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Frobenius Manifolds and Integrable Hierarchies

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Lecture 2.

Recall: Frobenius algebra $(A, \;< \;,\; >)$
A commutative associative algebra with a unity, $\dim A = n$,
$< \;,\; >$ **invariant** symmetric nondegenerate bilinear form

$$< a \cdot b, c > = < a, b \cdot c > \quad \text{for all} \quad a,\, b,\, c \in A.$$

The goal: describe n-parameter deformations of n-dimensional Frobenius algebras $(A_t, \ < \ , \ >)$

$$A_t = \mathrm{span}(\phi_1, \ldots, \phi_n), \quad t = (t^1, \ldots, t^n)$$

such that

$$\eta_{\alpha\beta} = <\phi_\alpha, \phi_\beta> = \text{constant}, \quad \phi_1 = \text{identity} \quad \text{for all } t,$$

$$<\phi_\alpha \cdot \phi_\beta, \phi_\gamma> = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

for some function $F(t)$,

$$\sum \left(A^\alpha_\beta t^\beta + B^\alpha \right) \frac{\partial F}{\partial t^\alpha} = (3-d)F(t) + \text{quadratic polynomial}$$

for some constants A^α_β, B^α, d satisfying $A^\alpha_1 = \delta^\alpha_1$.

Example 1. X smooth projective variety, $\dim_{\mathbb{C}} X = d$, $H^{\text{odd}}(X) = 0$, $\dim H^*(X) =: n$

Primary observables $\leftrightarrow$ cohomologies (graded Frobenius algebra)

$$\phi_1 = 1, \phi_2, \ldots, \phi_n \in H^*(X), \quad \phi_\alpha \in H^{2\,q_\alpha}(X)$$

$$< \phi_\alpha, \phi_\beta >= \int_X \phi_\alpha \wedge \phi_\beta$$

Deformation: use **Gromov - Witten invariants** of genus zero

Moduli spaces $X_{0,m,\beta}$ of stable genus 0 and degree β curves on X with m marked points

$$X_{0,m,\beta} := \left\{ f \,:\, \left(\mathbf{P}^1, x_1, \ldots, x_m\right) \to X, \quad f_*[\mathbf{P}^1] = \beta \in H_2(X;\mathbb{Z}) \right\}$$

$$\langle \phi_{\alpha_1} \ldots \phi_{\alpha_m} \rangle_{0,\beta} := \int_{[X_{0,m,\beta}]} ev_1^*(\phi_{\alpha_1}) \wedge \ldots \wedge ev_m^*(\phi_{\alpha_m})$$

Evaluation maps ev_i, $i = 1, \ldots, m$ given by

$$ev_i : X_{g,m,\beta} \to X, \quad f \mapsto f(x_i).$$

$$F(t) = \sum_m \frac{1}{m!} t^{\alpha_1} \ldots t^{\alpha_m} \sum_{\beta \in H_2(X;\mathbb{Z})} \langle \phi_{\alpha_1} \ldots \phi_{\alpha_m} \rangle_{0,\beta}$$

$$E = \sum (1 - q_\alpha) t^\alpha \partial_\alpha + \sum r_\gamma \partial_\gamma \quad \text{where} \quad c_1(X) = \sum r_\gamma \phi_\gamma \in H^2(X)$$

Particular case 1 $X = \mathbf{P}^1$

$$\phi_1 = 1 \in H^0(\mathbf{P}^1), \quad \phi_2 \in H^2(\mathbf{P}^1), \quad \int_{\mathbf{P}^1} \phi_2 = 1$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + e^{t_2}, \quad E = t_1 \partial_1 + 2\partial_2, \quad \partial_E F = 2F + t_1^2$$

$$\partial_1 \cdot \partial_1 = \partial_1, \quad \partial_1 \cdot \partial_2 = \partial_2, \quad \partial_2 \cdot \partial_2 = e^{t_2} \partial_1$$

Frobenius algebra = quantum cohomology

$$QH^*(\mathbf{P}^1) = \mathbb{C}[\phi]/\{\phi^2 = q\}, \quad q = e^{t_2}$$

Classical limit

$$\mathrm{Re}\, t_2 \to -\infty, \quad QH^*(\mathbf{P}^1) \to H^*(\mathbf{P}^1)$$

Particular case 2 $X = \mathbf{P}^2$

$$\phi_1 = 1 \in H^0(\mathbf{P}^2), \quad \phi_2 \in H^2(\mathbf{P}^2), \quad \phi_3 \in H^4(\mathbf{P}^2), \quad \int_{\mathbf{P}^2} \phi_3 = 1$$

$$F(t_1, t_2, t_3) = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + \sum_{k \geq 1} \frac{N_k}{(3k-1)!} t_3^{3k-1} e^{k\, t_2}$$

$$E = t_1 \partial_1 + 3\partial_2 - t_3 \partial_3, \quad \partial_E F = F + 3 t_1 t_2$$

$N_k =$ number of rational curves of degree k
on $\mathbf{P}^2$ passing through $3k-1$ points

E.g.,

$$N_1 = 1, \quad N_2 = 1, \quad N_3 = 12 \dots$$

Small quantum cohomology: $t_3 = 0$

Frobenius algebra $\mathbb{C}[\phi]/\{\phi^3 = q\}, \quad q = e^{t_2}$

Topological LG models (= singularity theory at $g = 0$)

Analytic function (e.g., polynomial) $f(z)$,

$$z = (z_1, \ldots, z_N) \in \mathbb{C}^N$$

isolated singularity at $z = 0$: $df|_{z=0} = 0$

Topological observables $\leftrightarrow$ elements of the Jacobi ring

$$\phi_1 = 1,\ \phi_2, \ldots, \phi_n \in Jac_f = \mathbb{C}[z_1, \ldots, z_N]/\{\partial_1 f(z), \ldots, \partial_N f(z)\}$$

$$n = \dim Jac_f = \text{Milnor number}$$

Inner product given by residue pairing

$$\langle \phi, \psi \rangle = \frac{1}{(2\pi i)^N} \int \frac{\phi(z)\psi(z)}{\partial_1 f(z) \ldots \partial_N f(z)}\, d^N z$$

Perturbed Frobenius algebra: same for the universal unfolding

$$f_t(z) = f(z) + \sum_{\alpha=1}^{n} \tilde{t}^{\alpha} \phi_{\alpha}(z)$$

$$\mathbb{C}[z_1, \ldots, z_N]/\{\partial_1 f_t(z), \ldots, \partial_N f_t(z)\}$$

Coefficients

$$\tilde{t}^{\alpha}(t^1, \ldots, t^n)$$

given via flat coordinates on the base of uniiversal unfolding

Particular case: A_n topological minimal models, $N = 1$

$$f(z) = z^{n+1}$$

$$f_t = z^{n+1} + a_1(t) z^{n-1} + \ldots + a_n(t)$$

Algebra

$$A_t = \mathbb{C}[z]/\{f_t'(z) = 0\}$$

Inner product

$$\langle \phi, \psi \rangle_t = \frac{1}{n+1} \mathrm{res}_{z=0} \frac{\phi(z)\,\psi(z)}{f_t'(z)}$$

Flat coordinates $t = t(a)$: expand the solution to

$$f_t(z) = k^{n+1}$$

$$z = k - \frac{1}{n+1}\left(\frac{t_n}{k} + \frac{t_{n-1}}{k^2} + \ldots + \frac{t_1}{k^n}\right) + \ldots$$

Coupling to gravity: integration over the space of metrics on $\Sigma \Rightarrow$ integration over the moduli space of conformal classes

$$\bar{\mathcal{M}}_{g,n} \ni (C_g, x_1, \ldots, x_n)$$

Gravitational descendents come from cocycles on $\bar{\mathcal{M}}_{g,n}$

Example 1 2D topological gravity

Matter sector $= \{ 1 \}$

Descendents = tautological classes in $H^*(\bar{\mathcal{M}}_{g,n})$

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\bar{\mathcal{M}}_{g,n})$$

$\mathcal{L}_i$ line bundle on $\bar{\mathcal{M}}_{g,n}$

the fiber over $(C_g, x_1, \ldots, x_n) \in \bar{\mathcal{M}}_{g,n}$ is $T^*_{x_i} C_g$

Total free energy

$$\mathcal{F} = \langle e^{\sum t_p \psi^p} \rangle = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g$$

$$\mathcal{F}_g = \sum \frac{1}{n!} t_{p_1} \ldots t_{p_n} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{p_1} \wedge \ldots \wedge \psi_n^{p_n}$$

Theorem (Witten - Kontsevich) The partition function of 2D topological gravity

$$Z = e^{\mathcal{F}}$$

is tau-function of the solution to the KdV hierarchy

$$u = \epsilon^2 \partial_x^2 \log Z, \quad x = t_0$$

specified by the string equation

$$L_{-1} Z = \partial_1 Z$$

$$L_{-1} = \sum_{p \geq 0} t_{p+1} \partial_p + \frac{1}{2\epsilon^2} t_0^2$$

KdV hierarchy

$$
\begin{aligned}
u_{t_0} &= u_x \\
u_{t_1} &= u\, u_x + \frac{\epsilon^2}{12} u_{xxx} \\
u_{t_2} &= \frac{1}{2} u^2 u_x + \frac{\epsilon^2}{12}(2u_x u_{xx} + u\, u_{xxx}) + \frac{\epsilon^4}{240} u^V \\
&\dots
\end{aligned}
$$

Lax operator

$$
L = \frac{\epsilon^2}{2}\partial_x^2 + u
$$

$$
\epsilon \frac{\partial L}{\partial t_k} = [L, A_k]\,, \qquad A_k = \left[\frac{(2k+1)!!}{2^k}\right]^{-1} \left(L^{\frac{2k+1}{2}}\right)_+
$$

Kontsevich - Witten tau-function

$$\mathcal{F} = \frac{1}{\epsilon^2}\left(\frac{t_0^3}{6} + \frac{t_0^3\,t_1}{6} + \frac{t_0^3\,t_1^2}{6} + \frac{t_0^3\,t_1^3}{6} + \frac{t_0^3\,t_1^4}{6} + \frac{t_0^4\,t_2}{24} + \frac{t_0^4\,t_1\,t_2}{8}\right.$$

$$\left. + \frac{t_0^4\,t_1^2\,t_2}{4} + \frac{t_0^5\,t_2^2}{40} + \frac{t_0^5\,t_3}{120} + \frac{t_0^5\,t_1\,t_3}{30} + \frac{t_0^6\,t_4}{720} + \dots\right)$$

$$+\left(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0\,t_2}{24} + \frac{t_0\,t_1\,t_2}{12} + \frac{t_0\,t_1^2\,t_2}{8} + \frac{t_0^2\,t_2^2}{24}\right.$$

$$\left. + \frac{t_0^2\,t_3}{48} + \frac{t_0^2\,t_1\,t_3}{16} + \frac{t_0^3\,t_4}{144} + \dots\right)$$

$$+\epsilon^2\left(\frac{7\,t_2^3}{1440} + \frac{7\,t_1\,t_2^3}{288} + \frac{29\,t_2\,t_3}{5760} + \frac{29\,t_1\,t_2\,t_3}{1440} + \frac{29\,t_1^2\,t_2\,t_3}{576} + \frac{5\,t_0 t_2^2 t_3}{144}\right.$$

$$+\frac{29\,t_0\,t_3^2}{5760} + \frac{29\,t_0\,t_1\,t_3^2}{1152} + \frac{t_4}{1152} + \frac{t_1\,t_4}{384} + \frac{t_1^2\,t_4}{192} + \frac{t_1^3\,t_4}{96} + \frac{11\,t_0\,t_2\,t_4}{1440}$$

$$\left. + \frac{11\,t_0\,t_1\,t_2\,t_4}{288} + \frac{17\,t_0^2\,t_3\,t_4}{1920} + \dots\right) + O(\epsilon^4).$$

Example 2 $\mathbf{P}^1$ sigma model coupled with topological gravity

$$\phi_1 = 1 \in H^0(\mathbf{P}^1), \quad \phi_2 \in H^2(\mathbf{P}^1), \quad \int_{\mathbf{P}^1} \phi_2 = 1$$

Total free energy

$$\mathcal{F} = \langle e^{\sum s_p \phi_1 \psi^p + t_p \phi_2 \psi^p} \rangle = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g$$

$$\mathcal{F}_g = \sum \frac{1}{n!} t_{\alpha_1, p_1} \ldots t_{\alpha_n, p_n}$$

$$\times \int_{\left[\mathbf{P}^1_{g,n,\beta}\right]} ev_1^* \phi_{\alpha_1} \wedge \psi_1^{p_1} \wedge \ldots \wedge ev_n^* \phi_{\alpha_n} \wedge \psi_n^{p_n}$$

$$t_{1,p} = s_p, \quad t_{2,p} = t_p$$

Here

$$\mathbf{P}^1_{g,n,\beta} = \left\{ f : (C_g, x_1, \ldots, x_n) \to \mathbf{P}^1, \quad \beta = \text{degree of the map } f \right\}$$

Theorem (B.D., Y.Zhang, using A.Okounkov and R.Pandharipande). The partition function is the tau-function

$$\tau = e^{\mathcal{F}}$$

$$u = \log \frac{\tau(s_0 + \epsilon)\tau(s_0 - \epsilon)}{\tau^2(s_0)}$$

$$v = \epsilon \frac{\partial}{\partial t_0} \log \frac{\tau(s_0 + \epsilon)}{\tau(s_0)}.$$

of the extended Toda hierarchy specified by the string equation

$$L_{-1}\tau = \partial_{s_1}\tau$$

$$L_{-1} = \sum_{p \geq 0} t_{p+1} \frac{\partial}{\partial t_p} + s_{p+1} \frac{\partial}{\partial s_p} + \frac{s_0 t_0}{\epsilon^2}$$

Difference Lax operator

$$L = \Lambda + v + e^u \Lambda^{-1}, \quad \Lambda = e^{\epsilon \partial_x}$$

$$\epsilon \frac{\partial L}{\partial t_k} = \frac{1}{(k+1)!} \left[(L^{k+1})_+, L \right]$$
$$\epsilon \frac{\partial L}{\partial s_k} = \frac{2}{k!} \left[\left(L^k (\log L - c_k) \right)_+ , L \right]$$
$$c_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$$

($\log L$ introduced by G.Carlet, B.D., Y.Zhang)

E.g.,

$$\epsilon \partial_{t_0} u = v(s_0) - v(s_0 + \epsilon)$$

$$\epsilon \partial_{t_0} v = e^{u(s_0+\epsilon)} - e^{u(s_0)}.$$

Eliminating v and defining

$$u_n = u(n\epsilon), \quad t = \frac{t_0}{\epsilon}$$

$\Rightarrow$ the standard **Toda lattice equation**

$$\ddot{u}_n = e^{u_{n-1}-u_n} - e^{u_n-u_{n+1}}.$$

$$
\begin{aligned}
\mathcal{F}_0 = & \frac{s_0^2\,t_0}{2!} + \frac{s_0^2\,s_1\,t_0}{2!} + \frac{s_0^3\,t_1}{3!} + \frac{s_0^3\,s_2\,t_0}{3!} + \frac{s_0^4\,t_2}{4!} + \frac{s_0^2\,s_1^2\,t_0}{2!} \\
& +2\frac{s_0^3\,s_1\,t_1}{3!} + \frac{s_0^4\,s_3\,t_0}{4!} + \frac{s_0^5\,t_3}{5!} + 3\frac{s_0^3\,s_1\,s_2\,t_0}{3!} + 3\frac{s_0^4\,s_2\,t_1}{4!} + 3\frac{s_0^4\,s_1\,t_2}{4!} \\
& +e^{t_0}\Bigg[1 - 2\,s_1 + 2\frac{s_1^2}{2!} - 2\,s_0 s_2 + s_1 t_0 + s_0 t_1 - 2\frac{{s_0}^2\,s_3}{2!} \\
& -2\frac{{s_1}^2\,t_0}{2!} + s_0\,s_2\,t_0 + \frac{{s_0}^2\,t_2}{2!} - 2\frac{{s_0}^2\,s_1\,s_3}{2!} - s_0\,s_1\,s_2\,t_0 + \frac{{s_0}^2\,s_3\,t_0}{2!} \\
& +2\frac{{s_1}^2\,{t_0}^2}{(2!)^2} - \frac{{s_0}^2\,s_2\,t_1}{2!} + s_0\,s_1\,t_0\,t_1 + \frac{{s_0}^2\,{t_1}^2}{2!} + \frac{{s_0}^2\,s_1\,t_2}{2!} + \frac{{s_0}^3\,t_3}{3!}\Bigg] \\
& +e^{2t_0}\Bigg[-\frac{3}{4}s_3 + \frac{1}{4}t_2 + \frac{5}{4}\frac{{s_2}^2}{2!} + \frac{3}{4}s_1\,s_3 + \frac{1}{4}s_3\,t_0 - \frac{3}{4}s_2\,t_1 + \frac{1}{2}\frac{{t_1}^2}{2!} \\
& -\frac{1}{4}s_1\,t_2 + \frac{1}{4}s_0\,t_3 + 2\,s_0\,s_2\,s_3 - \frac{3}{2}\frac{{s_2}^2\,t_0}{2!} - \frac{3}{2}\,s_1\,s_3\,t_0 - 2\,s_0\,s_3\,t_1 \\
& +\frac{1}{2}s_2\,t_0\,t_1 - s_0\,s_2\,t_2 + \frac{1}{2}s_1\,t_0\,t_2 + s_0\,t_1\,t_2 + 2\,s_0\,s_1\,s_2\,s_3 + 4\frac{{s_0}^2\,{s_3}^2}{(2!)^2} \\
& +\frac{s_1\,{s_2}^2\,t_0}{2!} - 3\,s_0\,s_2\,s_3\,t_0 + \frac{{s_2}^2\,{t_0}^2}{(2!)^2} + \frac{s_1\,s_3\,{t_0}^2}{2!} + \frac{s_0\,{s_2}^2\,t_1}{2!} \\
& -2\,s_0\,s_1\,s_3\,t_1 - s_1\,s_2\,t_0\,t_1 + s_0\,s_3\,t_0\,t_1 - 2\frac{s_0\,s_2\,{t_1}^2}{2!} + \frac{s_1\,t_0\,{t_1}^2}{2!}
\end{aligned}
$$

$$
\begin{aligned}
&+3\frac{s_0\,{t_1}^3}{3!} - s_0\,s_1\,s_2\,t_2 - 3\frac{{s_0}^2\,s_3\,t_2}{2!} + s_0\,s_2\,t_0\,t_2 + s_0\,s_1\,t_1\,t_2 \\
&\left.+2\frac{{s_0}^2\,{t_2}^2}{(2!)^2} - \frac{{s_0}^2\,s_2\,t_3}{2!} + \frac{s_0\,s_1\,t_0\,t_3}{2} + \frac{3}{2}\frac{{s_0}^2\,t_1\,t_3}{2!}\right] \\
&+e^{3t_0}\left[\frac{50}{27}\frac{{s_3}^2}{2!} - \frac{7}{9}\,s_3\,t_2 + \frac{1}{3}\frac{{t_2}^2}{2!} - \frac{2}{9}\,s_2\,t_3 + \frac{1}{6}t_1\,t_3 - 2\frac{{s_2}^2\,s_3}{2!} - \frac{14}{9}\frac{{s_3}^2\,t_0}{2!}\right. \\
&+2\,s_2\,s_3\,t_1 - 2\frac{s_3\,{t_1}^2}{2!} + \frac{{s_2}^2\,t_2}{2!} + \frac{1}{3}s_3\,t_0\,t_2 - s_2\,t_1\,t_2 + \frac{{t_1}^2\,t_2}{2!} - s_0\,s_3\,t_3 \\
&+\frac{1}{6}s_2\,t_0\,t_3 + \frac{1}{2}s_0\,t_2\,t_3 - 4\frac{s_0\,s_2\,{s_3}^2}{2!} + 5\frac{{s_2}^2\,s_3\,t_0}{2!} + 4\frac{s_1\,{s_3}^2\,t_0}{2!} + \frac{2}{3}\frac{{s_3}^2\,{t_0}^2}{(2!)^2} \\
&+8\frac{s_0\,{s_3}^2\,t_1}{2!} - 3\,s_2\,s_3\,t_0\,t_1 + \frac{s_3\,t_0\,{t_1}^2}{2!} + 3\,s_0\,s_2\,s_3\,t_2 - 2\frac{{s_2}^2\,t_0\,t_2}{2!} \\
&-2\,s_1\,s_3\,t_0\,t_2 - 5\,s_0\,s_3\,t_1\,t_2 + s_2\,t_0\,t_1\,t_2 - 2\frac{s_0\,s_2\,{t_2}^2}{2!} + \frac{s_1\,t_0\,{t_2}^2}{2!} \\
&+3\frac{s_0\,t_1\,{t_2}^2}{2!} + \frac{s_0\,{s_2}^2\,t_3}{2!} - s_0\,s_1\,s_3\,t_3 - \frac{1}{2}s_1\,s_2\,t_0\,t_3 + \frac{1}{2}s_0\,s_3\,t_0\,t_3 \\
&\left.-\frac{3}{2}s_0\,s_2\,t_1\,t_3 + \frac{1}{2}s_1\,t_0\,t_1\,t_3 + 2\frac{s_0\,{t_1}^2\,t_3}{2!} + \frac{1}{2}s_0\,s_1\,t_2\,t_3 + \frac{{s_0}^2\,{t_3}^2}{(2!)^2}\right] + \ldots
\end{aligned}
$$

$$\mathcal{F}_1 = \frac{s_1}{12} + \frac{{s_1}^2}{24} - \frac{{s_1}^3}{18} + \frac{s_0\,s_2}{12} + \frac{s_0\,s_1\,s_2}{6} + \frac{{s_0}^2\,s_3}{24} - \frac{t_0}{24} - \frac{s_1\,t_0}{24} - \frac{{s_1}^2\,t_0}{24}$$

$$-\frac{s_0\,s_2\,t_0}{24} - \frac{s_0\,s_1\,s_2\,t_0}{8} - \frac{{s_0}^2\,s_3\,t_0}{48}$$

$$-\frac{s_0\,t_1}{24} - \frac{s_0\,s_1\,t_1}{12} - \frac{{s_0}^2\,s_2\,t_1}{16} - \frac{{s_0}^2\,t_2}{48} - \frac{{s_0}^2\,s_1\,t_2}{16} - \frac{{s_0}^3\,t_3}{144}$$

$$-\frac{1}{96}e^{t_0}\,\Big[-4\,s_3\,t_0 + 8\,s_1\,s_3\,t_0 + 4\,{s_2}^2\,{t_0}^2 + 4\,s_2\,t_1 - 8\,s_1\,s_2\,t_1 + 8\,s_2\,t_0\,t_1$$

$$+24\,s_1\,s_2\,t_0\,t_1 + 8\,s_0\,s_3\,t_0\,t_1 + 8\,s_1\,s_2\,{t_0}^2\,t_1 + 16\,s_1\,{t_1}^2 - 24\,{s_1}^2\,{t_1}^2$$

$$+24\,s_0\,s_2\,{t_1}^2 + 4\,s_1\,t_0\,{t_1}^2 + 12\,{s_1}^2\,t_0\,{t_1}^2 + 12\,s_0\,s_2\,t_0\,{t_1}^2 + {s_1}^2\,{t_0}^2\,{t_1}^2$$

$$+4\,s_0\,{t_1}^3 + 16\,s_0\,s_1\,{t_1}^3 + 2\,s_0\,s_1\,t_0\,{t_1}^3 + {s_0}^2\,{t_1}^4 - 4\,t_2 + 4\,s_1\,t_2 + 8\,{s_1}^2\,t_2$$

$$+8\,s_0\,s_2\,t_2 + 8\,s_0\,s_2\,t_0\,t_2 + 8\,s_0\,t_1\,t_2 + 24\,s_0\,s_1\,t_1\,t_2 + 8\,s_0\,s_1\,t_0\,t_1\,t_2$$

$$+10\,{s_0}^2\,{t_1}^2\,t_2 + 4\,{s_0}^2\,{t_2}^2 - 4\,s_0\,t_3 + 8\,s_0\,s_1\,t_3 + 4\,{s_0}^2\,t_1\,t_3\Big]$$

$$+\frac{1}{48}e^{2\,t_0}\,\Big[4\,s_3\,{t_1}^2 - 2\,s_3\,t_0\,{t_1}^2 + 2\,s_2\,{t_1}^3 + 4\,s_2\,t_0\,{t_1}^3 + 16\,s_1\,{t_1}^4 + 3\,s_1\,t_0\,{t_1}^4$$

$$+3\,s_0\,t_1{}^5 - 14\,s_3\,t_2 - 4\,s_3\,t_0\,t_2 + 4\,s_2\,t_1\,t_2 + 4\,s_2\,t_0\,t_1\,t_2 - 2\,t_1{}^2\,t_2 + 18\,s_1\,t_1{}^2\,t_2$$

$$+3\,s_1\,t_0\,t_1{}^2\,t_2 + 7\,s_0\,t_1{}^3\,t_2 + 4\,t_2{}^2 - 8\,s_1\,t_2{}^2 + 4\,s_0\,t_1\,t_2{}^2 - 6\,s_2\,t_3 - 4\,s_2\,t_0\,t_3$$

$$+5\,t_1\,t_3 - 12\,s_1\,t_1\,t_3 - 4\,s_1\,t_0\,t_1\,t_3 - 6\,s_0\,t_1{}^2\,t_3 - 8\,s_0\,t_2\,t_3\Big]$$

$$+\frac{e^{3\,t_0}}{288}\left[29t_1{}^6 + 66t_1{}^4t_2 + 33t_1{}^2t_2{}^2 - 40t_2{}^3 + 6t_1{}^3t_3 - 48t_1t_2t_3 + 18t_3{}^2\right]$$
$$+\dots$$

$$\mathcal{F}_2 = \frac{7\,t_2}{5760} + \frac{7\,s_1\,t_2}{1920} + \frac{7\,s_0\,t_3}{5760} + e^{t_0}\left(\frac{29\,t_1{}^4}{11520} + \frac{t_1{}^2\,t_2}{80} + \frac{13\,t_2{}^2}{2880} + \frac{7\,t_1\,t_3}{1440}\right) + \dots$$

Small phase space:

$$t_p = s_p = 0, \quad p > 0$$

$$(\mathcal{F}_0)_{\mathrm{small}} = \frac{1}{2}s_0^2 t_0 + e^{t_0}.$$

Main message 1:

Theory of 2D TFTs = Theory of integrable systems

Frobenius manifold M - multiplication on tangent bundle

$$a, b \in T_x M \ \mapsto \ a \cdot b \in T_x M$$

Commutative, **associative**, with a unity e and with invariant symmetric nondegenerate bilinear form on T^*M

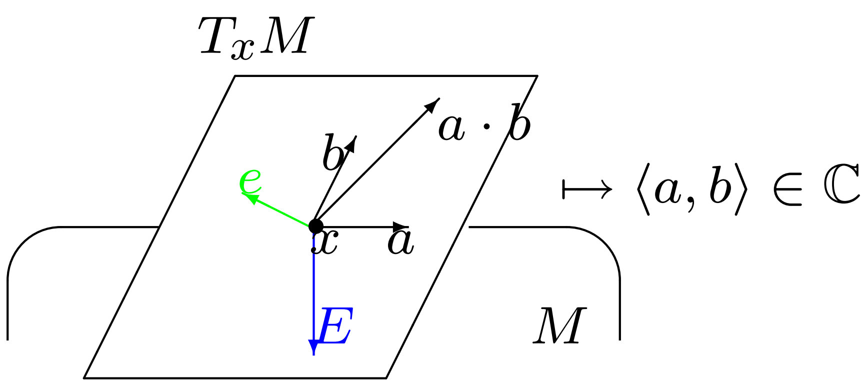
$$\langle a, b \rangle \in \mathbb{C}$$

that is constant in flat coordinates

$$\langle dx^i, dx^j \rangle = \eta^{ij}$$

Main axiom: the product in the flat coordinates given by triple derivatives of a function $F(x)$

$$\partial_i \cdot \partial_j = c_{ij}^k(x)\partial_k, \quad c_{ij}^k(x) = \eta^{kl}\frac{\partial^3 F(x)}{\partial x^i \partial x^j \partial x^l}$$



Other axioms:

- the unity e is constant in the flat coordinates

- quasihomogeneity: linear vector field E is defined

$$E = (A^i_j x^j + B^i)\partial_i$$

$$[e, E] = e, \quad \mathrm{Lie}_E F(x) = (3 - d)F(x) + \text{quadratic}$$

Recall examples

- $QH^*(\mathbf{P}^1)$, dim $= 2$, $d = 1$

$$F = \frac{1}{2} u\, v^2 + e^u$$

$$\langle \partial_u, \partial_v \rangle = 1$$

$$e = \partial_v, \qquad E = v\, \partial_v + 2\partial_u$$

- A_n singularity, dim $= n$, $d = \frac{n-1}{n+1}$

$$f_a(x) = x^{n+1} + a_1 x^{n-1} + \ldots + a_n \in M$$

algebra

$$T_aM = \mathbb{C}[x]/\{f_a'(x) = 0\}, \quad e = 1$$

metric on T_aM

$$\langle \frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j} \rangle = \frac{1}{n+1} \text{res}_{x=0} \frac{x^{n-i-j}}{f_a'(x)}$$

Euler vector field

$$E = \frac{1}{n+1}\left(2\, a_1 \frac{\partial}{\partial a_1} + 3\, a_2 \frac{\partial}{\partial a_2} + \ldots + (n+1)\, a_n \frac{\partial}{\partial a_n}\right)$$

Deformed connection on $M \times \mathbb{C}^*$ is flat

$$\tilde{\nabla}_a b = \nabla_a b + z\, a \cdot b, \qquad a,\; b, \in T_x M$$

$$\tilde{\nabla}_{\frac{d}{dz}} b = \partial_z b + E \cdot b + \frac{\mu}{z} b$$

$$\mu = \frac{d-2}{2} - \nabla E$$

Theorem $\tilde{\nabla}$ is flat connection

Proof: check compatibility of the equations for the horizontal sections $\omega = d\theta$

$$\partial_\alpha \partial_\lambda \theta = z\, c_{\alpha\lambda}^{\sigma}(t) \partial_\sigma \theta$$
$$\partial_z \partial_\alpha \theta = \mathcal{U}_\alpha^\sigma(t) \partial_\sigma \theta - \frac{\mu_\alpha}{z}\, \partial_\alpha \theta$$
$$\mu_\alpha = q_\alpha - \frac{d}{2}$$

E.g.,

$$\partial_\beta \ \left[\ \partial_\alpha \partial_\lambda \theta = z\, c_{\alpha\lambda}^{\sigma}(t) \partial_\sigma \theta\ \right] - \ (\alpha \leftrightarrow \beta)$$

yields

$$z\, \partial_\beta c_{\alpha\lambda}^\sigma + z^2 c_{\alpha\lambda}^\rho c_{\rho\beta}^\sigma = z\, \partial_\alpha c_{\beta\lambda}^\sigma + z^2 c_{\beta\lambda}^\rho c_{\rho\alpha}^\sigma$$

$$z\,\partial_\beta c^\sigma_{\alpha\lambda} + z^2 c^\rho_{\alpha\lambda} c^\sigma_{\rho\beta} = z\,\partial_\alpha c^\sigma_{\beta\lambda} + z^2 c^\rho_{\beta\lambda} c^\sigma_{\rho\alpha}$$

Comes from symmetry of 4th mixed derivatives of F

$$\partial_\beta c^\sigma_{\alpha\lambda} = \partial_\alpha c^\sigma_{\beta\lambda} = \eta^{\sigma\nu} \partial_\nu \partial_\alpha \partial_\beta \partial_\lambda F(t)$$

and from associativity of the algebra

$$c^\rho_{\alpha\lambda} c^\sigma_{\rho\beta} = c^\rho_{\beta\lambda} c^\sigma_{\rho\alpha}$$

Similarly compatibility with ∂_z (use quasihomogeneity)

Example For A_n singularity the oscillatory integrals

$$\theta(a;z) = z^{-1/2} \int e^{z\, f_a(x)}\, dx$$

are horizontal wrt $\tilde\nabla$

$$\tilde\nabla d\theta = 0$$

Semisimplicity: for generic $t \in M$ the algebra on $T_t M$ is *semisimple* (no nilpotents)

Theorem

Denote $\mathcal{U}(t) :=$ operator of multiplication by E on $T_t M$

The characteristic roots $u_1(t), \ldots, u_n(t)$

$$\det\left(\mathcal{U}(t) - u\right) = 0$$

are independent coordinates on M. In these coordinates the multiplication table takes the following canonical form

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i, \quad \partial_i = \frac{\partial}{\partial u_i}$$

(canonical coordinates)

In the canonical coordinates the metric is diagonal

$$\langle \, , \, \rangle = h_1^2(u)du_1^2 + \ldots + h_n^2(u)du_n^2$$

the unity and the Euler vector field become

$$e = \sum_{i=1}^{n} \partial_i$$

$$E = \sum_{i=1}^{n} u_i \partial_i$$

Normalized basis of idempotents

$$\pi_i = h_i^{-1}(u)\partial_i,$$

Transition matrix $\Psi = (\psi_{i\alpha})$

$$\partial_\alpha = \sum_{i=1}^{n} \psi_{i\alpha}(u)\pi_i$$

Unity

$$e = \sum_i \partial_i$$

Orthogonality

$$\Psi^T \Psi = \eta$$

So

$$h_i = \psi_{i1}$$

$$\partial_i = \eta^{\alpha\beta}\psi_{i1}\psi_{i\alpha}\partial_\beta$$

$$c_{\alpha\beta\gamma} = \sum_i \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}$$

Example A_n singularity, $f_a = x^{n+1} + a_1 x^{n-1} + \ldots + a_n$.

The canonical coordinates

$$u_i = u_i(a), \quad i = 1, \ldots, n$$

are the critical values of $f_a(x)$

$$u_i = f_a(x_i), \quad f_a'(x_i) = 0$$

Rewriting the equations for the horizontal sections of $\tilde{\nabla}$ in the canonical coordinates we arrive at

$$\partial_i Y = (z\, E_i + V_i)\, Y$$
$$(E_i)_{jk} = \delta_{ij}\delta_{ik}, \quad V_i^T = -V_i$$
$$\partial_z Y = \left(U + \frac{V}{z}\right) Y$$
$$U = \text{diag}\,(u_1, \dots, u_n), \quad V^T = -V$$

Flatness $\Rightarrow$

$$\partial_i V = [V, V_i] \equiv \{V, H_i(V; u)\}$$
$$V_i = \text{ad}_{E_i}\text{ad}_U^{-1}(V)$$

Isomonodromy deformations of

$$\partial_z Y = \left(U + \frac{V}{z}\right) Y$$

Hamiltonian time-dependent flows on $so(n)$ with quadratic Hamiltonians

$$H_i = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j}$$

Isomonodromy tau-function $\tau_I(u)$

$$d \log \tau_I(u) = \sum_{i=1}^{n} H_i du_i$$

Holonomy of the flat connection $\tilde{\nabla}$

$$\mathsf{P}\, e^{\oint_C \tilde{\nabla}}$$

does not change under deformations of the loop $C \Rightarrow$ isomonodromicity:

the monodromy of

$$\frac{d}{dz} Y = \left(U + \frac{V(u)}{z} \right) Y$$

in the complex z-plane **does not depend** on $u = (u_1, \ldots, u_n) \in M$, $u_i \neq u_j$ for $i \neq j$

Monodromy of

$$\frac{d}{dz}Y = \left(U + \frac{V}{z}\right) Y$$

consists of

- monodromy at the origin (μ, R)

$$Y_0 = \left(\Theta_0 + z\,\Theta_1 + O(z^2)\right) z^\mu z^R$$

$$\mu = \mathrm{diag}\,(\mu_1, \dots, \mu_n) \text{ eigenvalues of } V$$

R a nilpotent matrix, in the resonant case

$$R = R_1 + R_2 + \dots, \quad (R_k)_{\alpha\beta} \neq 0 \text{ only if } \mu_\alpha - \mu_\beta = k$$

- Stokes matrix at $z = \infty$. Half-planes Π_R, Π_L

$$Y_R(z),\ Y_L(z), \quad Y_{R/L} \sim \left(1 + O(\frac{1}{z})\right) e^{z\,U}$$

$$Y_R = Y_L S \text{ or} Y_R = Y_L S^T \text{ on two halfs of } \Pi_R \cap \Pi_L$$

S upper triangular matrix, $S_{ii} = 1$

- central connection matrix C

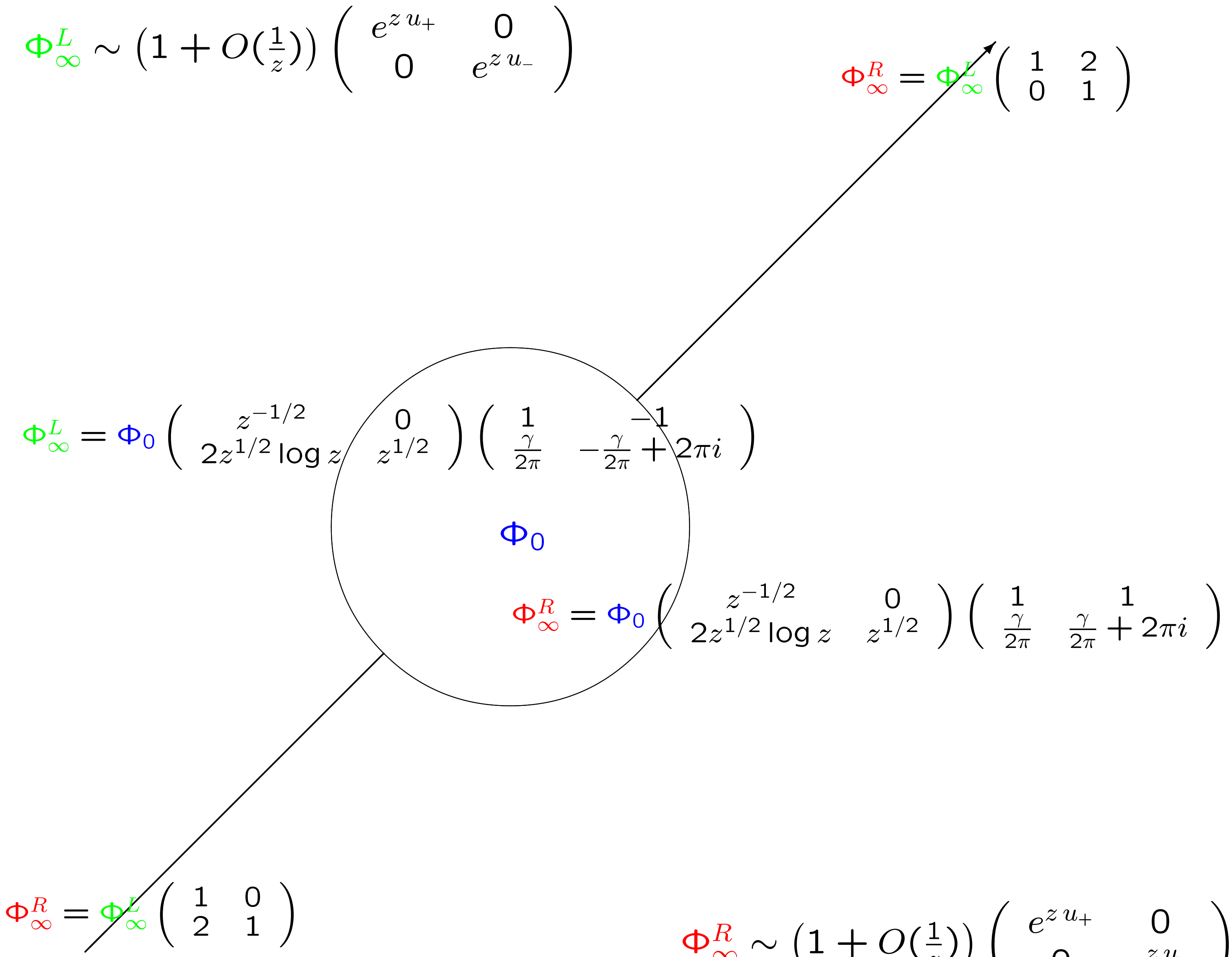
$$Y_0 = Y_R C$$

$$C^T S\, C = \eta\, e^{\pi i \mu} e^{\pi i R}$$

Theorem (B.D.) Monodromy data (μ, R, S, C) completely determine the semisimple Frobenius manifold

Reconstruction: by solving certain Riemann - Hilbert problem

Example Frobenius manifold $F = \frac{1}{2}v^2 u + e^u$



Here $u_\pm = v \pm 2e^{\frac{u}{2}}$

Main message 2:

Integrable hierarchies suitable for the needs of 2D TFT are parametrized by the same data (μ, R, S, C). They can be constructed by "quantization" of the Riemann - Hilbert problem