Sofia, September 28-30, 2006

Frobenius Manifolds and Integrable Hierarchies

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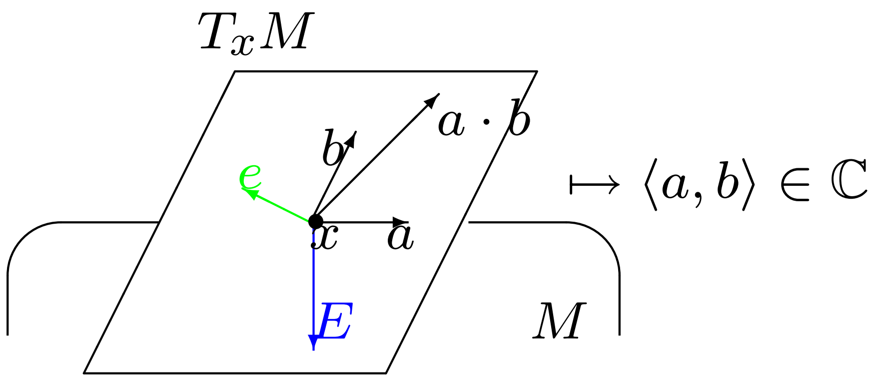
Lecture 3.



T_xM
$a \cdot b$
b
e
x
a
E
M
$\longmapsto \langle a, b \rangle \in \mathbb{C}$

Deformed connection on $M \times \mathbb{C}^*$ is flat

$$\tilde{\nabla}_a b = \nabla_a b + z\, a \cdot b, \qquad a,\ b, \in T_x M$$

$$\tilde{\nabla}_{\frac{d}{dz}} b = \partial_z b + E \cdot b + \frac{\mu}{z} b$$

$$\mu = \frac{d-2}{2} - \nabla E$$

Theorem $\tilde{\nabla}$ is flat connection

Proof: check compatibility of the equations for the horizontal sections $\omega = d\theta$

$$
\begin{aligned}
\partial_\alpha \partial_\lambda \theta &= z\, c^{\sigma}_{\alpha\lambda}(t) \partial_\sigma \theta \\
\partial_z \partial_\alpha \theta &= \mathcal{U}^{\sigma}_{\alpha}(t) \partial_\sigma \theta - \frac{\mu_\alpha}{z} \partial_\alpha \theta \\
\mu_\alpha &= q_\alpha - \frac{d}{2}
\end{aligned}
$$

E.g.,

$$\partial_\beta \; [\; \partial_\alpha \partial_\lambda \theta = z\, c^\sigma_{\alpha\lambda}(t) \partial_\sigma \theta \;] - \; (\alpha \leftrightarrow \beta)$$

yields

$$z\, \partial_\beta c^\sigma_{\alpha\lambda} + z^2 c^\rho_{\alpha\lambda} c^\sigma_{\rho\beta} = z\, \partial_\alpha c^\sigma_{\beta\lambda} + z^2 c^\rho_{\beta\lambda} c^\sigma_{\rho\alpha}$$

Comes from symmetry of 4th mixed derivatives of F

$$\partial_\beta c^\sigma_{\alpha\lambda} = \partial_\alpha c^\sigma_{\beta\lambda} = \eta^{\sigma\nu} \partial_\nu \partial_\alpha \partial_\beta \partial_\lambda F(t)$$

and from associativity of the algebra

$$c^\rho_{\alpha\lambda} c^\sigma_{\rho\beta} = c^\rho_{\beta\lambda} c^\sigma_{\rho\alpha}$$

Similarly compatibility with ∂_z (use quasihomogeneity)

Deformed flat coordinates (called J-functions by Givental): functions $\theta = \theta(x; z)$ on the universal covering of $M \times \mathbb{C}^*$ s.t.

$$\tilde{\nabla}\, d\theta(x; z) = 0.$$

Example For A_n singularity the oscillatory integrals

$$\theta(a; z) = z^{-1/2} \int e^{z\, f_a(x)}\, dx$$

are deformed flat coordinates

$$\tilde{\nabla} d\theta = 0$$

Semisimplicity: for generic $t \in M$ the algebra on $T_t M$ is *semisimple* (no nilpotents)

Theorem

Denote $\mathcal{U}(t) :=$ operator of multiplication by E on $T_t M$

The characteristic roots $u_1(t), \ldots, u_n(t)$

$$\det\left(\mathcal{U}(t) - u \cdot 1\right) = 0$$

are independent coordinates on M. In these coordinates the multiplication table takes the following canonical form

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i, \quad \partial_i = \frac{\partial}{\partial u_i}$$

(canonical coordinates)

In the canonical coordinates the metric is diagonal

$$\langle\ ,\ \rangle = h_1^2(u)du_1^2 + \ldots + h_n^2(u)du_n^2$$

the unity and the Euler vector field become

$$e = \sum_{i=1}^{n} \partial_i$$

$$E = \sum_{i=1}^{n} u_i \partial_i$$

Normalized basis of idempotents

$$\pi_i = h_i^{-1}(u)\partial_i,$$

Transition matrix $\Psi = (\psi_{i\alpha})$

$$\partial_\alpha = \sum_{i=1}^{n} \psi_{i\alpha}(u)\pi_i$$

Unity

$$e = \sum_i \partial_i$$

Orthogonality

$$\Psi^T \Psi = \eta$$

So

$$h_i = \psi_{i1}$$

$$\partial_i = \eta^{\alpha\beta}\psi_{i1}\psi_{i\alpha}\partial_\beta$$

$$c_{\alpha\beta\gamma} = \sum_i \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}$$

Example A_n singularity, $f_a = x^{n+1} + a_1 x^{n-1} + \ldots + a_n$.

The canonical coordinates

$$u_i = u_i(a), \quad i = 1, \ldots, n$$

are the critical values of $f_a(x)$

$$u_i = f_a(x_i), \quad f_a'(x_i) = 0$$

Rewriting the equations for the horizontal sections of $\tilde{\nabla}$ in the canonical coordinates we arrive at

$$\partial_i Y = (z\, E_i + V_i)\, Y$$
$$(E_i)_{jk} = \delta_{ij}\delta_{ik}, \quad V_i^T = -V_i$$
$$\partial_z Y = \left(U + \frac{V}{z}\right)\, Y$$
$$U = \mathrm{diag}\,(u_1, \dots, u_n), \quad V^T = -V$$

Flatness $\Rightarrow$

$$\partial_i V = [V, V_i] \equiv \{V, H_i(V; u)\}$$
$$V_i = \mathrm{ad}_{E_i} \mathrm{ad}_U^{-1}(V)$$

Isomonodromy deformations of

$$\partial_z Y = \left(U + \frac{V}{z}\right)\, Y$$

Hamiltonian time-dependent flows on $so(n)$ with quadratic Hamiltonians

$$H_i = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j}$$

Isomonodromy tau-function $\tau_I(u)$

$$d \log \tau_I(u) = \sum_{i=1}^{n} H_i du_i$$

Holonomy of the flat connection $\tilde{\nabla}$

$$\mathsf{P}\, e^{\oint_C \tilde{\nabla}}$$

does not change under deformations of the loop C $\Rightarrow$ isomonodromicity:

the monodromy of

$$\frac{d}{dz} Y = \left(U + \frac{V(u)}{z} \right) Y$$

in the complex z-plane **does not depend** on $u = (u_1, \ldots, u_n) \in M$, $u_i \neq u_j$ for $i \neq j$

Monodromy of

$$\frac{d}{dz}Y = \left(U + \frac{V}{z}\right) Y$$

consists of

- monodromy at the origin (μ, R)

$$Y_0 = \left(\Theta_0 + z\,\Theta_1 + O(z^2)\right) z^{\mu} z^{R}$$

$$\mu = \text{diag}\,(\mu_1, \dots, \mu_n) \text{ eigenvalues of } V$$

R a nilpotent matrix, in the resonant case

$$R = R_1 + R_2 + \dots, \quad (R_k)_{\alpha\beta} \neq 0 \text{ only if } \mu_\alpha - \mu_\beta = k$$

- Stokes matrix at $z = \infty$. Half-planes Π_R, Π_L

$$Y_R(z),\ Y_L(z), \quad Y_{R/L} \sim \left(1 + O(\frac{1}{z})\right) e^{z\,U}$$

$$Y_R = Y_L S \text{ or} Y_R = Y_L S^T \text{ on two halfs of } \Pi_R \cap \Pi_L$$

S upper triangular matrix, $S_{ii} = 1$

- central connection matrix C

$$Y_0 = Y_R C$$

$$C^T S\, C = \eta\, e^{\pi i \mu} e^{\pi i R}$$

Theorem (B.D.) Monodromy data (μ, R, S, C) completely determine the semisimple Frobenius manifold

Reconstruction: by solving certain Riemann - Hilbert problem

Integrable hierarchies

System of n 1+1 evolutionary PDEs ("vector fields" on the space of function)

$$u^i_t + A^i_j(u)u^j_x + \text{higher order derivatives} = 0, \quad i = 1, \ldots, n$$

Weak dispersion expansion: start from

$$u^i_t + F^i(u, u_x, u_{xx}, \ldots) = 0$$

Introduce **slow variables** $x \mapsto \epsilon\, x, \quad t \mapsto \epsilon\, t$

$$u^i_t + \frac{1}{\epsilon} F^i(u, \epsilon\, u_x, \epsilon^2 u_{xx}, \ldots)$$

$$= u^i_t + A^i_j(u)u^j_x + \epsilon \left(B^i_j(u)u^j_{xx} + \frac{1}{2} C^i_{jk}(u)u^j_x u^k_x \right) + \ldots = 0$$

Integrability:

Maximal Abelian subalgebras of Hamiltonian “vector fields”

Example. The simplest hyperbolic equation

$$v_t + a(v)v_x = 0$$

is a Hamiltonian PDE,

$$v_t + \{H, v(x)\} = v_t + \partial_x \tfrac{\delta H}{\delta v(x)} = 0, \quad H = \textstyle\int f(v)\, dx$$

$$\{v(x), v(y)\} = \delta'(x-y), \qquad f''(v) = a(v)$$

Any two such flows commute:

$$v_t + a(v)v_x = 0 \qquad (v_t)_s = (v_s)_t$$

$$v_s + b(v)v_x = 0$$

e.g., from commuting Hamiltonians $\{H_f, H_g\} = 0$,

$$H_f = \int f(v)\, dx, \quad H_g = \int g(v)\, dx, \quad f''(v) = a(v), \quad g''(v) = b(v)$$

This is a **complete family** of commuting Hamiltonians!

Hierarchy:

Commuting flows on the space of vector functions $(u^1(x), \ldots, u^n(x))$ organized in n infinite chains

$$t = (t^{\alpha,p}), \quad \alpha = 1, \ldots, n, \quad p = 0,\, 1,\, 2, \ldots$$

$$\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x}$$

Example Dispersionless KdV ($n = 1$)

$$\frac{\partial u}{\partial t^p} = \frac{u^p}{p!} u_x = \{u(x), H_p\}_1, \qquad H_p = \int h_p \, dx, \qquad h_p = \frac{u^{p+2}}{(p+2)!}$$

$$\{u(x), u(y)\}_1 = \delta'(x - y)$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x - y) + \frac{1}{2} u_x \delta(x - y)$$

Bihamiltonian recursion

$$\{\ .\ , H_{p-1}\}_2 = \left(p + \frac{1}{2}\right) \{\ .\ , H_p\}_1$$

Observe **strange symmetry**:

$$\frac{\partial h_{p-1}}{\partial t^q} = \frac{\partial h_{q-1}}{\partial t^p} = \partial_x \Omega_{pq}, \qquad \Omega_{pq} = \frac{u^{p+q+1}}{p!\, q!\, (p+q+1)}$$

Solutions given in implicit form

$$\sum_{p\geq 0} (t_p - c_p)\,\frac{u^p}{p!} = 0$$

arbitrary constants c_p

Tau-function of a solution $u = u(\mathbf{t})$, $\mathbf{t} = (t^0, t^1, \dots)$ defined by

$$\frac{\partial^2 \log \tau}{\partial t^p \partial t^q} = \Omega_{pq}(u(\mathbf{t})).$$

Explicitly

$$\log \tau = \frac{1}{2} \sum \Omega_{pq}(u)(t_p - c_p)(t_q - c_q)$$

Back to Frobenius manifolds.

Flat coordinates v^1, ..., v^n on M, $< dv^\alpha, dv^\beta >= \eta^{\alpha\beta}$. Introduce the Poisson bracket

$$\{v^\alpha(x), v^\beta(y)\} = \eta^{\alpha\beta}\,\delta'(x-y).$$

Let $\theta^{(1)}(v;z)$ and $\theta^{(2)}(v;z)$ be two deformed flat coordinates (recall, $\tilde\nabla\, d\theta^{(1,2)}(v;z) = 0$). Introduce the Hamiltonians

$$H_z^{(1,2)} := \int \theta^{(1,2)}(v;z)\,dx.$$

Theorem The Hamiltonians $H_z^{(1)}$, $H_w^{(2)}$ commute for arbitrary z, w.

Proof. By definition

$$\left\{H_z^{(1)}, H_w^{(2)}\right\} = \int \frac{\delta H_z^{(1)}}{\delta v^\alpha(x)} \eta^{\alpha\beta} \frac{\partial}{\partial x} \frac{\delta H_w^{(2)}}{\delta v^\beta(x)}\, dx$$

$$= \int \partial_\alpha \theta^{(1)}(v;z) \eta^{\alpha\beta} \partial_\beta \partial_\gamma \theta^{(2)}(v;w) v_x^\gamma\, dx$$

Using $\partial_\beta \partial_\gamma \theta^{(2)}(v;w) = w\, c_{\beta\gamma}^\nu(v) \partial_\nu \theta(v;w)$ rewrite

$$\left\{H_z^{(1)}, H_w^{(2)}\right\} = w \int < d\theta^{(1)} \cdot d\theta^{(2)}, v_x >\, dx$$

$$= \frac{w}{z+w} \int \partial_x < d\theta^{(1)}, d\theta^{(2)} > dx = 0$$

since $\partial_x < d\theta^{(1)}, d\theta^{(2)} >= (z+w) < d\theta^{(1)} \cdot d\theta^{(2)}, v_x >$.

Problem: to classify perturbations

$$u^i_t = A^i_j(u)u^j_x + \epsilon\left(B^i_j(u)u^j_{xx} + \frac{1}{2}C^i_{jk}(u)u^j_x u^k_x\right) + O(\epsilon^2)$$

$i = 1, \ldots, n$

Semisimplicity: characteristic roots of $(A^i_j(u))$ distinct for generic u

ϵ-expansion: Coefficient of ϵ^m is a polynomial in u_x, u_{xx}, $\ldots$, $u^{(m+1)}$ of the degree $m+1$

$$\deg u^{(k)} = k, \quad k \geq 1$$

Consider modulo Miura-type transformations

$$u^i \mapsto \tilde{u}^i = f_0^i(u) + \epsilon\, f_1^i(u; u_x) + \epsilon^2\, f_2^i(u; u_x, u_{xx}) + O(\epsilon^3)$$

$$\deg f_m^i(u; u_x, \ldots, u^{(m)}) = m, \quad \det\left(\frac{\partial f_0^i(u)}{\partial u^j}\right) \neq 0$$

Definition. The perturbation is called **trivial** if it can be eliminated by a Miura-type transformation

Simple example: Hamiltonian perturbations of Hopf equation

$$v_t + v\, v_x = 0$$

Theorem 2. Any Hamiltonian perturbation of Hopf equation remains **integrable** up to the order $O(\epsilon^4)$.

Step 1: classification. Any perturbation is equivalent, modulo $O(\epsilon^5)$, to one of the form

$$u_t + u\, u_x + \frac{\epsilon^2}{24}\left[2c\, u_{xxx} + 4c' u_x u_{xx} + c'' u_x^3\right] + \epsilon^4 \left[2p\, u_{xxxxx}\right.$$

$$\left. + 2p'(5u_{xx}u_{xxx} + 3u_x u_{xxxx}) + p''(7u_x u_{xx}^2 + 6u_x^2 u_{xxx}) + 2p''' u_x^3 u_{xx}\right] = 0$$

where $c = c(u)$, $p = p(u)$ are two arbitrary functions.

Main arguments:

• *rigidity* of the G-FZ Poisson bracket $\{v(x), v(y)\} = \delta'(x-y)$ (triviality of the Poisson cohomology, Getzler 2001)

So, it suffices to classify *deformations of the Hamiltonian*

$$H_0 = \frac{1}{6}\int v^3\, dx \quad \mapsto \quad H_\epsilon = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \dots$$

• Classify H_ϵ modulo *canonical transformations*

$$u \mapsto u + \epsilon\{u(x), F\} + \frac{\epsilon^2}{2}\{\{u(x), F\}, F\} + \dots$$

(time-ϵ shift generated by a Hamiltonian F).

So, any perturbation of H_0 is equivalent to

$$H_\epsilon = \int \left[\frac{u^3}{6} - \epsilon^2 \frac{c(u)}{24} u_x^2 + \epsilon^4 \, p(u) u_{xx}^2\right] dx + O(\epsilon^5)$$

for some functions $c(u)$, $p(u)$.

Step 2: deforming the entire commutative algebra. Define a deformations of the first integral $H_0^f = \int f(v)\, dx$ by

$$H_\epsilon^f = \int h_f \, dx$$

$$h_f = f - \frac{\epsilon^2}{24} c\, f''' u_x^2 + \epsilon^4 \left[\left(p\, f''' + \frac{c^2\, f^{(4)}}{480}\right) u_{xx}^2 - \left(\frac{c\, c''\, f^{(4)}}{1152} + \frac{c\, c'\, f^{(5)}}{1152} + \frac{c^2\, f^{(6)}}{3456} + \frac{p'\, f^{(4)}}{6} + \frac{p\, f^{(5)}}{6}\right) u_x^4\right]$$

Then

$$\{H_\epsilon^f, H_\epsilon^g\} = 0 \ \left(\text{mod } O(\epsilon^6)\right) \quad \text{for any} \quad f = f(u),\ g = g(u)$$

Multicomponent case: deformation theory of *bihamiltonian* hyperbolic PDEs.

• Uniqueness results ("bihamiltonian cohomology"): any deformation of an order n hyperbolic system depends on at most n arbitrary functions of 1 variable $c_1(u_1)$, . . . , $c_n(u_n)$

• Existence result (in progress): for the integrable hyperbolic systems associated with a semisimple Frobenius manifold the integrable deformation exists in all orders in ϵ for the particular choice

$$c_1 = c_2 = \ldots = c_n = 1$$

(integrable hierarchies of the topological type).

The construction uses “quantization” of the Riemann - Hilbert problem associated with the semisimple Frobenius manifold considered as a canonical transformation of the Givental symplectic space.

Corollary. For any n the total GW potential of $\mathbf{CP}^n$ is a tau function of a particular solution to an integrable hierarchy of the order $n+1$. (Uses Givental's proof of the Virasoro conjecture for $\mathbf{CP}^n$).

List of examples:

- $n = 1$ KdV

- $n = 2$ Boussinesq, (extended) Toda, (extended) nonlinear Schrödinger

- any $n \geq 1$ Drinfeld - Sokolov hierarchies of ADE type

Other hierarchies are new

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