Integrable Structures of the Gauge/String Correspondence

Gleb Arutyunov

Utrecht University

International School on
Modern Trends in
Mathematical Physics
24-30 September 2006

Part I. Superconformal Representation Theory

Conformal algebra in d-dimensions: SO(d,2)

Metric

$$\eta_{ab} = (-1, 1, \ldots, 1)\,, \qquad a, b = 0, \ldots, d-1$$

Generators

$$\begin{aligned} \mathrm{M_{ab}} \quad &-- \quad \text{Lorentz rotations} \;\; \mathrm{SO(d-1,1)} \\ \mathrm{P_a} \quad &-- \quad \text{translations} \\ \mathrm{K_a} \quad &-- \quad \text{conformal boosts} \\ \mathrm{D} \quad &-- \quad \text{dilatation} \end{aligned}$$

Algebra relations

$$\begin{aligned} [\mathrm{M_{ab}}, \mathrm{P_c}] &= \mathrm{i}(\eta_{ac}\mathrm{P_b} - \eta_{\mathrm{bc}}\mathrm{P_a})\,, \qquad [\mathrm{M_{ab}}, \mathrm{K_c}] = \mathrm{i}(\eta_{\mathrm{ac}}\mathrm{K_b} - \eta_{\mathrm{bc}}\mathrm{K_a})\,, \\ [\mathrm{M_{ab}}, \mathrm{M_{cd}}] &= \mathrm{i}(\eta_{\mathrm{ac}}\mathrm{M_{bd}} - \eta_{\mathrm{bc}}\mathrm{M_{ad}} - \eta_{\mathrm{ad}}\mathrm{M_{bc}} + \eta_{\mathrm{bd}}\mathrm{M_{ac}})\,, \\ [\mathrm{D}, \mathrm{P_a}] &= \mathrm{i}\mathrm{P_a}\,, \qquad [\mathrm{D}, \mathrm{K_a}] = \mathrm{i}\mathrm{K_a}\,, \\ [\mathrm{K_a}, \mathrm{P_b}] &= -2\mathrm{i}\mathrm{M_{ab}} - 2\mathrm{i}\eta_{\mathrm{ab}}\mathrm{D}\,. \end{aligned}$$

$$\mathrm{M}^\dagger_{\mathrm{ab}} = \mathrm{M_{ab}}\,, \quad \mathrm{P}^\dagger_{\mathrm{a}} = \mathrm{P_a}\,, \quad \mathrm{K}^\dagger_{\mathrm{a}} = \mathrm{K_a}\,, \quad \mathrm{D}^\dagger = \mathrm{D}\,.$$

Quasi-primary fields $\Phi_I(x)$

$$[\mathrm{P_a}, \Phi_\mathrm{I}(\mathrm{x})] = \mathrm{i}\partial_\mathrm{a}\Phi_\mathrm{I}(\mathrm{x}) \,, \qquad [\mathrm{D}, \Phi_\mathrm{I}(\mathrm{x})] = \mathrm{i}(\mathrm{x}^\mathrm{a}\partial_\mathrm{a} + \Delta)\Phi_\mathrm{I}(\mathrm{x})$$

$$[\mathrm{M_{ab}}, \Phi_\mathrm{I}(\mathrm{x})] = \mathrm{i}(\mathrm{x_a}\partial_\mathrm{b} - \mathrm{x_b}\partial_\mathrm{a})\Phi_\mathrm{I}(\mathrm{x}) + \Phi_\mathrm{J}(\mathrm{x})(\Lambda_\mathrm{ab})^\mathrm{J}{}_\mathrm{I}$$

$$[\mathrm{K_a}, \Phi_\mathrm{I}(\mathrm{x})] = \mathrm{i}(\mathrm{x}^2\partial_\mathrm{a} - 2\mathrm{x_a}\mathrm{x}^\mathrm{b}\partial_\mathrm{b} - 2\Delta\mathrm{x_a})\Phi_\mathrm{I}(\mathrm{x}) - 2\Phi_\mathrm{J}(\mathrm{x})(\Lambda_\mathrm{ab})^\mathrm{J}{}_\mathrm{I}\mathrm{x}^\mathrm{b}$$

form irreps of conformal group parametrized by the conformal weight Δ and the Lorentz spin s.

Operator-state correspondence

$$|\Phi_I\rangle = \Phi_I(0)|0\rangle \qquad \leftarrow \qquad \text{conformal state}$$

The action of algebra generators on a conformal state

$$\mathrm{K_a}|\Phi_\mathrm{I}\rangle = 0, \quad \mathrm{D}|\Phi_\mathrm{I}\rangle = \mathrm{i}\Delta|\Phi_\mathrm{I}\rangle, \quad \mathrm{M_{ab}}|\Phi_\mathrm{I}\rangle = |\Phi_\mathrm{J}\rangle(\Lambda_\mathrm{ab})^\mathrm{J}{}_\mathrm{I}$$

$$\mathrm{Coset} = \mathrm{SO(d,2)}/\{\mathrm{K_a}, \mathrm{D}, \mathrm{M_{ab}}\}$$

Space of the conformal irrep span by $\mathrm{P_a}$:

$$\mathrm{P_{a_1} \ldots P_{a_n} |\Phi_I\rangle}$$

Generating function of conformal states (physical field on space-time)

$$\begin{aligned}\Phi_I(x)|0\rangle &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^{\mathrm{n}}}{n!} x^{\mathrm{a}_1} \ldots x^{\mathrm{a_n}} \mathrm{P_{a_1} \ldots P_{a_n} |\Phi_I\rangle} \\ &= e^{-\mathrm{i}x^{\mathrm{a}}\mathrm{P_a}} |\Phi_I\rangle\end{aligned}$$

Remark: x^{a} is a formal (harmonic) variable. The states $|\Phi_I\rangle$ are not normalizable! Operator D is hermitian but has imaginary eigenvalues.

The standard treatment of unitary irreps is based on another coset:

$$\mathrm{SO(d,2)/SO(d) \times SO(2)}$$

Set of generators

$$\mathrm{M_{rs}} \Leftarrow \text{generators of } \mathrm{SO(d)}$$
$$\mathrm{E_r^{\pm} = M_{d+1,r} \pm iM_{0r}}\,, \qquad \mathrm{r = 1, \ldots, d}$$
$$\mathrm{H = -\tfrac{1}{2}(P_0 + K_0)} \Leftarrow \text{conformal Hamiltonian}$$

Algebra

$$
\begin{aligned}
[\mathrm{E}^-_{\mathrm{r}}, \mathrm{E}^+_{\mathrm{r}}] &= 2\delta_{\mathrm{rs}}\mathrm{H} - 2\mathrm{i}\mathrm{M}_{\mathrm{rs}}\,, \\
[\mathrm{H}, \mathrm{E}^{\pm}_{\mathrm{r}}] &= \pm\mathrm{E}^{\pm}_{\mathrm{r}}\,, \qquad [\mathrm{E}^+_{\mathrm{r}}, \mathrm{E}^+_{\mathrm{s}}] = 0
\end{aligned}
$$

Conformal states are related

$$
|\Psi_I\rangle = e^{-\frac{\pi}{2}\mathrm{M}_{0\mathrm{d}}}|\Phi_I\rangle\,, \qquad \mathrm{H}|\Psi_I\rangle = \Delta|\Psi_I\rangle
$$

so that

$$
e^{\frac{\pi}{4}(\mathrm{P}_0 - \mathrm{K}_0)}(-\mathrm{i}\mathrm{D})\mathrm{e}^{-\frac{\pi}{4}(\mathrm{P}_0 - \mathrm{K}_0)} = \mathrm{H}
$$

Conformal primary state is defined

$$
\mathrm{H}|\Psi_I\rangle = \Delta|\Psi_I\rangle\,, \qquad \mathrm{E}^-_{\mathrm{r}}|\Psi_I\rangle = 0
$$

and all its descendants span a basis of positive energy irrep:

$$
\mathrm{E}^+_{\mathrm{r}_1}\dots\mathrm{E}^+_{\mathrm{r}_\mathrm{n}}|\Psi_I\rangle
$$

Conformal states (normalizable) form a positive definite matrix:

$$
\langle\Psi_I|\Psi_J\rangle
$$

Superconformal algebra $\mathfrak{su}(2,2|\mathcal{N})$

New generators

$$
\begin{array}{ccl}
\mathrm{Q}^i_\alpha, \bar{\mathrm{Q}}_{i\dot\alpha}, \mathrm{S}^\alpha_i, \bar{\mathrm{S}}^{i\dot\alpha} & -- & \text{Supercharges}: \quad i = 1, \ldots, \mathcal{N} \\
\mathrm{R}^i{}_j & -- & \text{Internal (R}-\text{) symmetry } \mathrm{U}(\mathcal{N})
\end{array}
$$

Algebra relations (only essential ones)

$$
\begin{aligned}
\{\mathrm{Q}^i_\alpha, \bar{\mathrm{Q}}_{j\dot\alpha}\} &= 2\delta^i{}_j \mathrm{P}_{\alpha\dot\alpha}\,, \qquad \mathrm{P}_{\alpha\dot\alpha} = \sigma^{\mathrm{a}}_{\alpha\dot\alpha}\mathrm{P}_{\mathrm{a}} \\
\{\bar{\mathrm{S}}^{i\dot\alpha}, \mathrm{S}^\alpha_j\} &= 2\delta^i{}_j \mathrm{K}^{\dot\alpha\alpha}\,, \qquad \mathrm{K}^{\dot\alpha\alpha} = (\bar\sigma^{\mathrm{a}})^{\dot\alpha\alpha}\mathrm{K}_{\mathrm{a}} \\
\{\mathrm{Q}^i_\alpha, \mathrm{S}^\beta_j\} &= 4\delta^i{}_j(\mathrm{M}_\alpha{}^\beta - \frac{\mathrm{i}}{2}\delta_\alpha{}^\beta \mathrm{D}) - 4\delta_\alpha{}^\beta \mathrm{R}^{\mathrm{i}}{}_{\mathrm{j}} \\
[\mathrm{R}^i{}_j, \mathrm{R}^k{}_l] &= \delta^k{}_j \mathrm{R}^i{}_l - \delta^i{}_l \mathrm{R}^k{}_j
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathrm{R}^i{}_j, \mathrm{Q}^k_\alpha] &= \delta^k{}_j \mathrm{Q}^i_\alpha - \frac{1}{4}\delta^i{}_j \mathrm{Q}^k_\alpha \\
[\mathrm{R}^i{}_j, \mathrm{S}^\alpha_k] &= -\delta^i{}_k \mathrm{S}^\alpha_j + \frac{1}{4}\delta^i{}_j \mathrm{S}^\alpha_k
\end{aligned}
$$

For $\mathcal{N} = 4$ we can impose $\mathrm{R}^i{}_i \equiv 0$. Therefore R-symmetry is $\mathfrak{su}(4)$.

The Cartan matrix for $\mathfrak{su}(4)$: $K_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

The Chevalley basis for $\mathfrak{su}(4)$:

$$[H_i, H_j] = 0, \quad [E_i^+, E_j^-] = \delta_{ij} H_j \,, \quad [H_i, E_j^\pm] = \pm K_{ji} E_j^\pm$$

An irrep of $\mathfrak{su}(4)$ is defined by its highest weight state $|a_1, a_2, a_3\rangle$:

$$\begin{aligned} E_i^+ |a_1, a_2, a_3\rangle &= 0 \\ H_i |a_1, a_2, a_3\rangle &= a_i |a_1, a_2, a_3\rangle \,, \quad a_i \geq 0 \end{aligned}$$

Here $[a_1, a_2, a_3]$ are *Dynkin labels*.

"Old" generators $\mathrm{R}^i{}_j$ can be rewritten via the generators of the Chevalley basis, e.g.

$$\mathrm{R}^1{}_1 = \frac{1}{4}(3H_1 + 2H_2 + H_3) \,, \qquad \mathrm{R}^1{}_2 = E_1^+$$

Now we see that, e.g.,

$$[H_1, \mathrm{Q}^1_\alpha] = \mathrm{Q}^1_\alpha \,, \qquad [H_{2,3}, \mathrm{Q}^1_\alpha] = [E_i^+, \mathrm{Q}^1_\alpha] = 0$$

i.e. Q^1_α is the hws of irrep $[1, 0, 0]$.

Superconformal primary state

$$|\mathrm{hws}\rangle = |\Delta; s_1, s_2; a_1, a_2, a_3\rangle$$

$$\begin{array}{rcl} \Delta & -- & \text{Conformal dimension} \\ s_1, s_2 & -- & \text{Lorentz spins} \\ a_1, a_2, a_3 & -- & \text{Dynkin labels of the } \mathfrak{su}(4) \text{ irrep} \end{array}$$

$$\begin{aligned} \mathrm{K_a}|\mathrm{hws}\rangle &= 0, \quad \mathrm{D}|\mathrm{hws}\rangle = \mathrm{i}\Delta|\mathrm{hws}\rangle, \\ \mathrm{S}_i^\alpha|\mathrm{hws}\rangle &= \bar{\mathrm{S}}^{i\dot{\alpha}}|\mathrm{hws}\rangle = 0 \end{aligned}$$

Complete basis is generated from the ordered span

$$\prod_{i,j,\alpha,\dot{\alpha}} (\mathrm{Q}^i_\alpha)^{n_{i\alpha}} (\bar{\mathrm{Q}}_{j\dot{\alpha}})^{\bar{n}_{j\dot{\alpha}}} |\mathrm{hws}\rangle$$

Dimension of a generic long multiplet

$$\dim = 2^{16} \dim(a_1, a_2, a_3)(2s_1 + 1)(2s_2 + 1)$$

The space of states can be expanded into irreps of the Lorenz and $\mathfrak{su}(4)$ algebras.

BPS states

$$\mathrm{Q}^i_\alpha|\mathrm{hws}\rangle = 0, \qquad i=1 \quad \text{or} \quad i=1,2$$

Compute the anticommutator $\{\mathrm{Q}^i_\alpha, \mathrm{S}^\beta_j\}$ on $|\mathrm{hws}\rangle$:

$$\{\mathrm{Q}^i_\alpha, \mathrm{S}^\beta_j\} \quad = \quad 4\delta^i{}_j(\mathrm{M}_\alpha{}^\beta - \frac{\mathrm{i}}{2}\delta_\alpha{}^\beta \mathrm{D}) - 4\delta_\alpha{}^\beta \mathrm{R}^{\mathrm{i}}{}_{\mathrm{j}}$$

We have

$$\mathrm{M}_\alpha{}^\beta|\mathrm{hws}\rangle = 0 \rightsquigarrow \text{no spin!}$$

and

$$\left(\frac{1}{2}\Delta\delta^i{}_j - \mathrm{R}^i{}_j\right)|\mathrm{hws}\rangle = 0\,, \qquad i=1,2$$

One gets

$$\left(\frac{1}{2}\Delta - \mathrm{R}^1{}_1\right)|\mathrm{hws}\rangle = 0 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(3a_1 + 2a_2 + a_3)$$

$$\left(\frac{1}{2}\Delta - \mathrm{R}^2{}_2\right)|\mathrm{hws}\rangle = 0 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(-a_1 + 2a_2 + a_3)$$

Thus

$$i=1 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(3a_1 + 2a_2 + a_3)$$

$$i=1,2 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(2a_2 + a_3)$$

1. We derived that

$$Q^i_\alpha|\mathrm{hws}\rangle = 0, \qquad i = 1 \quad \text{or} \quad i = 1, 2$$

results into

$$i = 1 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(3a_1 + 2a_2 + a_3)$$
$$i = 1, 2 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(2a_2 + a_3)$$

2. Simultaneous imposition

$$\bar{Q}_{j\dot{\alpha}}|\mathrm{hws}\rangle = 0, \qquad j = 4 \quad \text{or} \quad j = 3, 4$$

gives

$$j = 4 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(a_1 + 2a_2 + 3a_3)$$
$$j = 3, 4 \qquad \rightsquigarrow \qquad \Delta = \frac{1}{2}(a_1 + 2a_2)$$

Intersection of 1) and 2) allows only two cases

- $\frac{1}{4}\mathrm{BPS}: \qquad [q, p, q], \quad \Delta = p + 2q$
- $\frac{1}{2}\mathrm{BPS}: \qquad [0, p, 0], \quad \Delta = p$

Shortening and unitary bounds of $\mathcal{N}=4$ SUSY

SUSY multiplets are classified by the hws:

$$|\text{hws}\rangle = |\Delta; s_1, s_2; p, k, q\rangle$$

There are three series of UIRs

$$\begin{array}{ll} \text{A)} & \Delta \geq 2 + s_1 + s_2 + p + k + q \quad \text{(continuous)} \\ \text{B)} & \Delta \geq 1 + s_1 + p + k + q, \quad s_2 = 0, \quad 1 + s_1 \leq \dfrac{q-p}{2} \\ \text{C)} & \Delta = 2p + k, \quad q = p \quad \text{(protected)} \end{array}$$

Conclusions to Part I

- Operators from Series C) are non-renormalized even in quantum theory (consequence of representation theory).

- Series A) and B) provide only the bounds. For states saturating the bounds further (more sophisticated) shortening conditions apply. They correspond to *semi-short operators*. Generically operators saturating the bounds in free theory are driven away in quantum theory.

- Among semi-short multiplets there are very peculiar protected ones.

Literature

Part I. Representation Theory

- G. Mack and A. Salam, "Finite Component Field Representations Of The Conformal Group," Annals Phys. **53** (1969) 174.
- G. Mack, "All Unitary Ray Representations Of The Conformal Group SU(2,2) With Positive Energy," Commun. Math. Phys. **55** (1977) 1.
- V. K. Dobrev and V. B. Petkova, "All Positive Energy Unitary Irreducible Representations Of Extended Conformal Supersymmetry," Phys. Lett. B **162** (1985) 127.
- J. Erdmenger and H. Osborn, "Conserved currents and the energy-momentum tensor in conformally invariant theories for general dimensions," hep-th/9605009.
- G. Arutyunov, S. Frolov and A. C. Petkou, "Operator product expansion of the lowest weight CPOs in $N = 4$ SYM(4) at strong coupling," hep-th/0005182
- F. A. Dolan and H. Osborn, "On short and semi-short representations for four dimensional superconformal symmetry," hep-th/0209056

Part II. Integrable Structure of Gauge Theory

Gauge theory

Content: $\phi^i, A_\mu, \psi_\alpha^r$
$i = 1, \ldots, 6$ and $r = 1, \ldots, 4$.

Action:

$$S = \frac{1}{g^2}\int d^4x \mathrm{Tr}\Big\{\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(D_\mu\phi^i)^2 + \bar{\psi}\Gamma^\mu D_\mu\psi - \frac{1}{4}\sum_{ij}[\phi^i,\phi^j]^2 - \frac{i}{2}\Gamma_i[\bar{\psi},\phi^i]\psi\Big\}$$

Invariance: Susy+conformal invariance $\rightsquigarrow$ $\mathfrak{psu}(2,2|4)$ algebra

Observables: All gauge-invariant operators transform in UIRs of superconformal group:

$$[\Delta, s_1, s_2; a_1, a_2, a_2]$$

Example:

$$\mathcal{O}(x) = \mathrm{Tr}\left(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}\right) + \ldots, \qquad \Phi_1 = \phi^1 + i\phi^2, \quad \text{etc}$$

realize irrep of SU(4) with labels $[J_2 - J_3, J_1 - J_2, J_2 + J_3]$

Anomalous dimension: $\mathcal{O}(x)$ are *composite*

$$\Delta_{\text{classical}} \Longrightarrow \Delta_{\text{classical}} + \Delta(\lambda, N)$$

Spin chains and anomalous dimensions

Renormalization of the composite operators:

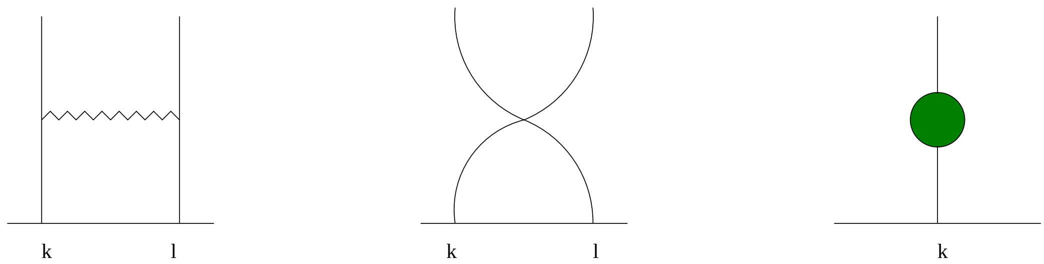
$$\mathcal{O}_I^{\rm ren} = Z_I{}^J \mathcal{O}_J\,, \qquad D = \frac{dZ}{d\ln\Lambda} Z^{-1}$$

Eigenvalues of D are anomalous dimensions.

Consider a scalar operator made of L elementary scalars

$$\mathrm{Tr}\left(\phi^{i_1}\phi^{i_2}\ldots\phi^{i_L}\right)$$

View it as the spin chain of length L where spin at every site can have 6 polarizations.



One-loop graphs

$$Z_{A\ldots i_k i_{k+1}\ldots}^{\ \ \ldots j_k j_{k+1}\ldots} = I - \frac{\lambda}{16\pi^2}\ln\Lambda\ \delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}}$$

$$Z_{B\ldots i_k i_{k+1}\ldots}^{\ \ \ldots j_k j_{k+1}\ldots} = I - \frac{\lambda}{16\pi^2}\ln\Lambda\ \left(2\delta_{i_k}^{j_{k+1}}\delta_{i_{k+1}}^{j_k} - \delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}} - \delta_{i_k i_{k+1}}\delta^{j_k j_{k+1}}\right)$$

$$Z_{C\ldots i_k i_{k+1}\ldots}^{\ \ \ldots j_k j_{k+1}\ldots} = I + \frac{\lambda}{8\pi^2}\ln\Lambda\ \delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}}$$

Valid only in the large N limit!

Sample calculation: Quartic Scalar Interaction

$$f^{ab\underline{e}}f^{cd\underline{e}}\phi_i^a\phi_j^b\phi_i^c\phi_j^d$$

$$\mathrm{Tr}\Big(\ldots\underbrace{\mathrm{T}^s\phi^s_{j_k}}_{k}\ldots\ldots\underbrace{\mathrm{T}^p\phi^p_{j_l}}_{l}\ldots\Big)$$

Using Jacobi identity:

$$f^{ab\underline{e}}f^{cd\underline{e}} = -f^{cb\underline{e}}f^{da\underline{e}} - f^{db\underline{e}}f^{ac\underline{e}}$$

We get

$$\delta_{j_kj_l}\mathrm{Tr}\Big(\ldots(f^{\underline{e}da}\phi_j^d\mathrm{T}^a)\ldots\ldots(f^{\underline{e}bc}\phi_j^b\mathrm{T}^c)\ldots\Big) =$$

$$= -\delta_{j_kj_l}\mathrm{Tr}\Big(\ldots\underbrace{[\mathrm{T}^e,\phi_j]}_{k}\ldots\ldots\underbrace{[\mathrm{T}^e,\phi_j]}_{l}\ldots\Big)$$

In the large N limit the leading contribution is only due to the term with $l = k + 1$!

We thus obtain the nearest-neighbor structure

$$N\delta_{j_kj_{k+1}}\mathrm{Tr}\Big(\ldots\underbrace{\phi_j}_{k}\underbrace{\phi_j}_{k+1}\ldots\Big) =$$

$$= N\delta_{j_kj_{k+1}}\delta^{i_ki_{k+1}}\mathrm{Tr}\Big(\ldots\phi_{i_k}\phi_{i_{k+1}}\ldots\Big)$$

Complete Z is

$$Z^{\cdots j_k j_{k+1} \cdots}_{\cdots i_k i_{k+1} \cdots} =$$
$$I + \frac{\lambda}{16\pi^2} \ln \Lambda \left(2\delta^{j_{k+1}}_{i_k} \delta^{j_k}_{i_{k+1}} - 2\delta^{j_k}_{i_k} \delta^{j_{k+1}}_{i_{k+1}} + \delta_{i_k i_{k+1}} \delta^{j_k j_{k+1}} \right)$$

Introducing the trace operator K and the permutation operator P:

$$K = \delta_{i_k i_{k+1}} \delta^{j_k j_{k+1}} , \qquad P = \delta^{j_{k+1}}_{i_k} \delta^{j_k}_{i_{k+1}}$$

the matrix of anomalous dimensions becomes

$$\mathrm{D}_{1-\mathrm{loop}} = g^2 \sum_{i=1}^{L} \left(I - P_{i,i+1} + \frac{1}{2} K_{i,i+1} \right)$$

where

$$g^2 = \frac{\lambda}{8\pi^2} = \frac{g^2_{\mathrm{YM}} N}{8\pi^2}$$

Let us define the dilatation operator as

$$\mathrm{D} = L + g^2 \mathbf{H}$$

where

$$\mathbf{H} = \sum_{i=1}^{L} \left(I - P_{i,i+1} + \frac{1}{2} K_{i,i+1} \right)$$

is the Hamiltonian of the SO(6) *integrable* spin chain!

Reduction to the $\mathfrak{su}(2) \subset \mathfrak{su}(4) \simeq \mathfrak{so}(6)$

One puts $K = 0$ and identifies

$$P = \frac{1}{2}\Big(I \otimes I + \sum \sigma^{\alpha} \otimes \sigma^{\alpha}\Big)$$

The Hamiltonian of the $\mathfrak{su}(2)$ spin chain is

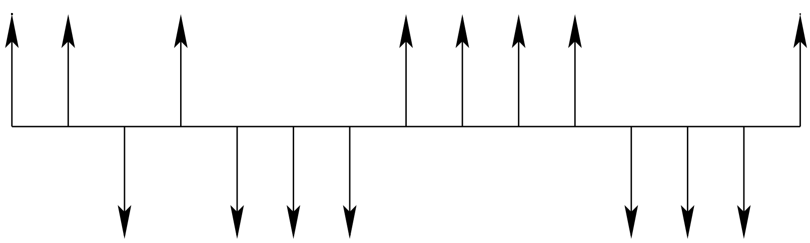
$$\mathbf{H} = \sum_{i=1}^{L} \Big(I - P_{i,i+1}\Big)$$

This is the famous $\mathrm{XXX}_{1/2}$ Heisenberg spin chain!

Alternatively one can find the dilatation operator by renormalizing the operators

$$\mathrm{Tr}(Z^{L-M}\Phi^{M}) + \ldots,$$

where Z and Φ are two complex scalars.



The Heisenberg spin chain. The Hamiltonian H acts as $2^L \times 2^L$ matrix, where L is the length of the chain. M is a number of magnons.

Quantum inverse scattering method

The Lax operator

$$\mathcal{L}_n(\varphi) = \frac{1}{\varphi + i} \left(\begin{array}{cc} \varphi + \frac{i}{2} + iS_n^3 & iS_n^- \\ iS_n^+ & \varphi + \frac{i}{2} - iS_n^3 \end{array} \right) , \qquad S^\alpha = \frac{1}{2}\sigma^\alpha$$

is a 2×2 matrix in the auxiliary two-dimensional space.

Monodromy around the chain

$$\mathrm{T}(\varphi) = \mathcal{L}_L(\varphi) \ldots \mathcal{L}_1(\varphi) = \left(\begin{array}{cc} A(\varphi) & B(\varphi) \\ C(\varphi) & D(\varphi) \end{array} \right)$$

Fundamental commutation relations

$$\mathrm{R}_{12}(\varphi - \varphi')\mathrm{T}_1(\varphi)\mathrm{T}_2(\varphi') = \mathrm{T}_2(\varphi')\mathrm{T}_1(\varphi)\mathrm{R}_{12}(\varphi - \varphi') \, ,$$

where

$$\mathrm{T}_1(\varphi) = \mathrm{T}(\varphi) \otimes I \, , \qquad \mathrm{T}_2(\varphi) = I \otimes \mathrm{T}(\varphi)$$

The Yang-Baxter equation

$$\mathrm{R}_{12}\mathrm{R}_{13}\mathrm{R}_{23} = \mathrm{R}_{23}\mathrm{R}_{13}\mathrm{R}_{12}$$

Explicit form of the R-matrix

$$\mathrm{R}_{12}(\varphi) = \frac{\varphi}{\varphi + i} I_{12} + \frac{i}{\varphi + i} P_{12} \equiv \mathcal{L}(\varphi) \, , \qquad \mathrm{R}_{12}(0) = P_{12}$$

Commuting charges

$$[\mathrm{tr}\mathrm{T}(\varphi), \mathrm{tr}\mathrm{T}(\varphi')] = 0$$

Monodromy at $\varphi = 0$:

$$\begin{aligned}\mathrm{trT}(0) &= \mathrm{Tr}\Big(P_{L,a}...P_{L,1}\Big) = \mathrm{Tr}\Big(P_{12}P_{23}...P_{L-1,L}P_{L,a}\Big) = \\ &= P_{12}P_{23}...P_{L-1,L} = \mathcal{U} \leftarrow \text{shift} \ \ \text{operator}\end{aligned}$$

Local commuting charges can be introduced as

$$\mathrm{trT}(\varphi) = \mathcal{U} \exp\Big(i \sum_{r=2}^{\infty} \varphi^{r-1} \mathbf{Q}_r \Big)$$

Simultaneous eigenstates of $\mathbf{Q}_k$

$$C(\varphi)\Omega = 0; \qquad A(\varphi)\Omega = a(\varphi)\Omega; \qquad D(\varphi)\Omega = d(\varphi)\Omega;$$

$$\Psi = B(\varphi_1)...B(\varphi_M)\Omega \, ,$$

the Bethe roots φ_k are found by solving the Bethe equations

$$\left(\frac{\varphi_k + \frac{i}{2}}{\varphi_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{\varphi_k - \varphi_j + i}{\varphi_k - \varphi_j - i} \, ,$$

$$\mathcal{U} = \prod_{j=1}^{M} \frac{\varphi_k + \frac{i}{2}}{\varphi_k - \frac{i}{2}} = \mathbf{I} \leftarrow \text{cyclisity} \ \ \text{of} \ \ \text{the} \ \ \text{trace!}$$

On the state with M magnons one gets

$$\exp\Big(i \sum_{r=2}^{\infty} \varphi^{r-1} \mathbf{Q}_r \Big) = \prod_{k=1}^{M} \frac{\varphi - \varphi_k - \frac{i}{2}}{\varphi - \varphi_k + \frac{i}{2}} + \Big(\frac{\varphi}{\varphi + i}\Big)^L \prod_{k=1}^{M} \frac{\varphi - \varphi_k + \frac{3i}{2}}{\varphi - \varphi_k + \frac{i}{2}}$$

For $\mathbf{Q}_r$ with $r \leq L$ the second term on the rhs can be put to zero.

One gets

$$\mathbf{Q}_r = \sum_{k=1}^{M} \mathbf{q}_r(\varphi_k) \,,$$

where $\mathbf{q}_r(\varphi_k)$ are local excitation charges

$$\mathbf{q}_r(\varphi_k) = \frac{i}{r-1} \left(\frac{1}{\left(\varphi_k + \frac{i}{2}\right)^{r-1}} - \frac{1}{\left(\varphi_k - \frac{i}{2}\right)^{r-1}} \right)$$

An eigenvalue of the Hamiltonian is

$$\mathbf{H} = \mathbf{Q}_2 = \sum_{k=1}^{M} \mathbf{q}_2(\varphi_k) = \sum_{k=1}^{M} \frac{1}{\varphi_k^2 + \frac{1}{4}}$$

Particle interpretation of the Bethe Ansatz: a magnon is a particle with momentum p_k

$$\exp(ip_k) = \frac{\varphi_k + \frac{i}{2}}{\varphi_k - \frac{i}{2}}$$

The Bethe Ansatz reads

$$\exp(iLp_k) = \prod_{\substack{j=1 \\ j\neq k}}^{M} \mathrm{S}(p_k, p_j) \,, \qquad \sum_{k=1}^{M} p_k = 0$$

Here $\mathrm{S}(p_k, p_j)$ is the S-matrix for pairwise scattering of local excitations with momenta p_k:

$$\mathrm{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} \,, \qquad \varphi(p) = \tfrac{1}{2} \cot(\tfrac{1}{2} p)$$

Thermodynamic Limit (TDL)

$M, L \to \infty$ with $\frac{M}{L} = \alpha$ fixed,

$$\omega^2 = \frac{g^2}{2L^2} \equiv \frac{\lambda}{16\pi^2 L^2} \quad \text{fixed}$$

One rescale the roots $\varphi_k \to L\varphi_k$ and takes the $\log$ of the Bethe equations:

$$L \log\left(\frac{\varphi_k + \frac{i}{2L}}{\varphi_k - \frac{i}{2L}}\right) + 2\pi i n = \sum_{j\neq k}^{M} \log\left(\frac{\varphi_k - \varphi_j + \frac{i}{L}}{\varphi_k - \varphi_j - \frac{i}{L}}\right), \qquad L \to \infty$$

One finds

$$\pi n + \frac{1}{2\varphi_k} = \frac{1}{L}\sum_{j\neq k}^{M} \frac{1}{\varphi_k - \varphi_j} \equiv \fint_{\mathbf{C}} \frac{d\varphi'}{\varphi_k - \varphi'} \left(\frac{1}{L}\sum_{j=1}^{M} \delta(\varphi' - \varphi_j)\right)$$

One introduces the distribution density $\rho(\varphi)$

$$\rho(\varphi) = \frac{1}{L}\sum_{j=1}^{M} \delta(\varphi - \varphi_j), \qquad \int_{\mathbf{C}} d\varphi\, \rho(\varphi) = \frac{M}{L}$$

and gets the integral Bethe equations

$$\fint_{\mathbf{C}} d\varphi' \frac{\rho(\varphi')\,\varphi}{\varphi' - \varphi} = \frac{1}{2} + \pi n_{\mathbf{C}}\, \varphi, \qquad \int_{\mathbf{C}} d\varphi \frac{\rho(\varphi)}{\varphi} = 0.$$

Charges in the TDL limit. Resolvent.

If we rescale the roots as $\varphi_k \longrightarrow L\varphi_k$ and take the limit $L \longrightarrow \infty$ we find

$$\mathbf{q}_r(\varphi_k) = \frac{L^{-r}}{\varphi_k^r}$$

The total charges

$$\mathbf{Q}_r \quad = \quad \sum_{k=1}^{M} \mathbf{q}_r(\varphi_k) = \int \mathrm{d}\varphi \sum_{k=1}^{M} \mathbf{q}_r(\varphi)\delta(\varphi - \varphi_k) =$$

$$= L^{-r+1} \underbrace{\int_{\mathbf{C}} d\varphi \frac{\rho(\varphi)}{\varphi^r}}_{\text{moments of the density}}$$

In the TDL limit one defines the rescaled charges

$$\mathbf{q}_r(\varphi_k) \longrightarrow L^r \mathbf{q}_r(\varphi_k)\,, \qquad \mathbf{Q}_r \longrightarrow L^{r-1}\mathbf{Q}_r$$

Resolvent – the generating function of commuting charges

$$\mathbf{G}(\varphi) = \int_{\mathbf{C}} d\varphi' \rho(\varphi') \frac{\varphi'}{\varphi' - \varphi} = \sum_{k=0}^{\infty} \mathbf{Q}_k \varphi^k$$

It is an analytic function on the complex plane with cuts

$$\mathbf{Q}_0 = \alpha, \qquad \mathbf{Q}_1 = 0 \leftarrow \text{total momentum}, \qquad \mathbf{Q}_2 \leftarrow \text{energy}$$

Integrability and degeneracy of the spectrum

There exists a distinguished (third) charge

$$\mathbf{Q}_3(g) = \sum_{k=0}^{\infty} \mathbf{Q}_{3,2k-2}\, g^{2k-2}$$

There exists discrete charge – the parity $\mathbf{P}$:

$$\mathbf{P}\mathrm{Tr}(Z^3\Phi^2 Z\Phi) = \mathrm{Tr}(\Phi Z\Phi^2 Z^3)$$

One has

$$[\mathrm{D}, \mathbf{Q}_3] = 0 = [\mathbf{P}, \mathrm{D}], \qquad \text{but} \qquad [\mathbf{P}, \mathbf{Q}_3] \neq 0 \quad !!!$$

$\mathbf{P}$ and $\mathbf{Q}_3$ cannot be simultaneously diagonalized $\rightsquigarrow$ degeneracy of the spectrum.

Example of degenerate pair:

$$\begin{aligned}
\mathcal{O}_+^{[3,1,3]} &= 2\mathrm{Tr}(\Phi\Phi\Phi ZZZZ) - 3\mathrm{Tr}(\Phi\Phi Z\Phi ZZZ) + 2\mathrm{Tr}(\Phi\Phi ZZ\Phi ZZ) \\
&- 3\mathrm{Tr}(\Phi\Phi ZZZ\Phi Z) + 2\mathrm{Tr}(\Phi Z\Phi Z\Phi ZZ) \\
\mathcal{O}_-^{[3,1,3]} &= -\mathrm{Tr}(\Phi\Phi Z\Phi ZZZ) + \mathrm{Tr}(\Phi\Phi ZZZ\Phi Z)
\end{aligned}$$

Existence of degenerate pairs is a highly efficient criterion to check integrability at higher loops!

Higher Loops

Closed subsectors for the dilatation operator D:

- $\mathfrak{su}(2)$. Operators with $[p,q,p]$ and $\Delta_0 = 2p+q$ can be made only from two complex scalars Z and Φ. Classically they are $\frac{1}{4}$-BPS states. Operators:

  $$\mathrm{Tr}(Z^{L-M}\Phi^M) + \ldots,$$

  where Z and Φ are two complex scalars.

- $\mathfrak{sl}(2)$ Operators $\mathrm{Tr}(\mathcal{D}^M Z^L)+\ldots$, where $\mathcal{D} = \mathcal{D}_1 + i\mathcal{D}_2$

- $\mathfrak{su}(1,1)$ The Dynkin content: $[0,0,L]$ and $\Delta_0 = \frac{3}{2}L + M$. Operators $\mathrm{Tr}(\mathcal{D}^M \Psi^L) + \ldots$.

- $\mathfrak{su}(2|3)$: Three complex scalars, two complex fermions. Classically they are $\frac{1}{8}$-BPS states. Operators:

  $$\mathrm{Tr}(\Phi_1^{n_1}\Phi_2^{n_2}\Phi_3^{n_3}\Psi_1^{n_4}\Psi_2^{n_5}) + \ldots,$$

The dilatation operator on the $\mathfrak{su}(2)$ subsector (up to three loops)

$$
\begin{aligned}
\mathrm{D}_{1\ell} &= I - P_{i,i+1} \\
\mathrm{D}_{2\ell} &= -\frac{3}{2}I + 2P_{i,i+1} - \frac{1}{2}P_{i,i+2} \\
\mathrm{D}_{3\ell} &= 5I - 7P_{i,i+1} + 2P_{i,i+2} \\
&- \frac{1}{2}(P_{i,i+3}P_{i+1,i+2} - P_{i,i+2}P_{i+1,i+3})
\end{aligned}
$$

This was obtained by careful analysis of the Feynman graphs + susy input

Extremely important feature: the anomalous dimensions and the higher charges obey the "BMN scaling":

$$\Delta = \Delta\Big(\omega^2, L\Big) \quad \text{is finite when } L \to \infty$$

where

$$\omega^2 = \frac{g^2}{2L} = \frac{\lambda}{16\pi^2 L^2} - -\text{the BMN coupling}$$

There is an heuristic all-loop asymptotic Bethe ansatz but its formulation will be postponed till string theory section

Literature

Part II. Gauge Theory and Comparison

- J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," hep-th/0212208.
- N. Beisert, C. Kristjansen and M. Staudacher, "The dilatation operator of N = 4 super Yang-Mills theory," hep-th/0303060;
- N. Beisert, "The complete one-loop dilatation operator of N = 4 super Yang-Mills theory," hep-th/0307015.
- N. Beisert and M. Staudacher, "The N = 4 SYM integrable super spin chain," hep-th/0307042.
- N. Beisert, "Higher loops, integrability and the near BMN limit," hep-th/0308074.
- N. Beisert, "The su(2|3) dynamic spin chain," hep-th/0310252.
- G. Arutyunov and M. Staudacher, "Matching higher conserved charges for strings and spins," 0310182
- G. Arutyunov and M. Staudacher, "Two-loop commuting charges and the string / gauge duality," hep-th/0403077.
- N. Beisert, V. Dippel and M. Staudacher, "A novel long range spin chain and planar N = 4 super Yang-Mills," hep-th/0405001.

Part III. Integrable Structure of String Theory

String sigma model (classical bosonic)

- The action is a sum of SO(2,4) and SO(6) sigma models

$$I = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma (L_S + L_{AdS}) \ ,$$

where

$$\begin{aligned} L_S &= -\frac{1}{2}\partial_a X_M \partial^a X_M + \frac{1}{2}\Lambda(X_M X_M - 1)\,, \\ L_{AdS} &= -\frac{1}{2}\eta_{MN}\partial_a Y_M \partial^a Y_N + \frac{1}{2}\tilde{\Lambda}(\eta_{MN} Y_M Y_N + 1) \end{aligned}$$

$$X_M, \quad M = 1, \ldots, 6; \qquad Y_M, \quad M = 0, \ldots, 5$$

are the the embedding coordinates; Λ and $\tilde{\Lambda}$ are Lagrange multipliers.

- Virasoro constraints
- Periodicity conditions (closed strings)

$$X_M(\sigma + 2\pi) = X_M(\sigma) \ , \qquad Y_M(\sigma + 2\pi) = Y_M(\sigma) \ .$$

- Cartan sates with only three non-vanishing components

$$J_1 = J_{12}, \quad J_2 = J_{34}, \quad J_3 = J_{56}$$

of the SO(6) angular momentum are relevant

Reduction to the Neumann system. Rigid strings

The simplest case: string rotates in S^5 and is trivially embedded in AdS_5 as $Y_5 + iY_0 = e^{i\kappa\tau}$ with $Y_1, ..., Y_4 = 0$.

Ansatz for periodic motion with three J_i non-zero:

$$X_1 + iX_2 = x_1(\sigma)\, e^{iw_1\tau} ,$$
$$X_3 + iX_4 = x_2(\sigma)\, e^{iw_2\tau} ,$$
$$X_5 + iX_6 = x_3(\sigma)\, e^{iw_3\tau} ,$$

where

$$X_M^2 = 1 \quad \Longrightarrow \quad \sum_{i=1}^{3} x_i^2 = 1 .$$

Sigma model evolution equations reduce to the 1-d ("mechanical") system (σ is time now !)

$$x_i'' = -w_i^2 x_i - x_i \sum_{j=1}^{3} \left(x_j'^2 - w_j^2 x_j^2 \right) .$$

This is the famous *Neumann dynamical system*

The crucial point is the existence of the additional (to the energy H) integral of motion:

$$F_i = x_i^2 + \sum_{j \neq i} \frac{(x_i \pi_j - x_j \pi_i)^2}{w_i^2 - w_j^2} , \qquad \sum_{i=1}^{3} F_i = 1 ,$$

$$H = \frac{1}{2} \sum_{i=1}^{3} (x_i'^2 + w_i^2 x_i^2) = \frac{1}{2} \sum_{i=1}^{3} w_i^2 F_i .$$

Sigma model solitons (rigid strings) are described in terms of periodic solutions of the Neumann system

Angular momentum components (spins):

$$J_i = \sqrt{\lambda} w_i \int_0^{2\pi} \frac{d\sigma}{2\pi} x_i^2(\sigma)$$

From here

$$w_i \equiv w_i(J_1, J_2, J_3)$$

Space-time energy E:

$$E = \sqrt{\lambda}\kappa$$

The only non-trivial Virasoro constraint is (dot and prime are derivatives over τ and σ)

$$\frac{E^2}{\lambda} = \dot{X}_M \dot{X}_M + X'_M X'_M \,.$$

As a consequence of this relation the energy is a function of the SO(6) spins:

$$E = E(J_1, J_2, J_3; \lambda)\,.$$

The general three spin solutions are described in terms of the hyperelliptic functions. Elliptic two spin solutions ($x_3(\sigma) = 0$) correspond to folded or circular strings.

Folded string solution

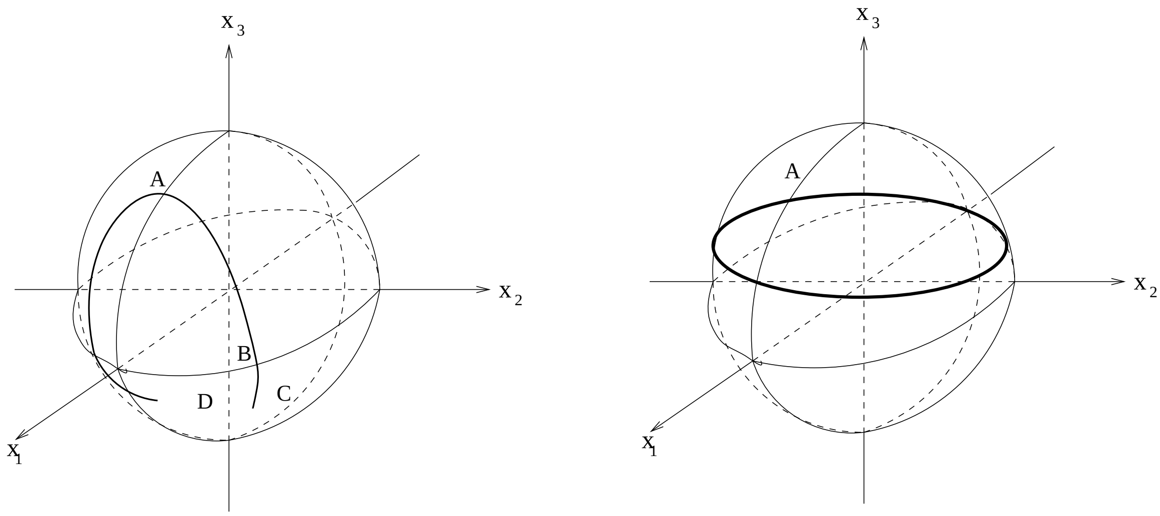
$$x_1(\sigma) = \mathsf{dn}\Big(\sigma\sqrt{w_{21}^2},\ t\Big)\ , \quad x_2(\sigma) = \sqrt{t}\ \mathsf{sn}\Big(\sigma\sqrt{w_{21}^2},\ t\Big)$$

and

$$\frac{\pi}{2}\sqrt{w_{21}^2} = \mathrm{K}(t) \qquad\qquad w_{ij}^2 \equiv w_i^2 - w_j^2$$

Solving for w's in terms of spins J_i, the modulus t can be further found from

$$\left(\frac{J_2}{\mathrm{K}(t) - \mathrm{E}(t)}\right)^2 - \left(\frac{J_1}{\mathrm{E}(t)}\right)^2 = \frac{4}{\pi^2}\lambda\,.$$



Folded and circular rigid strings (generically hyperelliptic).

Gauge and String Theories Have the Same Integrable Structure up to Two Loops

Bäcklund Transform of Rigid (Neumann) Strings

[Arutyunov and Staudacher, hep-th/0310182, hep-th/0403077]

Classical String Bethe Equation

[Kazakov, Marshakov, Minahan and Zarembo, hep-th/0402207]

Ferromagnetic Sigma Model

[Kruczenski, hep-th/0311203; Kruczenski and Tseytlin, hep-th/0406189]

Quantum String Bethe Equations

Classical String Bethe Equation

[Kazakov, Marshakov, Minahan and Zarembo, hep-th/0402207]

Gauge Theory Asymptotic Bethe Ansatz

[Beisert, Dippel and Staudacher, hep-th/0405001]

Strings on $\mathbb{R} \times S^3 \approx \mathbb{R} \times$ SU(2)

$$Y_1 = \ldots = Y_4 = X_5 = X_6 = 0; \qquad Y_5 + iY_6 = \exp(i\kappa\tau)$$

Combine

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} \in \mathrm{SU}(2)$$

and make the $\mathfrak{su}(2)$-current

$$A_\tau = g^\dagger \partial_\tau g, \qquad A_\sigma = g^\dagger \partial_\sigma g$$

Construct the φ-dependent matrices U and V:

$$\begin{aligned} U &= \frac{\varphi}{1-\varphi^2} A_\tau + \frac{1}{1-\varphi^2} A_\sigma \\ V &= -\frac{1}{1-\varphi^2} A_\tau - \frac{\varphi}{1-\varphi^2} A_\sigma \end{aligned}$$

Zero-curvature (integrability) condition

$$\partial_\tau U - \partial_\sigma V + [U, V] = 0$$

Introducing $D_\tau = \partial_\tau - V$ and $D_\sigma = \partial_\sigma - U$:

$$[D_\tau, D_\sigma] = 0$$

which is a compatibility condition for

$$D_\tau \Psi = 0\,, \qquad D_\sigma \Psi = 0$$

Monodromy matrix

$$\mathsf{T}(\varphi) = \mathsf{P}\exp\int_0^{2\pi} U(\sigma,\varphi)\mathrm{d}\varphi$$

Quasi-momentum

$$p(\varphi) = \arccos\left(\frac{1}{2}\mathrm{Tr}\ \mathrm{T}(\varphi)\right)$$

Asymptotics

$$p(\varphi) \overset{\varphi\to\pm1}{\Longrightarrow} -\frac{\pi\kappa}{\varphi\pm1}$$

Spectral density and the resolvent

$$p(\varphi) + \frac{\pi\kappa}{\varphi-1} + \frac{\pi\kappa}{\varphi+1} = \int_{\mathbf{C}} \frac{d\varphi'\rho_{\,\mathsf{s}}(\varphi')}{\varphi-\varphi'} \equiv \mathbf{G}(\varphi)$$

Classical String Bethe Equation

$$\fint_{\mathrm{C}} \frac{d\varphi' \rho_{\mathrm{s}}(\varphi')}{\varphi - \varphi'} = \frac{1}{2}\frac{1}{\sqrt{\varphi^2 - 4\omega^2}} + \pi n_\nu +$$

$$+ \omega^2 \int_{\mathrm{C}} \frac{d\varphi' \rho_{\mathrm{s}}(\varphi')}{\sqrt{\varphi^2 - 4\omega^2}\sqrt{\varphi'^2 - 4\omega^2}} \times$$

$$\times \frac{\varphi - \sqrt{\varphi^2 - 4\omega^2} - \varphi' + \sqrt{\varphi'^2 - 4\omega^2}}{(\varphi + \sqrt{\varphi^2 - 4\omega^2})(\varphi' + \sqrt{\varphi'^2 - 4\omega^2}) - 4\omega^2}$$

CSBE

Spectral density $\rho_{\mathrm{s}}(\varphi)$ of a finite-gap solution of the string sigma-model is normalized as

$$\int_{\mathrm{C}} d\varphi \; \rho_{\mathrm{s}}(\varphi) = \frac{M}{L} = \alpha \, .$$

Effective coupling constant ω is $\omega^2 = \frac{g^2}{2L^2}$.

The integers n_ν are winding numbers.

What is the quantum (discrete) analog of the CSBE? This should describe the *quantum* corrections of the string sigma model.

Quantum String Bethe Equation

The BE we propose to describe the leading quantum effects for strings in the $\mathfrak{su}(2)$ sector

$$\exp(iLp_k) = \prod_{\substack{j=1\\ j\neq k}}^{M} \mathrm{S}(p_k, p_j)\,, \qquad \sum_{k=1}^{M} p_k = 0$$

Here $\mathrm{S}(p_k, p_j)$ is the S-matrix for pairwise scattering of local excitations with momenta p_k

$$\mathrm{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} \times$$
$$\times \exp\left(2i \sum_{r=0}^{\infty} \left(\frac{g^2}{2}\right)^{r+2} \mathfrak{q}_{r+2}(p_{[k}) \, \mathfrak{q}_{r+3}(p_{j]})\right)$$

QSBE

[Arutyunov, Frolov, and Staudacher, hep-th/0406256]

$$\mathrm{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i}$$

Gauge theory ABAE

[Beisert, Dippel and Staudacher, hep-th/0405001]

Here the phase function $\varphi(p)$ is

$$\varphi(p) = \tfrac{1}{2}\cot(\tfrac{1}{2}p)\sqrt{1 + 8g^2\sin^2(\tfrac{1}{2}p)}\ ,$$

or inverse

$$p := p(\varphi)$$

Relation between g and the 't Hooft coupling λ:

$$g^2 = \frac{\lambda}{8\pi^2}$$

The charges $\mathbf{q}_r(p)$ are

$$\mathbf{q}_r(p) = g^{-r+1}\frac{2\sin(\frac{r-1}{2}p)}{r-1}\left(\frac{\sqrt{1 + 8g^2\sin^2(\frac{1}{2}p)}\ -\ 1}{2g\sin(\frac{1}{2}p)}\right)^{r-1}$$

In particular,

$$\begin{aligned}
\mathbf{q}_1(p) &= p \quad \Leftarrow \quad \text{Momentum} \\
\mathbf{q}_2(p) &= \frac{1}{g^2}\left(\sqrt{1 + 8g^2\sin^2(\tfrac{1}{2}p)} - 1\right) \Leftarrow \text{Energy}
\end{aligned}$$

The total charge of the M excitations given by the sum of individual charges

$$\mathbf{Q}_r = \sum_{k=1}^{M} \mathbf{q}_r(p_k)$$

The energy of string states in the global AdS coordinates is

$$\mathbf{E}(g) = L + g^2 \mathbf{Q}_2$$

Two limiting cases

- Thermodynamic Limit (TDL):
  $M, L \to \infty$ with $\frac{M}{L} = \alpha$ fixed, $\omega^2 = \frac{g^2}{2L^2}$ fixed

- BMN Limit:
  $M = 2, 3, \ldots$ finite, $L = M + J \to \infty$, $\lambda' = \frac{\lambda}{J^2}$ fixed

Motivation

In the TDL the charge densities $\mathbf{q}_r(\varphi)$ become

$$\mathbf{q}_r(\varphi) = \frac{1}{\sqrt{\varphi^2 - 4\omega^2}} \frac{1}{\left(\frac{1}{2}\varphi + \frac{1}{2}\sqrt{\varphi^2 - 4\omega^2}\right)^{r-1}}$$

The total commuting charges are

$$\mathbf{Q}_r = \int_{\mathbf{C}} d\varphi \; \rho_{\mathrm{s}}(\varphi) \; \mathbf{q}_r(\varphi)$$

Crucial observation:

The CSBE can be cast into the form $\quad (\omega^2 = \frac{g^2}{2L^2})$

$$⨍_{\mathbf{C}} \frac{d\varphi' \rho_{\mathrm{s}}(\varphi')}{\varphi - \varphi'} = \pi n_\nu + \frac{1}{2}\mathbf{q}_1(\varphi) + \sum_{r=0}^{\infty} \omega^{2r+4} \mathbf{q}_{[r+3}(\varphi) \mathbf{Q}_{r+2]}$$

Discrete version

$$\prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} =$$

$$= \exp\left(iLp_k + 2i \sum_{r=0}^{\infty} \left(\frac{g^2}{2}\right)^{r+2} \mathbf{q}_{[r+3}(p_k)\mathbf{Q}_{r+2]} \right)$$

To obtain the integral Bethe equations in the TDL introduce the distribution density

$$\rho(\varphi) = \frac{1}{L} \sum_{k=1}^{M} \delta(\varphi - \varphi(p_k))\,, \qquad \int_{\mathbf{C}} d\varphi\ \rho(\varphi) = \frac{M}{L}$$

Taking the $\log$ of the discrete equation and passing to the TDL we obtain the CSBE.

Near-BMN Limit

The BMN limit:

$$M \to \text{finite}, \quad L = M + J \sim J \to \infty, \quad \lambda' = \frac{\lambda}{J^2} \to \text{fixed}$$

Scaling in the BMN limit

$$p_k \sim 1/J, \qquad \varphi(p) \sim J$$

No scattering of local excitations in the BMN limit.
BEs and their solution

$$e^{ip_k J} = 1 \quad \to \quad p_k = \frac{2\pi n_k}{J}, \qquad \sum_{k=1}^{M} n_k = 0$$

Anomalous dimension

$$\Delta = \Delta_{\mathrm{g}} = J + \sum_{k=1}^{M} \sqrt{1 + \lambda' \, n_k^2}$$

coincides with the BMN energy formula.

The result is the same for both QSBE and the gauge theory ABE.

The near-BMN limit:

One expands (for non-coinciding n_k)

$$p_k = \frac{2\pi n_k}{J} + \frac{p_k^{(2)}}{J^2}$$

and works out the leading large J behavior of the phases

$$\varphi(p_k) = \frac{J}{2\pi n_k}\sqrt{1+\lambda' n_k^2} + \mathcal{O}(J^0)$$

and the charges

$$\mathbf{q}_r(p_k) = \frac{2\pi n_k}{J}\left[\frac{4\pi}{\lambda' J}\left(\sqrt{1+\lambda' n_k^2}-1\right)\right]^{r-1} + \mathcal{O}(J^{-r-1})$$

The momentum shift

$$\frac{p_k^{(2)}}{2\pi} = -\sum_{\substack{j=1\\ j\neq k}}^{M} \frac{n_k^2\sqrt{1+\lambda' n_j^2} + n_j^2\sqrt{1+\lambda' n_k^2}}{n_k - n_j}$$

Energy formula for M excitations from QSBE

$$\Delta = J + \sum_{k=1}^{M} \sqrt{1+\lambda' n_k^2} - \frac{\lambda'}{J} \sum_{\substack{k,j=1 \\ j\neq k}}^{M} \frac{n_k}{n_k - n_j} \left(n_j^2 + n_k^2 \sqrt{\frac{1+\lambda' n_j^2}{1+\lambda' n_k^2}} \right)$$

Reproduces exactly (!) the cases $M = 2, 3$ of direct quantization near plane-wave background by Callan, Lee, McLoughlin, Schwarz, Swanson and Wu, hep-th/0307032,0404007,0405153

Anom. dimension for M excitations from ABAE

$$\Delta_{\mathrm{g}} = J + \sum_{k=1}^{M} \sqrt{1+\lambda' n_k^2} - \frac{\lambda'}{J} \sum_{k=1}^{M} \frac{M\, n_k^2}{\sqrt{1+\lambda' n_k^2}} - \frac{\lambda'}{J} \sum_{\substack{k,j=1 \\ j\neq k}}^{M} \frac{2 n_k^2 n_j}{n_k^2 - n_j^2} \left(n_j + n_k \sqrt{\frac{1+\lambda' n_j^2}{1+\lambda' n_k^2}} \right)$$

Strong Coupling Limit $\lambda \to \infty$

We expect to find the famous asymptotics $\sqrt[4]{\lambda}$

Simplest case: $M = 2$

In the large g limit one has

$$\mathbf{q}_r(p) \to g^{-r+1} \frac{\sin(\frac{1}{2}(r-1)p)}{(r-1)} 2^{\frac{1}{2}(r+1)}$$

The phase function $\varphi(p) \to \infty$ when $g \to \infty$ limit.

The QSBE reduces to

$$Lp - 8\sqrt{2}\, g\, \cos(\tfrac{1}{2}p)\, \log(\cos(\tfrac{1}{2}p)) = 0$$

The momentum p has an expansion

$$p = \frac{p_0}{\sqrt{g}} + \frac{p_1}{g} + \ldots; \qquad L \ll \sqrt[4]{\lambda}$$

Solution

$$p_0 = 2^{\frac{1}{4}}\sqrt{n\pi}.$$

The leading large λ asymptotics of Δ is

$$\Delta = 2\left(n^2\lambda\right)^{\frac{1}{4}}$$

"Gauge theory" operators are dual to string modes with masses: $m^2 = 4n\sqrt{\lambda}$!

<u>Generic case of M excitations</u>

$$p_k : \qquad \sum_{k=1}^{M} p_k = \sum_{k=1}^{m} p_k^{+} + \sum_{k=m+1}^{M} p_k^{-} = 0; \qquad \text{Re}\, p_k^{\pm} > 0$$

The scaling dimension is

$$\Delta = 2\left(\left(\sum_{k=1}^{m} n_k\right)^2 \lambda\right)^{\frac{1}{4}}$$

"Gauge theory" operators are dual to string modes with masses: $m^2 = 4n\sqrt{\lambda}$, $n = \sum_{k=1}^{m} n_k$

Where is the Gauge/String Correspondence: Interpolating Bethe Ansatz?

$$\mathsf{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} \times$$
$$\times \quad \exp\left(2i \sum_{r=0}^{\infty} c_r(g, L)\, \mathfrak{q}_{r+2}(p_{[k}) \, \mathfrak{q}_{r+3}(p_{j]})\right)$$

From the gauge theory point of view the role of $c_r(g, L)$ is to hopefully account for the wrapping interactions

The required properties:

- $c_r(g, L) \to 0$ in the asymptotic limit, $L \to \infty$ and g is held finite;
- $c_r(g, L) \sim \mathcal{O}(g^{2(L-1)})$ in the perturbative gauge theory, $g \ll 1$ and L is finite;
- $c_r(g, L) \to \left(\frac{g^2}{2}\right)^{r+2}$ in the limit $L, g \to \infty$ and $\frac{g}{L}$ is held finite;
- $c_r(g, L) \to \left(\frac{g^2}{2}\right)^{r+2}$ in the strong coupling limit, $g \to \infty$ and $1 \ll L \ll \sqrt{g}$.

Literature

Part III. String Theory and Comparison

- D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from N $= 4$ super Yang Mills," hep-th/0202021.
- S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "A semi-classical limit of the gauge/string correspondence," hep-th/0204051.
- S. Frolov and A. A. Tseytlin, "Multi-spin string solutions in $AdS_5 \times S^5$," hep-th/0304255.
- G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, "Spinning strings in $AdS_5 \times S^5$ and integrable systems," hep-th/0307191
- G. Arutyunov, J. Russo and A. A. Tseytlin, "Spinning strings in $AdS_5 \times S^5$: New integrable system relations," hep-th/0311004
- C. G. . Callan, H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson and X. Wu, "Quantizing string theory in $AdS_5 \times S^5$: Beyond the pp-wave," hep-th/0307032.
- T. McLoughlin and I. Swanson, "N-impurity superstring spectra near the pp-wave limit," hep-th/0407240.
- V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, "Classical / quantum integrability in AdS/CFT," hep-th/0402207.
- G. Arutyunov, S. Frolov and M. Staudacher, "Bethe ansatz for quantum strings," hep-th/0406256.

Problems

- Neither the *complete dilatation operator* nor the corresponding Bethe Ansatz are known beyond one loop. Length of the chain is not conserved.

- What happens to integrability when the number of loops start to exceed the (classical) length of the chain? No insight on the structure of *wrapping interactions*.

- The BMN scaling is proved rigorously in gauge theory *only up to 3 loops*.

- How integrability is related to conformality of the theory?

- The classical string Bethe equations for the whole PSU(2,2|4) sigma-model are yet unknown.

- How to *derive* the classical string Bethe Ansatz from actual quantization of string?

- Get $1/J^2$ correction to the string energy and compare to predictions of the string Bethe Ansatz