On the Renormalization Group Flow

in Two Dimensional Superconformal Models

by **Changrim Ahn** and **Marian Stanishkov**

1. Introduction

In this paper we extend the results of the paper [1] to the case of **supersymmetric minimal models SM_p, $p \to \infty$, perturbed by the least relevant fields**. The first order corrections were already obtained time ago in [2] . There it was argued that there exists an IR fixed point of the RG flow which coincides with the minimal superconformal model SM_{p-2}. In the paper [1] (see also [3]) the β function, the fixed point and the matrix of anomalous dimensions of certain fields was obtained up to second order in perturbation theory. That extends the famous results of A.Zamolodchikov [4] . **Calculation up to second order is always a challenge even in two dimensions**. The problem is that one needs the corresponding **four-point function** which is not known exactly even in two dimensions. Fortunately, in the scheme proposed in [1] (which is an extension of that proposed by Zamolodchikov in [4]) one needs the value of this function **up to 0'th order in the small parameter** $\epsilon = \frac{2}{p+2}$.

Basic ingredients for the computation of the correlation functions in two dimensions are the **conformal blocks**. In the last years an exact relation between the latter and the instanton partition functions of certain $N = 2$ super YM theories in four dimensions, the so called **AGT correspondence**, was established (see e.g. [5-8]) For the $N = 1$ superconformal theories that motivated the computation of the recurrence relation for the conformal blocks of the NS [9-11] and Ramond [12-13] fields of the theory. It was shown in [14-16] that these conformal blocks coincide with the instanton partition functions of super YM theories in certain spaces. With these basic ingredients in hand we computed here the four-point functions up to the desired order.

The other difficulty arise in the **regularization of the integrals** involved. We follow here the regularization proposed in [1] and show that it works perfectly in our case.

One can possibly further consider the more general case of $SU(2)$ coset models. It was shown time ago [17] that the structure constants and conformal blocks (basic ingredients for the calculation) for these theories can be obtained from just the usual minimal models by certain projected tensor product (this was recently generalized for the super-Liouville theory [18]). On that basis also a generalized AGT relation was proposed [19-21].

The paper [1] was also motivated by an alternative approach to the perturbed minimal models, the so called RG domain wall [22] . The comparison gives a perfect agreement with the perturbative calculations to second order. Moreover it was found there that the eigenvectors expressing the fields of the IR CFT do not receive ϵ-corrections and it was speculated that they are exact. We obtained the same result in the supersymmetric case. Moreover, the aforementioned eigenvectors are exacly the same as in the $N = 0$ minimal models. One can speculate that probably this result is universal for all the coset models perturbed by the least relevant field.

This paper is organized as follows.

In Section 2 we present the $N = 1$ SM_p theory perturbed by the last component of the superfield $\Phi_{1,3}$. The basic ingredients necessary for the calculations in the second order of the perturbation theory are presented. We refer for the details to the original paper [1] .

In Section 3 we give some details needed for the computation of the conformal blocks in the NS sector. We mention also the important issue of the normalization of the fields.

Section 4 is devoted to the computation of the beta function and the IR fixed point. It is confirmed that it coincides up to second order with the model SM_{p-2}.

The matrix of anomalous dimensions for some components of the superfields $\Phi_{n,n\pm2}$ and $D\bar{D}\Phi_{n,n}$ was computed in Section 5. It is in perfect agreement with the first order result in [2] . The same is proved also for the first component $\phi_{n,n}$.

In Section 6 we explain how to compute the mixed conformal blocks of the (last components of) NS and Ramond fields. They are necessary for the calculations of the anomalous dimensions for the Ramond fields which is also presented there. The results are again in agreement with the conjectured RG flow to SM_{p-2}.

2. The theory

In this paper we consider a **minimal superconformal theory** SM_p perturbed by the least relevant field. This theory is invariant under $N = 1$ superconformal algebra with central charge $\hat{c} = 1 - \frac{8}{p(p+2)}$, $p = 3, 4, ...$ It contains a discrete set of primary fields divided into a NS and Ramond sectors depending on their supersymmetric transformations. The set of dimensions is labeled by two integers:

$$\Delta_{n,m} = \frac{((p+2)n - pm)^2 - 4}{8p(p+2)}(+\frac{1}{16})$$

and $m - n$ =even (odd) for NS (R) sectors respectively (the addition in brackets is for the Ramond fields). The fields in the NS sector are organized in superfields:

$$\Phi(z, \bar{z}, \theta, \bar{\theta}) = \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}\tilde{\phi}$$

The first (and the last) components of the spinless superfield of dimensions $\Delta = \bar{\Delta}$ ($\Delta + \frac{1}{2} = \bar{\Delta} + \frac{1}{2}$) are expressed as a product of "chiral fields" depending on z and $\bar{z}$ respectively. We use below for these chiral components the same notations ϕ and $\tilde{\phi}$. By supersymmetry, if we fix the two-point function of the first component ϕ to one then the second component's two-point function is $(2\Delta)^2$. Since in the renormalization procedure it is assumed that these functions are all equal to one, we have to normalize the second component $\tilde{\phi} \to \frac{1}{2\Delta}\tilde{\phi}$.

We will consider the superminimal model SM_p with $p \to \infty$ perturbed by the **least relevant field** $\tilde{\phi} = \tilde{\phi}_{1,3}$ of dimension $\Delta = \Delta_{1,3} + \frac{1}{2} = 1 - \epsilon$, $\epsilon = \frac{2}{p+2} \to 0$:

$$H(x) = H_0(x) + \lambda\tilde{\phi}(x)$$

It is obvious that this theory is also supersymmetric, since the perturbation can be written as a covariant super-integral over the superfield $\Phi_{1,3}$.

The two-point function of arbitrary fields up to second order is then given by:

$$< \phi_1(x)\phi_2(0) >_\lambda =< \phi_1(x)\phi_2(0) >_0 -\lambda \int < \phi_1(x)\phi_2(0)\tilde{\phi}(y) >_0 d^2y+$$
$$+ \frac{\lambda^2}{2} \int < \phi_1(x)\phi_2(0)\tilde{\phi}(x_1)\tilde{\phi}(x_2) >_0 d^2x_1 d^2x_2$$

where ϕ_1, ϕ_2 can be first or last components of a superfield or Ramond fields of dimensions Δ_1, Δ_2. The first order corrections were considered in [2] , here we will focus on the second order.

One can **use the conformal transformation properties** of the fields to bring the double integral to the form:

$$\begin{aligned}&\int < \phi_1(x)\phi_2(0)\tilde{\phi}(x_1)\tilde{\phi}(x_2) >_0 d^2x_1 d^2x_2 = \\ &= (x\bar{x})^{2-\Delta_1-\Delta_2-2\Delta} \int I(x_1) < \tilde{\phi}(x_1)\phi_1(1)\phi_2(0)\tilde{\phi}(\infty) >_0 d^2x_1\end{aligned} \tag{2.1}$$

where

$$I(x) = \int |y|^{2(a-1)}|1-y|^{2(b-1)}|x-y|^{2c} d^2y$$

and $a = 2\epsilon + \Delta_2 - \Delta_1$,$b = 2\epsilon + \Delta_1 - \Delta_2$, $c = -2\epsilon$. It is well known that the integral for $I(x)$ can be expressed in terms of hypergeometric functions:

$$\begin{aligned}I(x) = &\frac{\pi\gamma(b)\gamma(a+c)}{\gamma(a+b+c)}|F(1-a-b-c, -c, 1-a-c, x)|^2+ \\ &+ \frac{\pi\gamma(1+c)\gamma(a)}{\gamma(1+a+c)}|x^{a+c}F(a, 1-b, 1+a+c, x)|^2\end{aligned} \tag{2.2}$$

This form is useful for evaluating $I(x)$ near 0. Using the transformation properties of the hypergeometric functions, (2.2)can be rewritten as a function of $1-x$ and $\frac{1}{x}$ which is suitable for the investigation of $I(x)$ around the points 1 and ∞ respectively.

It is clear that **the integral (2.1)is singular**. We follow the regularization procedure proposed in [1] . It consists basically in cutting discs in the two-dimensional surface of radius l ($\frac{1}{l}$) around singular points 0, 1 (∞): $D_{l,0} = \{x \in C, |x| < l\}$, $D_{l,1} = \{x \in C, |x-1| < l\}$, $D_{l,\infty} = \{x \in C, |x| > 1/l\}$. Here l**:** $0 < l_0 < l < 1$ **is some intermediate scale,** l_0 **is the ultraviolet cut-off**. Clearly l should be canceled in the calculations and should not appear in the final result. Outside these discs one can safely integrate the correlation function (in the limit $\epsilon \to 0$), we call that "safe region" Ω_{l,l_0}. It is useful to do this integration in **radial coordinates**. Since the correlation function exhibits poles only at the points 0 and 1 the phase integration can be performed by using residue theorem and the resulting rational integral in the radial direction is straightforward. **Near the singular points one can use the OPE**. In doing that it turns out that we count twice **two lens-like regions** around the point 1 so we have to subtract those integrals. The latter are computed in [1] and we refer to that article for the explicit formulas, as well as for a more detailed explanation.

3. Computation of the conformal blocks in the NS - sector

Let us start with the correlation function that enters in the integral (2.1). The basic ingredients for the computation of the four-point correlation functions are the conformal blocks. These are quite complicated objects in general and close formula was not known. Recently it was argued that they coincide (up to factors) with the instanton partition function of certain $N=2$ YM theories on ALE spaces [9-11]. It was found a recurrence relation for the conformal blocks and proved that the mentioned partition function obey the same. Here we need the expressions for **the first few levels conformal blocks in order to have a guess** for the limit $\epsilon \to 0$, i.e. close to a free theory with $\hat{c} = 1$.

The chiral components of the fields obey the OPEs:

$$\begin{aligned}
\phi_1(x)\phi_2(0) &= x^{\Delta-\Delta_1-\Delta_2}\sum_{N=0}^{\infty}x^N C_N\phi_\Delta(0)\\
\tilde{\phi}_1(x)\phi_2(0) &= x^{\Delta-\Delta_1-\Delta_2-1/2}\sum_{N=0}^{\infty}x^N \tilde{C}_N\phi_\Delta(0)\\
\phi_1(x)\tilde{\phi}_2(0) &= x^{\Delta-\Delta_1-\Delta_2-1/2}\sum_{N=0}^{\infty}x^N \tilde{C}'_N\phi_\Delta(0)\\
\tilde{\phi}_1(x)\tilde{\phi}_2(0) &= x^{\Delta-\Delta_1-\Delta_2-1}\sum_{N=0}^{\infty}x^N C'_N\phi_\Delta(0)
\end{aligned} \tag{3.1}$$

where C_N's are polynomials in the generators of the superconformal algebra L_{-k} and $G_{-\alpha}$ $(k,\alpha>0)$ (with coefficients depending on the dimensions Δ, Δ_1, Δ_2 which we omitted)of dimension N usually called **chain vectors**. Here N runs over all nonnegative integers and half-integers. In our case of supersymmetric minimal models SM_p which one is relevant depend on the fusion rules.

Acting by positive mode generators on the both sides of these OPEs and using the super-conformal transformation properties gives the chain equations for L's:

$$L_k C_N = (\Delta + k\Delta_1 - \Delta_2 + N - k)C_{N-k}$$

(here C is any of of the chain vectors with the corresponding dimensions of the fields) and for G's:

$$\begin{aligned}
G_k C_N &= \tilde{C}_{N-k}\\
G_k \tilde{C}_N &= (\Delta + 2k\Delta_1 - \Delta_2 + N - k)C_{N-k}\\
G_k \tilde{C}'_N &= C'_{N-k}\\
G_k C'_N &= (\Delta + 2k\Delta_1 - \Delta_2 + N - k - \tfrac{1}{2})\tilde{C}'_{N-k}
\end{aligned} \tag{3.2}$$

where everywhere $k>\frac{1}{2}$, and in addition:

$$\begin{aligned}
G_{\frac{1}{2}}\tilde{C}'_N &= 2\Delta_2 C_{N-\frac{1}{2}} + C'_{N-\frac{1}{2}}\\
G_{\frac{1}{2}}C'_N &= -2\Delta_2\tilde{C}_{N-\frac{1}{2}} + (\Delta+\Delta_1-\Delta_2+N-1)\tilde{C}'_{N-\frac{1}{2}}
\end{aligned}$$

There are two independent constants at zero level in the OPEs (3.1), the other two are expressible through them:

$$\tilde{C}'_0 = -\tilde{C}_0, \qquad C'_0 = (\Delta-\Delta_1-\Delta_2)C_0$$

The above chain relations could be solved order by order. As mentioned before, in [10] ,[11] a recursion relation for the chain vectors was also found. We give here as an example and for further use the first terms for C':

$$\begin{aligned} C'_{\frac{1}{2}} &= -\frac{\Delta+\Delta_1+\Delta_2-\frac{1}{2}}{2\Delta}\tilde{C}_0 G_{-\frac{1}{2}} \\ C'_1 &= \frac{\Delta+\Delta_1-\Delta_2}{2\Delta} C_0 L_{-1} \end{aligned} \tag{3.3}$$

With the chain vectors in hand the conformal blocks are readily obtained. Presenting them as a vectors in the basis of L's and G's the conformal block can be expressed as:

$$F(\Delta,\Delta_i) = \sum_{N=0}^{\infty} x^N F_N = \sum_{N=0}^{\infty} x^N C_N(\Delta,\Delta_3,\Delta_4) S_N^{-1} C_N(\Delta,\Delta_1,\Delta_2)$$

where S_N is the Shapovalov matrix at level N. What of C_N's appear depends on the external fields involved.

The conformal blocks are in general quite complicated objects. Fortunately, in view of the renormalization scheme and the regularization of the integrals, **we need to compute them here only up to zero'th order in ϵ**. This simplifies significantly the problem.

Once the conformal blocks are known the correlation function of spinless fields for our SM_p models is written as:

$$\sum_n C_n |F(\Delta_n,\Delta_i)|^2$$

where the **range of n depends on the fusion rules and C_n is the corresponding structure constant**. Let us stress that the various structure constants that appear are connected. If we call the structure constant of the fist components C and that of two first and one last component $\tilde{C}$ then for the remaining ones we have:

$$\begin{aligned} &< \tilde{\phi}_1(\infty)\tilde{\phi}_2(1)\phi_3(0) > = (\Delta_3-\Delta_1-\Delta_2)^2 C_{(1)(2)(3)} \\ &< \tilde{\phi}_1(\infty)\tilde{\phi}_2(1)\tilde{\phi}_3(0) > = (\tfrac{1}{2}-\Delta_1-\Delta_2-\Delta_3)^2 \tilde{C}_{(1)(2)(3)} \end{aligned} \tag{3.4}$$

The structure constants $C_{(1)(2)(3)}$ and$\tilde{C}_{(1)(2)(3)}$ were obtained in [23] . We have to keep in mind also that our last components are normalized by $1/2\Delta$. In what follows we **compute the conformal blocks up to sufficiently high level and then check also the crossing symmetry and the behavior near the singular points** 1 **and** ∞.

4. β-function and fixed point

For the computation of the β-function in second order we need the four-point function of the perturbing field. Here we present the more general function $< \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(\infty) >$. There are three "channels" (or intermediate fields) in the corresponding conformal block: two even, corresponding to the identity and $\phi_{1,5}$ and one odd - to $\tilde{\phi}$ itself. Following the procedure we explained above, we get the following expression for this correlation function:

$$< \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(\infty) >= \\ = \left|\frac{(1-2x+7/3x^2-4/3x^3+1/3x^4)}{x^2(1-x)^2}\right|^2 + \frac{2(n+3)}{3(n+1)}\left|\frac{(1-3/2x+3/2x^2-1/2x^3)}{x(1-x)^2}\right|^2 \\ + \frac{(3+n)(4+n)}{18n(1+n)}\left|\frac{(1-x+x^2)}{(1-x)^2}\right|^2$$

We checked explicitly the crossing symmetry and the $x \to 1$ limit of this function. The function that enters the integral is obtained by the conformal transformation $x \to 1/x$ (explicit formula is presented below):

$$< \tilde{\phi}(x)\tilde{\phi}_{n,n+2}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}(\infty) >= \\ = \left|\frac{(1-4x+7x^2-6x^3+3x^4)}{3x^2(-1+x)^2}\right|^2 + \frac{2(n+3)}{3(n+1)}\left|\frac{(-1+3x-3x^2+2x^3)}{2x^2(-1+x)^2}\right|^2 \\ + \frac{(3+n)(4+n)}{18n(1+n)}\left|\frac{(1-x+x^2)}{x^2(-1+x)^2}\right|^2 \tag{4.1}$$

In order to compute the β-function and the fixed point to second order we just have to integrate the above function with $n = 1$.

The integration over the safe region far from the singularities yields ($I(x) \sim \frac{\pi}{\epsilon}$):

$$\int_{\Omega_{l,l_0}} I(x) < \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) > d^2x = \\ = -\frac{35\pi^2}{24\epsilon} + \frac{2\pi^2}{\epsilon l^2} + \frac{\pi^2}{2\epsilon l_0^2} - \frac{16\pi^2 \log l}{3\epsilon} - \frac{8\pi^2 \log 2l_0}{3\epsilon}$$

and **we omitted the terms of order l or l_0/l.**

We have to subtract the integrals over the lens-like regions since they would be accounted twice. We need to expand the function around 1 and compute the integrals using the formulas in [1] . Here is the result of that integration:

$$\frac{\pi^2}{\epsilon}\left(-\frac{1}{l^2} + \frac{1}{2l_0^2} + \frac{61}{24} - \frac{8}{3}\log(\frac{l}{2l_0})\right)$$

Next we have to compute the integrals near the singular points $0, 1$ and ∞. For that purpose we can use the OPE of the fields and take the appropriate limit of $I(x)$.

Near the point 0 the relevant OPE is:

$$\tilde{\phi}(x)\tilde{\phi}(0) = (x\bar{x})^{-2(\Delta_{1,3}+\frac{1}{2})}(1 + ...) + \hat{C}^{(1,3)}_{(1,3)(1,3)}(x\bar{x})^{-(\Delta_{1,3}+\frac{1}{2})}(\tilde{\phi}(0) + ...)$$

The channel $\phi_{1,5}$ gives after integration a term proportional to $\frac{l}{l_0}$ and we neglect it. The structure constant is the one of the normalized second components of a superfield. Here and below we denote these normalized structure constants by $\hat{C}$. The correct value therefore is computed as:

$$\hat{C}^{(1,3)}_{(1,3)(1,3)} = \frac{(\frac{1}{2} - 3\Delta_{1,3})^2}{(2\Delta_{1,3})^3}\tilde{C}^{(1,3)}_{(1,3)(1,3)} = \frac{2}{\sqrt{3}} - 2\sqrt{3}\epsilon$$

to first order in ϵ. The value of $I(x)$ near 0 can be found by taking the limit in (2.2)written in terms of $1/x$ (explicit form is given in [1]). Finally one gets:

$$\int_{D_{l,0}\ \ D_{l_0,0}} I(x) < \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) > d^2x =$$
$$= -\frac{\pi^2}{l^2\epsilon} + \frac{8\pi^2}{3\epsilon^2} - \frac{16\pi^2}{\epsilon} + \frac{8}{3}\frac{\pi^2 \log l}{\epsilon}$$

Note that the integral near 1 gives obviously the same result. So we have just to add the above result twice. To compute the integral near infinity lets first pass to variable $1/x \sim 0$ and use the OPE. The necessary connection is:

$$< \phi_1(x)\phi_2(0)\phi_3(1)\phi_4(\infty) >= (x\bar{x})^{-2\Delta_1} < \phi_1(1/x)\phi_4(0)\phi_3(1)\phi_2(\infty) > \qquad (4.2)$$

for any fields ϕ_i. Also, at this point $I(x) \sim \frac{\pi}{\epsilon}(x\bar{x})^{-2\epsilon}$. Calculation goes in the same way:

$$\int_{D_{l,\infty}\ \ D_{l_0,\infty}} I(x) < \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) > d^2x =$$
$$= -\frac{\pi^2}{l^2\epsilon} + \frac{4\pi^2}{3\epsilon^2} - \frac{8\pi^2}{\epsilon} + \frac{8\pi^2 \log l}{3\epsilon}$$

Finally putting altogether we obtain for the finite part of the integral:

$$\frac{20\pi^2}{3\epsilon^2} - \frac{44\pi^2}{\epsilon}$$

Here we want to mention that we follow the renormalization scheme proposed in [1] . Therefore we already **omitted the terms proportional to $\frac{1}{l_0^{2-4\epsilon}}$ which could be canceled by an appropriate counterterm in the action**.

Taking into account also the first order term (proportional to the above structure constant and computed in[2]) the final result (up to second order)for the two-point function of the perturbing field is:

$$
\begin{aligned}
G(x,\lambda) =&< \tilde{\phi}(x)\tilde{\phi}(0) >_\lambda= \\
&= (x\bar{x})^{-2+2\epsilon}(1-\lambda\frac{4\pi}{\sqrt{3}}\left(\frac{1}{\epsilon}-3\right)(x\bar{x})^{\epsilon}+ \\
&+\frac{\lambda^2}{2}\left(\frac{20\pi^2}{3\epsilon^2}-\frac{44\pi^2}{\epsilon}\right)(x\bar{x})^{2\epsilon}+...)
\end{aligned}
$$

Following the renormalization scheme [1] we introduce a **renormalized coupling** g **and a renormalized field** $\tilde{\phi}^g = \partial_g H$ (as $\tilde{\phi} = \partial_\lambda H$, and assume that $\tilde{\phi}^g = B(\lambda)\tilde{\phi}$, i.e. multiplicative renormalization). The two-point function of the latter $G(x,g) =< \tilde{\phi}^g(x)\tilde{\phi}^g(0) >$ is **normalized** as $G(1,g) = 1$. The renormalizability of the theory implies that:

$$\Theta(x) = \epsilon\lambda\tilde{\phi}(x) = \beta(g)\tilde{\phi}^g(x)$$

where Θ is the trace of the energy-momentum tensor and $\beta(g)$ is the β-function. From here we have:

$$
\begin{aligned}
\partial_\lambda g &= \sqrt{G(1,\lambda)} \\
\beta(g) &= \epsilon\lambda\frac{\partial g}{\partial\lambda} = \epsilon\lambda\sqrt{G(1,\lambda)}
\end{aligned}
$$

One can invert this and compute the bare coupling constant in terms of g and then the β-function. In our case this gives:

$$\lambda = g + g^2\frac{\pi}{\sqrt{3}}(\frac{1}{\epsilon}-3) + g^3\frac{\pi^2}{3}(\frac{1}{\epsilon^2}-\frac{5}{\epsilon}) + ... \tag{4.3}$$

and

$$\beta(g) = \epsilon g - g^2\frac{\pi}{\sqrt{3}}(1-3\epsilon) + ...$$

Note that it is important to keep in mind that we assume that the coupling constant λ (and g) is small, of order $\sim\epsilon$, and to keep only the relevant terms in the above calculations.

The above β-function has a zero:

$$g^* = \frac{\sqrt{3}}{\pi}\epsilon(1+\epsilon) \tag{4.4}$$

which is a non-trivial infrared fixed point. It coincides with the minimal supercoformal model SM_{p-2} as can be seen from the formula for the central charge:

$$c^* - c = -(\frac{2}{3})12\pi^2 \int_0^{g^*} \beta(g)dg = -4\epsilon^3 - 12\epsilon^4 + ...$$

up to this order in ϵ.

The **anomalous dimension** of the perturbing field:

$$\Delta^* = 1 - \partial_g \beta(g)|_{g^*} = 1 + \epsilon + 2\epsilon^2 + ...$$

matches the dimension of the second component of the superfield $\Phi_{3,1}^{p-2}$ of SM_{p-2}.

5. Mixing of the super-fields in the NS sector

Note that the choice of the perturbing field (i.e. second component of a super-field) guarantees the preservation of **super-symmetry along the RG flow**. The dimensions (close to $(1/2, 1/2)$) and the fusion rules of the **super-fields $\Phi_{n,n\pm2}$ and $D\bar{D}\Phi_{n,n}$ (D is the covariant super-derivative) suggest that they mix along the RG-trajectory**. We will compute the corresponding matrix of anomalous dimensions for the second components, the mixing of the first ones is a consequence of the super-symmetry. For this purpose we need to compute the matrix of the two-point functions involved and the corresponding integrals.

5.1. Function $< \tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(0) >_\lambda$

The corresponding function in the second order of the perturbation was already written above (4.1). The integration over the safe region (far from the singularities) goes in the same way as before. The result is:

$$\int_{\Omega_{l,l_0}} I(x) < \tilde{\phi}(x)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(0)\tilde{\phi}(\infty) > d^2x =$$
$$= \frac{(n+2)\pi^2}{6\epsilon n l_0^2} + \frac{2(1+2n)\pi^2}{3n\epsilon l^2} -$$
$$- \frac{4(1+5n+2n^2)\pi^2 \log l}{3\epsilon(n+n^2)} - \frac{4(1+n)\pi^2 \log 2l_0}{3n\epsilon} - \frac{(18+43n+9n^2)\pi^2}{24\epsilon(n+n^2)}$$

Also the integration over the lens-like regions is similar and gives:

$$\frac{\pi^2}{24n(1+n)\epsilon}[\frac{-8(1+n)(2+n)}{l^2}+\frac{4(1+n)(2+n)}{l_0^2}+(46+n(53+23n))+$$
$$+32(1+n)^2(\log l-\log 2l_0)]$$

In taking the integral around zero one should be more careful. It turns out that one should take into account also the descendents since they contribute nontrivial singular terms. Explicitly we have for the OPE:

$$\tilde{\phi}(x)\tilde{\phi}_{n,n+2}(0)=(x\bar{x})^{-(\Delta_{1,3}+1/2)}\hat{C}^{(n,n+2)}_{(1,3)(n,n+2)}\tilde{\phi}_{n,n+2}(0)+(x\bar{x})^{\Delta_{n,n}-\Delta_{n,n+2}-\Delta_{1,3}-1}\hat{C}^{(n,n)}_{(1,3)(n,n+2)}$$
$$(1+\frac{\Delta_{1,3}+\Delta_{n,n}-\Delta_{n,n+2}}{2\Delta_{n,n}}xL_{-1})(1+\frac{\Delta_{1,3}+\Delta_{n,n}-\Delta_{n,n+2}}{2\Delta_{n,n}}\bar{x}\bar{L}_{-1})\phi_{n,n}(0) \tag{5.1}$$

(the coefficient in front of L_{-1} is obtained from the chain relations (3.3)). In a CFT L_{-1} acts as a derivative and therefore:

$$<L_{-1}\phi_{n,n}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}(\infty)>=(\Delta_{n,n}+\Delta_{n,n+2}-\Delta_{1,3})<\phi_{n,n}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}(\infty)>$$

Let us stress again that the structure constants needed for the calculation are the "normalized" ones:

$$\begin{aligned}\hat{C}^{(n,n+2)}_{(1,3)(n,n+2)} &= \frac{(\frac{1}{2}-\Delta_{1,3}-2\Delta_{n,n+2})^2}{2\Delta_{1,3}(2\Delta_{n,n+2})^2}\tilde{C}^{(n,n+2)}_{(1,3)(n,n+2)} \\ &= \frac{(3+n)^2}{3(1+n)^2}-\frac{2(2+n)(3+n)^2\epsilon}{3(1+n)^2} \\ \hat{C}^{(n,n)}_{(1,3)(n,n+2)} &= \frac{(\Delta_{n,n}-\Delta_{1,3}-\Delta_{n,n+2})^2}{2\Delta_{1,3}2\Delta_{n,n+2}}C^{(n,n)}_{(1,3)(n,n+2)}=\sqrt{\frac{n+2}{3n}}\end{aligned} \tag{5.2}$$

Taking the same $I(x)$ as before and integrating gives for the channel $\phi_{n,n}$:

$$-\frac{(2+n)\pi^2}{3\epsilon nl^2}+\frac{2(-1+n)^2(2+n)(5+n)\pi^2}{3\epsilon^2 n(1+n)^2(3+n)^2}+$$
$$+\frac{(2+n)\pi^2(-4(-1+n)(-1+23n+9n^2+n^3)+4(-1+n)^2(3+n)^2\log l)}{6\epsilon n(1+n)^2(3+n)^2}$$

The channel $\tilde{\phi}_{n,n+2}$ is simpler, here it is sufficient to take $I(x)$ just to order 1 and no descendants:

$$\frac{2(3+n)^2\pi^2}{3(1+n)^2\epsilon^2}-\frac{(3+n)^2\pi^2(8+4n-2\log l)}{3(1+n)^2\epsilon}$$

The integrals around 1 are obviously the same, so the total contribution is twice the sum of above two terms.

Computation around infinity is almost the same as the one for the β-function, just we put the correct structure constants:

$$-\frac{\pi^2}{\epsilon l^2}+\frac{2(3+n)\pi^2}{3(1+n)\epsilon^2}-\frac{2\pi^2((3+n)(5+n)-(2n+6)\log l)}{3(1+n)\epsilon}$$

Finally, combining all the terms we get:

$$-\frac{2\pi^2(-20-143n-121n^2-33n^3-3n^4)}{3n(1+n)(3+n)^2\epsilon^2}-\frac{2\pi^2(5+n)(8+151n+143n^2+45n^3+5n^4)}{3n(1+n)(3+n)^2\epsilon}$$

(note that although the various integrals differ from that of [1] the final result is very similar, this will be also the case with the next integrals).

5.2. Function $<\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(0)>_\lambda$

Our calculation gives for the relevant four-point function in this case (order 0 in ϵ):

$$<\tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(\infty)>=\frac{1}{3}\sqrt{\frac{(-4+n^2)}{n^2}}\left|\frac{1}{(1-x)^2}(1-x+x^2)\right|^2$$

(only channel $\phi_{1,5}$ appears here). Transforming $x\to\frac{1}{x}$ according to (4.2) one obtains for the function in the integral (2.1):

$$<\tilde{\phi}(x)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(0)\tilde{\phi}(\infty)>=\frac{1}{3}\sqrt{\frac{(-4+n^2)}{n^2}}\left|\frac{1}{x^2(1-x)^2}(1-x+x^2)\right|^2$$

(note that this is different from [1]). For the integral over the safe region we need $I(x)$ which can be extracted from (2.2):

$$I(x)=-\frac{4\pi}{(n^2-4)\epsilon}$$

and the result is:

$$\int_{\Omega_{l,l_0}}I(x)<\tilde{\phi}(x)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(0)\tilde{\phi}(\infty)>d^2x=$$

$$=-\frac{4\pi^2}{3\epsilon n\sqrt{-4+n^2}l^2}-\frac{2\pi^2}{3\epsilon n\sqrt{-4+n^2}l_0^2}+\frac{\pi^2(9+16\log(2ll_0))}{6\epsilon n\sqrt{-4+n^2}}$$

Expanding around 1 and taking the integrals in the lens-like regions gives:

$$-\frac{\pi^2(23-\frac{8}{l^2}+\frac{4}{l_0^2}+16\log\frac{l}{2l_0})}{6en\sqrt{-4+n^2}}$$

(this term is to be subtracted).

The integral around the point 0 is very similar to that in the previous section. The only difference is that we take $\Delta_{n,n-2}$ instead of $\Delta_{n,n+2}$ in (5.1). Also, in the computation of the appropriate approximation of $I(x)$ we have to expand the hypergeometric functions for the channel $\phi_{n,n}$ up to order x. The computation for the channel $\tilde{\phi}_{n,n-2}$ is the same as above. Finally, we need the structure constant:

$$\hat{C}^{(n,n)}_{(1,3)(n,n-2)}=\frac{(\Delta_{n,n}-\Delta_{1,3}-\Delta_{n,n-2})^2}{2\Delta_{1,3}2\Delta_{n,n-2}}C^{(n,n)}_{(1,3)(n,n-2)}=\sqrt{\frac{n-2}{3n}}$$

At the end we get:

$$\int_{D_{l,0}\ D_{l_0,0}} I(x)<\tilde{\phi}(x)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(0)\tilde{\phi}(\infty)>d^2x=$$

$$=\frac{4\pi^2}{3\epsilon^2 n(-9+n^2)\sqrt{-4+n^2}}(10+\epsilon\frac{(-9+n^2)}{l^2}-2\epsilon(1+n^2)-2\epsilon(-9+n^2)\log l)$$

In principle one should compute also the integral around 1. Just as in the case of [1] it turns out to be the same as those around 0. So its enough to take the above result twice. Also, the integral around ∞ here is not singular, so we neglect it. It remains to collect the integrals computed above:

$$\frac{80(1-2\epsilon)\pi^2}{3\epsilon^2 n(-9+n^2)\sqrt{-4+n^2}}$$

Lets mention again that even though the individual integrals are different, the finite result is similar to that of [1] .

5.3. Function $<\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0)>_\lambda$

The function to integrate turns out to be almost the same as in [1] :

$$<\tilde{\phi}(x)\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0)\tilde{\phi}(\infty)>=\frac{2}{3}\sqrt{\frac{n+2}{n}}|x|^{-2}$$

and I(x) exactly coincides. Consequently the result of the integration over the safe region is almost the same:

$$\int_{\Omega_{l,l_0}} I(x)<\tilde{\phi}(x)\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0)\tilde{\phi}(\infty)>d^2x=$$

$$=\frac{8\epsilon\pi^2}{3(5+n)}\sqrt{\frac{2+n}{n}}((-5+3n)\log l+(1+n)\log 2l_0)$$

Same is true also for the lens-like integration:

$$-\frac{8\epsilon(1+n)\pi^2}{3(5+n)}\sqrt{\frac{2+n}{n}}\log\frac{l}{2l_0}$$

The OPE needed for computation around 0 was written above (5.1). One has to arrange the corresponding dimensions and structure constants from (5.2). So the contribution from the region near 0 is:

$$\int_{D_{l,0}\ D_{l_0,0}} I(x) < \tilde{\phi}(x)\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0)\tilde{\phi}(\infty) > d^2x =$$
$$= -\frac{4(-1+n)\sqrt{\frac{2+n}{n}}\pi^2}{3(3+n)(5+n)}(1+(n+(n+3)\log l)\epsilon)-$$
$$-\frac{2(n+3)\pi^2\sqrt{\frac{2+n}{n}}}{3(5+n)}(1+3\epsilon+2\epsilon\log l)$$

(this is not the same as [1] because of the different dimensions). Surprisingly the computation around the point 1 again gives a result identical to that around 0. So we have to add again twice the above contribution.

To compute the contribution from the region near ∞ we perform again the $x \to 1/x$ map (4.2). The necessary structure constants are already written above and we take the appropriate (up to ϵ^2) approximation for $I(x)$. The result is:

$$\int_{D_{l,\infty}\ D_{l_0,\infty}} I(x) < \tilde{\phi}(x)\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0)\tilde{\phi}(\infty) > d^2 =$$
$$= -\frac{8(-3+n)\pi^2}{3(n+5)}\sqrt{\frac{n+2}{n}}(1+(2+n+2\log l)\epsilon)$$

Combining all the terms:

$$-\frac{4(-1+n)\pi^2}{3(3+n)(5+n)}\sqrt{\frac{n+2}{n}}(11+3n+\epsilon(1+n)(9+2n))$$

(note that this is different from [1]).

5.4. Function $< \phi_{n,n}(1)\phi_{n,n}(0) >_\lambda$

Finally we need the function $< \tilde{\phi}(x)\phi_{n,n}(1)\phi_{n,n}(0)\tilde{\phi}(\infty) >$. It happens that this function coincides exactly with the one reported in [1] (**logarithmic instead of polynomial**)

. Therefore almost all integrals are the same. The only exception is the integral around ∞ due to the different structure constant:

$$\hat{C}^{(n,n)}_{(1,3)(n,n)}\hat{C}^{(1,3)}_{(1,3)(1,3)} = \frac{(-1+n^2)\epsilon^2}{6n}(1-2\epsilon)$$

(we took care of the normalization). With this, our final result is slightly different:

$$\frac{(-1+n^2)\pi^2}{6}(1+\epsilon)$$

At the end of this subsection we want to note that **the dimension of the first component $\phi_{n,n}$ is close to zero. Therefore it doesn't mix with other fields**. We present here the computation of its anomalous dimension. Taking into account also the first order contribution, the final result for the two-point function is:

$$\begin{aligned}
G_n(x,\lambda) &=< \phi_{n,n}(x)\phi_{n,n}(0) >_\lambda= \\
&= (x\bar{x})^{-2\Delta_{n,n}}(1-\lambda(\frac{\sqrt{3}\pi}{6}(-1+n^2)\epsilon(1+3\epsilon))(x\bar{x})^{\epsilon}+ \\
&+ \frac{\lambda^2}{2}(\frac{\pi^2}{6}(1+\epsilon)(-1+n^2))(x\bar{x})^{2\epsilon}+...)
\end{aligned}$$

Computation of the anomalous dimension goes in exactly the same way as for the perturbing field. We get:

$$\begin{aligned}
\Delta^g_{n,n} &= \Delta_{n,n} - \frac{\epsilon\lambda}{2}\partial_\lambda G_n(1,\lambda) = \\
&= \Delta_{n,n} + \frac{\sqrt{3}\pi g}{12}\epsilon^2(1+3\epsilon)(-1+n^2) - \frac{\pi^2 g^2}{12}\epsilon^2(-1+n^2)
\end{aligned}$$

(we again kept the appropriate terms of order $\epsilon \sim g$). Then at the fixed point (4.4):

$$\Delta^{g^*}_{n,n} = \frac{(-1+n^2)(\epsilon^2+3\epsilon^3+7\epsilon^4+...)}{8}$$

which exactly coincides (to this order in ϵ) with the dimension of the first component of the superfield $\Phi^{(p-2)}_{n,n}$ of the model SM_{p-2}.

5.5. Matrix of anomalous dimensions

Let us describe briefly the **renormalization scheme** of [1] . It is a variation of that originally proposed by Zamolodchikov [4] . The renormalized fields are expressed through the bare ones by:

$$\phi^g_\alpha = B_{\alpha\beta}(\lambda)\phi_\beta$$

(here ϕ could be first or last component). The matrix of anomalous dimensions Γ appears in the Callan-Symanzik equation for the renormalized two-point function $G^g_{\alpha\beta}(x) =< \phi^g_\alpha(x)\phi^g_\beta(0) >= BGB^T$, subject to the normalization condition $G^g_{\alpha\beta}(1) = \delta_{\alpha\beta}$:

$$(x\partial_x - \beta(g)\partial_g)G^g_{\alpha\beta} + \sum_{\rho=1}^{2}(\Gamma_{\alpha\rho}G^g_{\rho\beta} + \Gamma_{\beta\rho}G^g_{\alpha\rho}) = 0$$

It is expressed through the matrix B:

$$\Gamma = B\hat{\Delta}B^{-1} - \epsilon\lambda B\partial_\lambda B^{-1} \tag{5.3}$$

where $\hat{\Delta} = diag(\Delta_1, \Delta_2)$ is a diagonal matrix of the bare dimensions. The matrix B itself is computed from the matrix of the bare two-point functions we computed using the normalization condition and requiring the matrix Γ to be symmetric. Exact formulas can be found in [1] , here we present our results for the supersymmetric case.

We computed above some of the entries of the 3×3 matrix of two-point functions in the second order. This matrix is obviously symmetric. It turns out also that the remaining functions $< \tilde{\phi}_{n,n-2}(1)\tilde{\phi}_{n,n-2}(0) >_\lambda$ and $< \phi_{n,n}(1)\tilde{\phi}_{n,n-2}(0) >_\lambda$ can be obtained from the computed ones $< \tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(0) >_\lambda$ and $< \phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0) >_\lambda$ by just taking $n \to -n$. Lets denote for convenience the **basis of fields**:

$$\begin{aligned}
\phi_1 &= \tilde{\phi}_{n,n+2} \\
\phi_2 &= (2\Delta_{n,n}(2\Delta_{n,n+1}))^{-1}\partial\bar{\partial}\phi_{n,n} \\
\phi_3 &= \tilde{\phi}_{n,n-2}
\end{aligned}$$

where we normalized the field ϕ_2 so that its bare two-point function is 1. Its straightforward to correct the functions involving ϕ_2 taking into account the derivatives and the normalization.

Let us write the matrix of the 2-point functions up to second order in the perturbation expansion as:

$$\begin{aligned}
G_{\alpha,\beta}(x,\lambda) &=< \phi_\alpha(x)\phi_\beta(0) >_\lambda= \\
&= (x\bar{x})^{-\Delta_\alpha-\Delta_\beta}(\delta_{\alpha,\beta} - \lambda C^{(1)}_{\alpha,\beta}(x\bar{x})^\epsilon + \frac{\lambda^2}{2}C^{(2)}_{\alpha,\beta}(x\bar{x})^{2\epsilon} + ...)
\end{aligned}$$

The two-point functions in first order are proportional to the structure constants [4] :

$$C^{(1)}_{\alpha,\beta} = \hat{C}_{(1,3)(\alpha)(\beta)}\frac{\pi\gamma(\epsilon + \Delta_\alpha - \Delta_\beta)\gamma(\epsilon - \Delta_\alpha + \Delta_\beta)}{\gamma(2\epsilon)} \tag{5.4}$$

(obviously symmetric). Collecting all the dimensions and structure constants our calculations give:

$$
\begin{aligned}
C^{(1)}_{1,1} &= -\frac{2(3+n)(-1+2\epsilon+\epsilon n)\pi}{\sqrt{3}\epsilon(1+n)} \\
C^{(1)}_{1,2} &= \frac{8(-1+\epsilon)\sqrt{\frac{2+n}{n}}\pi}{\sqrt{3}\epsilon(1+n)(3+n)} \\
C^{(1)}_{1,3} &= 0 \\
C^{(1)}_{2,2} &= \frac{8\pi}{\sqrt{3}(-1+n^2)\epsilon} - \frac{4(1+n^2)\pi}{\sqrt{3}(-1+n^2)} \\
C^{(1)}_{2,3} &= \frac{8(-1+\epsilon)\sqrt{\frac{-2+n}{n}}\pi}{\sqrt{3}\epsilon(-3+n)(-1+n)} \\
C^{(1)}_{3,3} &= \frac{-2(-3+n)(-1+2\epsilon-\epsilon n)\pi}{\sqrt{3}\epsilon(-1+n)}
\end{aligned}
$$

for the first order terms, and for the second order:

$$
\begin{aligned}
C^{(2)}_{1,1} &= -\frac{2(-20-143n-121n^2-33n^3-3n^4)\pi^2}{3n(1+n)(3+n)^2\epsilon^2} - \\
&- \frac{2(5+n)(8+151n+143n^2+45n^3+5n^4)\pi^2}{3n(1+n)(3+n)^2\epsilon} \\
C^{(2)}_{1,2} &= -\frac{16\sqrt{\frac{2+n}{n}}(11+3n)\pi^2}{3(1+n)(3+n)(5+n)\epsilon^2} + \frac{16\sqrt{\frac{2+n}{n}}(57+18n+n^2)\pi^2}{3(1+n)(n+3)(n+5)\epsilon} \\
C^{(2)}_{1,3} &= \frac{80(1-2\epsilon)\pi^2}{3\epsilon^2 n(-9+n^2)\sqrt{-4+n^2}} \\
C^{(2)}_{2,2} &= \frac{32\pi^2}{3(-1+n^2)\epsilon^2} - \frac{8(19+n^2)\pi^2}{3(-1+n^2)\epsilon} \\
C^{(2)}_{2,3} &= -\frac{16\sqrt{\frac{-2+n}{n}}(-11+3n)\pi^2}{3(-1+n)(-3+n)(-5+n)\epsilon^2} - \frac{16\sqrt{\frac{-2+n}{n}}(57-18n+n^2)\pi^2}{3(-1+n)(-3+n)(-5+n)\epsilon} \\
C^{(2)}_{3,3} &= -\frac{2(-20+143n-121n^2+33n^3-3n^4)\pi^2}{3n(-1+n)(-3+n)^2\epsilon^2} + \\
&+ \frac{2(-5+n)(8-151n+143n^2-45n^3+5n^4)\pi^2}{3n(-1+n)(-3+n)^2\epsilon}
\end{aligned}
$$

Now we can apply the renormalization procedure of [1] and obtain the matrix of anomalous dimensions (5.3) . Bare coupling constant λ is expressed through g by (4.3) and the bare dimensions, up to order ϵ^2, are:

$$\begin{aligned}
\Delta_1 &= 1 - \frac{n+1}{2}\epsilon + \frac{1}{8}(-1+n^2)\epsilon^2 \\
\Delta_2 &= 1 + \frac{1}{8}(-1+n^2)\epsilon^2 \\
\Delta_3 &= 1 + \frac{n-1}{2}\epsilon + \frac{1}{8}(-1+n^2)\epsilon^2
\end{aligned}$$

The result is:

$$\begin{aligned}
\Gamma_{1,1} &= \Delta_1 - \frac{(3+n)(-1+\epsilon(2+n))\pi g}{\sqrt{3}(1+n)} + \frac{4g^2\pi^2(2+n)}{3(1+n)} \\
\Gamma_{1,2} = \Gamma_{2,1} &= -\frac{(-1+\epsilon)(-1+n)\sqrt{\frac{2+n}{3n}}\pi g}{(1+n)} + \frac{2g^2(-1+n)\sqrt{\frac{2+n}{n}}\pi^2}{3(1+n)} \\
\Gamma_{1,3} = \Gamma_{3,1} &= 0 \\
\Gamma_{2,2} &= \Delta_2 - \frac{2\sqrt{3}\pi(-2+\epsilon+\epsilon n^2)g}{3(-1+n^2)} + \frac{2g^2(3+n^2)\pi^2}{3(-1+n^2)} \\
\Gamma_{2,3} = \Gamma_{3,2} &= -\frac{(-1+\epsilon)\sqrt{\frac{-2+n}{3n}}(1+n)\pi g}{(-1+n)} \\
\Gamma_{3,3} &= \Delta_3 + \frac{(1+\epsilon(-2+n))(-3+n)\pi g}{\sqrt{3}(-1+n)} + \frac{4g^2\pi^2(-2+n)}{3(-1+n)}
\end{aligned}$$

We need the value of this matrix at the fixed point (4.4):

$$\begin{aligned}
\Gamma^{g^*}_{1,1} &= 1 + \frac{(20-4n^2)\epsilon}{8(1+n)} + \frac{(39-n-7n^2+n^3)\epsilon^2}{8(1+n)} \\
\Gamma^{g^*}_{1,2} = \Gamma^{g^*}_{2,1} &= \frac{(-1+n)\sqrt{\frac{2+n}{n}}\epsilon(1+2\epsilon)}{n+1} \\
\Gamma^{g^*}_{1,3} = \Gamma^{g^*}_{3,1} &= 0 \\
\Gamma^{g^*}_{2,2} &= 1 + \frac{4\epsilon}{-1+n^2} + \frac{(65-2n^2+n^4)\epsilon^2}{8(-1+n^2)} \\
\Gamma^{g^*}_{2,3} = \Gamma^{g^*}_{3,2} &= \frac{\sqrt{\frac{-2+n}{n}}(1+n)\epsilon(1+2\epsilon)}{n-1} \\
\Gamma^{g^*}_{3,3} &= 1 + \frac{(-5+n^2)\epsilon}{2(-1+n)} + \frac{(-39-n+7n^2+n^3)\epsilon^2}{8(-1+n)}
\end{aligned}$$

The **eigenvalues** of this matrix (up to order ϵ^2):

$$\Delta_1^{g^*} = 1 + \frac{1+n}{2}\epsilon + \frac{7+8n+n^2}{8}\epsilon^2$$
$$\Delta_2^{g^*} = 1 + \frac{-1+n^2}{8}\epsilon^2$$
$$\Delta_3^{g^*} = 1 + \frac{1-n}{2}\epsilon + \frac{7-8n+n^2}{8}\epsilon^2$$

coincide up to this order with dimensions $\Delta_{n+2,n}^{(p-2)}+1/2$,$\Delta_{n,n}^{(p-2)}+1$ and $\Delta_{n-2,n}^{(p-2)}+1/2$ of the model SM_{p-2}. **The corresponding (normalized) eigenvectors should be identified with the fields of this model**:

$$\tilde{\phi}_{n+2,n}^{(p-2)} = \frac{2}{n(1+n)}\phi_1^{g^*} + \frac{2\sqrt{\frac{2+n}{n}}}{1+n}\phi_2^{g^*} + \frac{\sqrt{-4+n^2}}{n}\phi_3^{g^*}$$
$$\phi_2^{(p-2)} = -\frac{2\sqrt{\frac{2+n}{n}}}{1+n}\phi_1^{g^*} - \frac{-5+n^2}{1+n^2}\phi_2^{g^*} + \frac{2\sqrt{\frac{n-2}{n}}}{n-1}\phi_3^{g^*}$$
$$\tilde{\phi}_{n-2,n}^{(p-2)} = \frac{\sqrt{-4+n^2}}{n}\phi_1^{g^*} - \frac{2\sqrt{\frac{-2+n}{n}}}{-1+n}\phi_2^{g^*} + \frac{2}{(-1+n)n}\phi_3^{g^*}$$

We used as before the notation $\tilde{\phi}$ for the last component of the corresponding superfield and:

$$\phi_2^{(p-2)} = \frac{1}{2\Delta_{n,n}^{p-2}(2\Delta_{n,n}^{p-2}+1)}\partial\bar{\partial}\phi_{n,n}^{(p-2)}$$

is the normalized derivative of the corresponding first component. We mention that just as in the case of $N=0$ minimal models these **eigenvectors are finite as** $\epsilon \to 0$. Moreover, they are **exactly the same combinations**!! (Probably this is a common property of all coset models perturbed by the least relevant field ?)

As we mentioned in the begining of this section the corresponding first components of $\Phi_{n,n\pm 2}$ and the last component of $\Phi_{n,n}$ will be also mixed along the RG flow in a way analogous to above. This is because of the preservation of the supersymmetry by a perturbation with the last component of a superfield. So we do not present a separate calculation for them.

6. Mixing of the fields in the Ramond sector

6.1. Computation of the conformal blocks in the Ramond sector

We want to mention that the computation of the conformal blocks in the Ramond sector is more involved. A way of computing them was recently proposed [12] . There, the first few levels conformal blocks were also shown to coincide (up to prefactors) with the instanton partition function of certain $N = 2$ YM theories in four dimensions establishing a generalized AGT correspondence.

Following [12] one can compute NS-R conformal blocks only for a special choice of the points. After that we can get the function necessary for the integration in the second order by using its conformal transformation properties.

The difficulties arise because of the **branch cut** in the OPE of Ramond fields with the supercurrent:

$$G(z)R^{\epsilon}(0) = \frac{\beta R^{-\epsilon}(0)}{z^{\frac{3}{2}}} + \frac{G_{-1}R^{\epsilon}(0)}{z^{\frac{1}{2}}} \tag{6.1}$$

where $\beta = \sqrt{\Delta - \frac{\hat{c}}{16}}$, $\epsilon = \pm 1$, and therefore one cannot obtain the usual commutation relations. Here the Ramond field R^{ϵ} is obviosly double degenerated because of the zero mode of G in this sector. We also used, as it is custom, the same notation ϵ as for our small parameter to distinguish the fields. It will not be missleading.

The difficulty can be removed in the following way. Consider the OPE between NS and Ramond fields:

$$\begin{aligned} \phi_1(x)R_2^{\epsilon}(0) &= x^{\Delta-\Delta_1-\Delta_2} \sum_{N=0}^{\infty} x^N C_N^{\epsilon} R_{\Delta}^{\epsilon}(0) \\ \tilde{\phi}_1(x)R_2^{\epsilon}(0) &= x^{\Delta-\Delta_1-\Delta_2-\frac{1}{2}} \sum_{N=0}^{\infty} x^N \tilde{C}_N^{\epsilon} R_{\Delta}^{-\epsilon}(0) \end{aligned} \tag{6.2}$$

Here N runs over nonnegative integers as G's have integer valued modes in the Ramond sector. Applying G_0 on the both sides of (6.2)and taking into account (6.1)one obtains:

$$\begin{aligned} G_0 C_N^{\epsilon} &= \tilde{C}_N^{\epsilon} + \beta_2 C_N^{-\epsilon} \\ G_0 \tilde{C}_N^{\epsilon} &= (\Delta - \Delta_2 + N) C_N^{\epsilon} - \beta_2 \tilde{C}_N^{-\epsilon} \end{aligned} \tag{6.3}$$

From the consistency conditions it follows that $\tilde{C}_0^{\epsilon}$ is not independent:

$$\tilde{C}_0^{\epsilon} = \beta C_{\epsilon} - \beta_2 C_{-\epsilon}$$

where we denoted $C_{\epsilon} = C_0^{\epsilon}$.

Acting with G_k with $k > 0$ gives the **chain relations**:

$$\begin{aligned} G_k C_N^{\epsilon} &= \tilde{C}_{N-k}^{\epsilon} \\ G_k \tilde{C}_N^{\epsilon} &= (\Delta + 2k\Delta_1 - \Delta_2 + N - k) C_{N-k}^{\epsilon} \end{aligned} \tag{6.4}$$

and L_k acts as usually, with the appropriate dimensions (see [12] for the details).

One has to **solve these chain relations order by order or to use the recursion formulae**. Then the conformal block for the function $< N(x)R(0)N(1)R(\infty) >$ (N here stays for the first or the last component of a NS field) is obtained in the same way as in the NS case:

$$F(x, \Delta, \Delta_i) = \sum_{N=0}^{\infty} x^N C_N(\Delta, \Delta_3, \Delta_4) S_N^{-1} C_N(\Delta, \Delta_1, \Delta_2)$$

where C_N could be actually C_N or $\tilde{C}_N$ depending on the function in consideration. Finally the real 2D correlation function is constructed as:

$$< N(x)R(0)N(1)R(\infty) >= \sum_n C_n |F_n(x)|^2$$

C_n are the structure constants and the **range of n is dictated by the fusion rules**. **The function that enters the integral is then obtained using the conformal transformation**.

As already mentioned, the conformal block in general is very complicated. Fortunately, here its sufficient to compute the finite term as $\epsilon \to 0$. We did the computation for the functions below up to high order and then check the behavior near the singular points. It turns out also that the **two-point function do not depend on which of the fields R^ϵ are involved**. So we drop the subscript from our notations in what follows.

6.2. Function $< R_{n,n+1}(1)R_{n,n+1}(0) >_\lambda$

Our calculation for the corresponding second order function gives:

$$< \tilde{\phi}(x) R_{n,n+1}(0) R_{n,n+1}(\infty) \tilde{\phi}(1) >= \frac{n^2-1}{12n^2} \left| \frac{1}{x(1-x)^2}(1 + \frac{n}{n+1}x - \frac{1}{n+1}x^2) \right|^2 +$$
$$+ \frac{(2+n)^2}{48n^2} \left| \frac{1}{x(1-x)^2}(1 + \frac{2n}{n+2}x + \frac{n-2}{n+2}x^2) \right|^2 + \frac{n+3}{12(n+1)} \left| \frac{1}{(1-x)^2}(1+x) \right|^2$$

To obtain the function that enters the integral we use the conformal transformation properties. One can easily get:

$$< \tilde{\phi}(x) R_{n,n+1}(0) R_{n,n+1}(1) \tilde{\phi}(\infty) >= (x\bar{x})^{-2\Delta_{1,3}-1} < \tilde{\phi}(\frac{x-1}{x}) R_{n,n+1}(0) R_{n,n+1}(\infty) \tilde{\phi}(1) >=$$
$$= \frac{n^2-1}{12n^2} \left| \frac{(2x-1)(nx+1)}{(n+1)x(x-1)} \right|^2 + \frac{(2+n)^2}{48n^2} \left| \frac{(2x-1)(n(2x-1)+2)}{(n+2)x(x-1)} \right|^2 + \frac{n+3}{12(n+1)} \left| \frac{2x-1}{x} \right|^2 \tag{6.5}$$

We first integrate over the safe region, where $I(x) \sim \pi/\epsilon$. The result is:

$$
\begin{aligned}
&\int_{\Omega_{l,l_0}} I(x) < \tilde{\phi}(x) R_{n,n+1}(0) R_{n,n+1}(1) \tilde{\phi}(\infty) >= \\
&= \frac{\pi^2}{\epsilon l^2} - \frac{\pi^2(20+13n)\log l}{24\epsilon n} - \frac{\pi^2(4+5n)\log 2l_0}{24\epsilon n} - \frac{\pi^2}{2\epsilon}
\end{aligned}
$$

From this we have to subtract the lens-like region integral:

$$
\frac{\pi^2(5n+4)}{24n\epsilon}(\log l - \log 2l_0)
$$

Next we proceed with the calculation of the integrals near the singular points. Near 0 (obviously near 1 the result is exactly the same) we use the OPE:

$$
\begin{aligned}
\tilde{\phi}(0) R_{n,n+1}(0) = {} & (x\bar{x})^{-\Delta_{1,3}-1/2} \tilde{C}^{(n,n+1)}_{(1,3)(n,n+1)} R_{n,n+1}(0) + \\
& + (x\bar{x})^{\Delta_{n,n-1}-\Delta_{n,n+1}-\Delta_{1,3}-1/2} \tilde{C}^{(n,n-1)}_{(1,3)(n,n+1)} R_{n,n-1}(0) + \\
& + (x\bar{x})^{\Delta_{n,n+3}-\Delta_{n,n+1}-\Delta_{1,3}-1/2} \tilde{C}^{(n,n+3)}_{(1,3)(n,n+1)} R_{n,n+3}(0)
\end{aligned}
\tag{6.6}
$$

We can approximate here $I(x) \sim \pi/\epsilon - \pi \log |x|^2$, the necessary structure constants read:

$$
\begin{aligned}
(\tilde{C}^{(n,n+1)}_{(1,3)(n,n+1)})^2 &= \frac{-(2+n)^2(-1+\epsilon(-2+4n))}{48n^2} \\
(\tilde{C}^{(n,n-1)}_{(1,3)(n,n+1)})^2 &= \frac{(1+2\epsilon)(-1+n^2)}{12n^2}
\end{aligned}
$$

We remind again that all the structure constants involving the field $\tilde{\phi}$ should be divided by $2\Delta_{1,3}$ and we keep in what follows the same notation $\tilde{C}$.

So the result of the integration is:

$$
\begin{aligned}
&\int_{D_{l,0}\ D_{l_0,0}} I(x) < \tilde{\phi}(x) R_{n,n+1}(0) R_{n,n+1}(1) \tilde{\phi}(\infty) >= \\
&= \frac{\pi^2(28+40n+12n^2+n^3)}{24\epsilon^2 n(2+n)^2} - \frac{\pi^2(4+24n+36n^2+15n^3+2n^4)}{12\epsilon n(2+n)^2} + \frac{\pi^2(4+5n)\log l}{24\epsilon n}
\end{aligned}
$$

As usual, around infinity we make the transformation $x \to 1/x$ and then take $x \to 0$. The structure constant that appear is:

$$
\hat{C}^{(1,3)}_{(1,3)(1,3)} \tilde{C}^{(n,n+1)}_{(1,3)(n,n+1)} = \frac{(2+n)(1-2\epsilon-2\epsilon n)}{6n}
$$

and we found:

$$\int_{D_{l,\infty}\ D_{l_0,\infty}} I(x) < \tilde{\phi}(x)R_{n,n+1}(0)R_{n,n+1}(1)\tilde{\phi}(\infty) >=$$
$$= -\frac{\pi^2}{l^2\epsilon} + \frac{(2+n)\pi^2}{6n\epsilon^2} - \frac{(2+n)\pi^2(1+n-\log l)}{3n\epsilon}$$

Finally lets collect all the terms to obtain:

$$\frac{\pi^2(44+64n+24n^2+3n^3-8\epsilon(1+n)(5+14n+7n^2+n^3))}{12\epsilon^2 n(2+n)^2}$$

6.3. Function $< R_{n,n-1}(1)R_{n,n+1}(0) >_\lambda$

The calculation of the four-point function with the perturbing fields is done in the same way:

$$< \tilde{\phi}(x)R_{n,n+1}(0)\tilde{\phi}(1)R_{n,n-1}(\infty) >= \frac{\sqrt{n^2-1}}{12n}\left|\frac{1}{x(1-x)}(1+x)\right|^2$$

Performing the same transformation as in (6.5), the integrand becomes:

$$< \tilde{\phi}(x)R_{n,n+1}(0)R_{n,n-1}(1)\tilde{\phi}(\infty) >= \frac{\sqrt{n^2-1}}{12n}\left|\frac{2x-1}{x(1-x)}\right|^2$$

This function is almost the same as in the $N = 0$ case but we should make again the calculations because the various structure constants and dimensions of the fields that appear are different. The integration over the safe region and lens-like regions are exactly the same, the results being:

$$\frac{8\sqrt{n^2-1}\pi^2(5\log l+\log 2l_0)}{3n\epsilon(-16+n^2)}$$

and

$$-\frac{8\sqrt{n^2-1}\pi^2(\log l-\log 2l_0)}{3n\epsilon(-16+n^2)}$$

respectively.

For the calculation around 0 we use the same OPE that appeared in the previous subsection (6.6)(without the last line because of the fusion rules). Most of the structure constants are presented above, we need also:

$$\tilde{C}^{(n,n-1)}_{(1,3)(n,n-1)} = \frac{-2+n}{4\sqrt{3}n}(1+(1+2n)\epsilon)$$

With this, the result is:

$$\int_{D_{l,0}\ D_{l_0,0}} I(x) < \tilde{\phi}(x) R_{n,n+1}(0) R_{n,n-1}(1) \tilde{\phi}(\infty) >=$$
$$= -\frac{2\sqrt{n^2-1}\pi^2(-28+n^2+2\epsilon(4+5n^2)+4\epsilon(-4+n^2)\log l)}{3n\epsilon^2(64-20n^2+n^4)}$$

The integral around 1 gives the same result in exactly he same way as for $N = 0$ so we have to take this term twice.

At the end, the calculation around ∞ with our structure constants gives:

$$\int_{D_{l,\infty}\ D_{l_0,\infty}} I(x) < \tilde{\phi}(x) R_{n,n+1}(0) R_{n,n-1}(1) \tilde{\phi}(\infty) >=$$
$$= \frac{16\sqrt{n^2-1}\pi^2(-1+2\epsilon-2\epsilon\log l)}{3n\epsilon^2(-16+n^2)}$$

We collect now all the terms and obtain the final result in the second order:

$$\frac{4\pi^2\sqrt{n^2-1}(44-5n^2-2\epsilon(20+n^2))}{3\epsilon^2 n(-16+n^2)(-4+n^2)}$$

6.4. Matrix of anomalous dimensions

The functions we computed above are enough since just as in the case of NS fields it turns out that the other two: $< R_{n,n-1}(1)R_{n,n-1}(0) >_\lambda$ and $< R_{n,n+1}(1)R_{n,n-1}(0) >_\lambda$ can be obtained from $< R_{n,n+1}(1)R_{n,n+1}(0) >_\lambda$ and $< R_{n,n-1}(1)R_{n,n+1}(0) >_\lambda$ by just changing $n \to -n$. Lets introduce again a **basis:** $R_1 = R_{n,n+1}$, $R_2 = R_{n,n-1}$. From the general formula (5.4)and the bare dimensions of the fields:

$$\Delta_1 = \frac{3}{16} - (\frac{1}{8} + \frac{n}{4})\epsilon + \frac{1}{8}(-1+n^2)\epsilon^2$$
$$\Delta_2 = \frac{3}{16} + (-\frac{1}{8} + \frac{n}{4})\epsilon + \frac{1}{8}(-1+n^2)\epsilon^2$$

we get for the 2×2 matrix of two-point functions in the first order:

$$C^{(1)}_{1,1} = \frac{(2+n)\pi}{2\sqrt{3}n\epsilon} - \frac{(2+n)(-1+2n)\pi}{2\sqrt{3}n}$$
$$C^{(1)}_{1,2} = C^{(1)}_{2,1} = -\frac{4\sqrt{-1+n^2}\pi(1+\epsilon)}{\sqrt{3}n(-4+n^2)\epsilon}$$
$$C^{(1)}_{2,2} = \frac{(-2+n)\pi}{2\sqrt{3}n\epsilon} + \frac{(-2+n)(1+2n)\pi}{2\sqrt{3}n}$$

According to our calculations, the second order matrix is given by:

$$C^{(2)}_{1,1} = \frac{(44+64n+24n^2+3n^3)\pi^2}{12\epsilon^2 n(2+n)^2} - \frac{2(1+n)(5+14n+7n^2+n^3)\pi^2}{3\epsilon n(2+n)^2}$$
$$C^{(2)}_{1,2} = C^{(2)}_{2,1} = \frac{4\sqrt{n^2-1}(44-5n^2)\pi^2}{3\epsilon^2 n(-16+n^2)(-4+n^2)} - \frac{8\sqrt{n^2-1}(20+n^2)\pi^2}{3\epsilon n(-16+n^2)(-4+n^2)}$$
$$C^{(2)}_{2,2} = \frac{(-44+64n-24n^2+3n^3)\pi^2}{12\epsilon^2(-2+n)^2 n} + \frac{2(-1+n)(-5+14n-7n^2+n^3)\pi^2}{3\epsilon(-2+n)^2 n}$$

Now, following the same procedure as in the NS case we get for the matrix of anomalous dimensions (we keep the terms up to order $\epsilon^2 \sim g^2$):

$$\Gamma_{1,1} = \Delta_1 - \frac{g(2+n)(-1-\epsilon+2\epsilon n)\pi}{4\sqrt{3}n} + \frac{g^2\pi^2}{4}$$
$$\Gamma_{1,2} = \Gamma_{2,1} = \frac{(1+\epsilon)g\sqrt{-1+n^2}\pi}{2\sqrt{3}n}$$
$$\Gamma_{2,2} = \Delta_2 + \frac{g(-2+n)(1+\epsilon+2\epsilon n)\pi}{4\sqrt{3}n} + \frac{g^2\pi^2}{4}$$

Therefore at the fixed point (4.4):

$$\Gamma^{g^*}_{1,1} = \frac{3}{16} + \frac{(4+n-2n^2)\epsilon}{8n} + \frac{(8+n-4n^2+n^3)\epsilon^2}{8n}$$
$$\Gamma^{g^*}_{1,2} = \Gamma^{g^*}_{2,1} = \frac{\sqrt{-1+n^2}\epsilon(1+2\epsilon)}{2n}$$
$$\Gamma^{g^*}_{2,2} = \frac{3}{16} + \frac{(-4+n+2n^2)\epsilon}{8n} + \frac{(-8+n+4n^2+n^3)\epsilon^2}{8n}$$

It remains to compute the **eigenvalues** of this matrix (up to order ϵ^2):

$$\Delta^{g^*}_1 = \frac{3}{16} + (\frac{1}{8}+\frac{n}{4})\epsilon + \frac{1}{8}(1+4n+n^2)\epsilon^2$$
$$\Delta^{g^*}_2 = \frac{3}{16} + (\frac{1}{8}-\frac{n}{4})\epsilon + \frac{1}{8}(1-4n+n^2)\epsilon^2$$

As expected, they coincide with the dimensions of the Ramond fields $\Delta^{(p-2)}_{n+1,n}$ and $\Delta^{(p-2)}_{n-1,n}$ of the model SM_{p-2}. The corresponding fields are expressed as a (normalized) linear combination:

$$R^{(p-2)}_{n+1,n} = \frac{1}{n}R^{g^*}_1 + \frac{\sqrt{n^2-1}}{n}R^{g^*}_2$$
$$R^{(p-2)}_{n-1,n} = -\frac{\sqrt{n^2-1}}{n}R^{g^*}_1 + \frac{1}{n}R^{g^*}_2$$

Two comments are in order: first we mention that just as in the NS (and $N=0$)case the above combinations **do not depend on** ϵ and second, again the linear combination is exactly **the same as for the** $N=0$ **theory**. **One can speculate that these are universal for all the perturbed coset theories**.

References

[1] R. Poghossian, JHEP **1401** (2014) 167; arXiv:hep-th/1303.3015.

[2] R. Poghossian, Sov. J. Nucl. Phys. **48** (1988) 763.

[3] A. Pghosyan, H. Poghosyan, JHEP **1310** (2013) 131; arXiv:hep-th/1305.6066.

[4] A. Zamolodchikov, Sov. J. Nucl. Phys. **46** (1987) 1090.

[5] L. Alday, D. Gaiotto, Y. Tachikawa,Lett. Math. Phys. **91** (2010) 167; arXiv:hep-th/0906.3219.

[6] N. Nekrasov, Adv. Theor. Math. Phys. **7** (2004) 831; hep-th/0206161.

[7] R. Poghossian, JHEP **0912** (2009) 038; arXiv:hep-th/0909.3412.

[8] A. Marshakov, A. Mironov, A. Morozov, Theor. Math. Phys. **164** (2010) 831; arXiv:hep-th/09073946.

[9] A. Belavin, V. Belavin, A. Neveu, Al. Zamolodchikov, Nucl. Phys. **B784** (2007) 202; hep-th/0703084.

[10] V. Belavin, Theor. Math. Phys. **152** (2007) 1275.

[11] L. Hadasz, Z. Jaskolski, P. Suchanek, JHEP **0703** (2007) 032; hep-th/0611266.

[12] A. Belavin, B. Mukhametzhanov, JHEP **1301** (2013) 178; arXiv:hep-th/1210.7454.

[13] L. Hadasz, Z. Jaskolski, P. Suchanek, JHEP **0811** (2008) 060; arXiv:hep-th/0810.1203.

[14] A. Belavin, B. Feigin, JHEP **1107** (2011) 079; arXiv:hep-th/1105.5800.

[15] A. Belavin, V. Belavin, M. Bershtein, JHEP **1109** (2011) 117; arXiv:hep-th/1106.4001.

[16] G. Bonelli, K. Maruyoshi, A. Tanzini, JHEP **1108** (2011) 056; arXiv:hep-th/1106.2505.

[17] C. Crnkovic, R. Paunov, G. Sotkov, M. Stanishkov, Nucl. Phys. **B336** (1990) 637.

[18] L. Hadasz, Z. Jaskolski, hepth1312.4520.

[19] A. Belavin, M. Bershtein, B. Feigin, A. Litvinov, G. Tarnopolsky, Comm. Math. Phys. **319** (2013) 269; arXiv:hep-th/1111.2803.

[20] V. Alba, V. Fateev, A. Litvinov, G. Tarnopolsky, Lett. Math. Phys. **98** (2011) 33; arXiv:hep-th/1012.1312.

[21] M. Afimov, G. Tarnopolsky, JHEP **1202** (2012) 036; arXiv:hep-th/1110.5628.

[22] D. Gaiotto, JHEP **1212** (2012) 103; arXiv:hep-th/1201.0767.

[23] A. Zamolodchikov, R. Poghossian, Sov. J. Nucl. Phys. **47** (1988) 929.