} \\
= \delta_{x_1, x_2} \delta(\gamma_1 - \gamma_2) \Theta(\gamma_1) |t|^{-2x_1} f_1(\mu) f_2(\gamma_1) \exp(-2\gamma_1 \ln(1 + \mu r/t)/\mu) \\
= \delta_{x_1, x_2} \delta(\gamma_1 - \gamma_2) \Theta(\gamma_1) f_1(\mu) f_2(\gamma_1) |t|^{-2x_1} (1 + \mu r/t)^{-2\gamma_1/\mu}. \tag{24}
\]

where in the third line two delta functions where recognised, and \(f_1, f_2\) contain unspecified dependencies on \(\mu\) and \(\gamma_1\), respectively.\(^5\) In the case \(\lambda < 0\), we have in quite an analogous way

\[
F = \delta_{x_1, x_2} \delta(\gamma_1 - \gamma_2) \Theta(-\gamma_1) |t|^{-2x_1} f_1(\mu) f_2(-\gamma_1) \exp(-2\gamma_1 \ln(1 + \mu r/t)/\mu) \\
= \delta_{x_1, x_2} \delta(\gamma_1 - \gamma_2) \Theta(-\gamma_1) f_1(\mu) f_2(-\gamma_1) |t|^{-2x_1} (1 + \mu r/t)^{-2\gamma_1/\mu}. \tag{25}
\]

The meaning of the signs of \(\lambda\), is carefully explained in [20]. Under convention that \(\mu > 0\) we have always \(\gamma_1 r/t = |\gamma_1 r/t| > 0\), independently of the sign of \(\lambda\). Therefore, we can always write for the time-space argument

\[
\mu_{x_1} \gamma_{x_1} r / t = \mu_{x_1} \gamma_{x_1} r / t \\
\left| r / t \right| = \mu_{x_1} \gamma_{x_1} r / t. \tag{26}
\]

(if \(\gamma_1 \neq 0\) and we have identified the source of the non-analyticity in the two-point function. Eqs. (24,25,26) combine to give our main result.

**Theorem 1.** With the convention that \(\mu = \mu_1 = \mu_2 > 0\), and if \(\nu_1 + \nu_2 > \frac{1}{2}\), the two-point correlator, co-variant under the representation (8), enhanced by (12), of the extended meta-conformal algebra \(\mathfrak{conf}(2)\), reads up to normalisation

\[
C_{12}(t, r) = \langle \phi(t, r) \phi(0, 0) \rangle = \delta_{x_1, x_2} \delta_{\gamma_1, \gamma_2} |t|^{-2x_1} \left(1 + \frac{\mu}{\gamma_1} \left| \frac{\gamma_1 r}{t} \right| \right)^{-2\gamma_1/\mu}. \tag{27}
\]

\(^5\)An eventual shift \(\zeta_+ \rightarrow \zeta_+ + c\), see (19), can be absorbed into the re-definition \(\hat{\mathcal{F}}(\gamma_+) e^{-\gamma_+ c} \rightarrow \hat{\mathcal{F}}(\gamma_+)\).
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This form has the correct symmetry $C_{12}(t, r) = C_{21}(-t, -r)$ under permutation of the scaling operators of a correlator. For $\gamma_1 > 0$ and $x_1 > 0$, the correlator decays to zero for $t \to \pm \infty$ or $r \to \pm \infty$.

In the limit $\mu \to 0$, the extended meta-conformal algebra (5,13) contracts to the extended altern-Virasoro algebra $\mathfrak{alv}(1)$, whose maximal finite-dimensional sub-algebra is the extended conformal Galilean algebra $\mathfrak{cga}(1) = \mathfrak{cga}(1) \oplus \mathbb{C} \mathcal{N}$. We recover as a special limit case:

**Theorem 2.** [14] If $\nu_1 + \nu_2 > \frac{1}{2}$, the two-point correlator, co-variant under the extended conformal Galilean algebra $\mathfrak{cga}(1)$, reads (up to normalisation)

\[
C_{12}(t, r) = \langle \phi_1(t, r) \phi_2(0, 0) \rangle = \delta_{x_1, x_2} \delta_{\gamma_1, \gamma_2} |t|^{-2x_1} \exp \left( - \left| \frac{2\gamma_1 r}{t} \right| \right).
\]

Any treatment of the $\mathfrak{cga}$ which neglects this non-analyticity cannot be correct.

5 Conclusion

Summarising, we have shown that for time-space meta-conformal invariance, as well as for its $\mu \to 0$ limit conformal galilean invariance (or BMS-invariance), the co-variant two-point correlators are given by eqs. (27,28) and are explicitly *non-analytic* in the temporal-spatial variables. Any form of the Ward identities which implicitly assumes such an analyticity cannot be correct. In our construction of physically sensible Ward identities, we extended the Cartan sub-algebra to a higher rank. The extra generator $\mathcal{N}$ provides an important ingredient in the demonstration that in direct space, the meta-conformally and galilean-conformally covariant two-point correlators rather are *distributions*.

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References


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