Thermal Correlators, Intertwining Operators, and Rindler-AdS/CFT

Satoshi Ohya
Institute of Quantum Science, Nihon University, Tokyo

Based on:
SO, `arXiv:1309.2939`

Introduction

Introduction

- Conformal symmetry is powerful enough to constrain possible forms of correlation functions.
- Indeed, up to overall normalization factors, 2- and 3-point functions are completely fixed by $SO(2,d)$ conformal symmetry in any spacetime dimension $d \geq 1$ **[Polyakov '70]**:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \delta_{\Delta_1\Delta_2}\frac{c_{\Delta_1}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{c_{\Delta_1\Delta_2\Delta_3}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}}$$

- Conformal constraints work well in coordinate space.
- Then, what about conformal constraints in momentum space?

Introduction

- Correlation functions in momentum space are directly related to physical observables. For example, imaginary part of momentum-space retarded 2-point function gives the spectral density of critical systems.
- Hence it would be desirable to understand how conformal symmetry constrains the possible forms of momentum-space correlators.
- Conceptually, there is no problem: momentum-space correlators are just given by Fourier transforms of position-space correlators.
- Practically, however, there is a problem: Fourier transforms of position-space correlators are generally hard, especially in the case of finite-temperature CFT.
- Indeed, in spite of its simplicity in coordinate space, 3-point functions in momentum space are known to be very complicated.
  - In fact, Fourier transform of 3-point functions in finite-temperature CFT_2 was first computed in 2014! **[Becker-Cabrera-Su '14]**
  - The study of conformal constraints in momentum space is still ongoing. **[Corianò-Delle Rose-Mottola-Serino '13] [Bzowski-McFadden-Skenderis '13]**

Introduction

- Today I will present a simple Lie-algebraic approach to compute momentum-space 2-point functions in finite-temperature CFT.

- For the sake of simplicity I shall focus on finite-temperature CFT_1. One of my targets is the (positive-frequency) 2-point Wightman function in frequency space

$$\tilde{G}^+_\Delta(\omega) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t} G^+_\Delta(t) \quad \text{with} \quad G^+_\Delta(t) = c_\Delta \left(\frac{\pi T}{\sinh(\pi T(t - i\epsilon))} \right)^{2\Delta}$$

  where T is the temperature. I shall present an algebraic method to compute $\tilde{G}^+_\Delta(\omega)$ (as well as other 2-point functions such as retarded/advanced 2-point functions) without performing the Fourier transform.

- The keys to my approach are:
  - 1-dimensional conformal algebra $\mathfrak{so}(2,1)$ in the basis in which the $SO(1,1)$ generator becomes diagonal; and
  - Killing vectors of 2-dimensional anti-de Sitter spacetime in the Rindler coordinates (Rindler-AdS_2).
- I shall also discuss an intertwining operator approach to finite-temperature 2-point functions in frequency space.

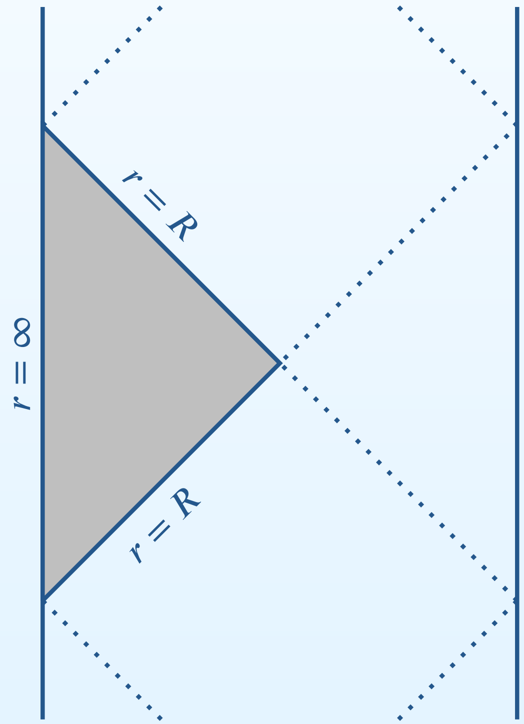
Plan of the talk

Holographic Perspective: Rindler-AdS_2/CFT_1

Rindler-AdS_2

- Rindler-AdS_2 (*aka* AdS_2 black hole **[Spradlin-Strominger '99]**) is a portion of AdS_2; it is just a single **Rindler wedge** of AdS_2 and described by the following metric:

$$ds^2_{\text{Rindler-AdS}_2} = -\left(\frac{r^2}{R^2} - 1\right) dt^2 + \frac{dr^2}{r^2/R^2 - 1}, \quad r \in (R, \infty)$$



- ◦ AdS_2 is topologically an infinite strip.
- ◦ Rindler-AdS_2 covers only a part of the whole AdS_2.
  - $r = R$: Rindler horizon
  - $r = \infty$: AdS_2 boundary
- ◦ CFT_1 dual to Rindler-AdS_2 lives on the boundary $r = \infty$ and has the temperature $T = 1/2\pi R$.

Rindler-AdS_2

- Rindler-AdS_2 (*aka* AdS_2 black hole **[Spradlin-Strominger '99]**) is a portion of AdS_2; it is just a single **Rindler wedge** of AdS_2 and described by the following metric:

$$ds^2_{\text{Rindler-AdS}_2} = -\left(\frac{r^2}{R^2} - 1\right) dt^2 + \frac{dr^2}{r^2/R^2 - 1}, \quad r \in (R, \infty)$$

- For the following discussions it is convenient to introduce a new coordinate system (t, x) via

$$r = R \coth(x/R), \quad x \in (0, \infty)$$

in which the metric becomes conformally flat:

$$ds^2_{\text{Rindler-AdS}_2} = \frac{-dt^2 + dx^2}{\sinh^2(x/R)}$$

- **Below I will work in the units $R = 1$ (i.e., $2\pi T = 1$).**

AdS$_2$ and $SO(2,1)$

- 1-dimensional conformal group $SO(2,1)$, which is the isometry of AdS$_2$, contains three distinct one-parameter subgroups:
  - compact rotation group $SO(2)$
  - noncompact Euclidean group $E(1)$
  - noncompact Lorentz group $SO(1,1)$
- Correspondingly, there exist three distinct classes of static AdS$_2$ coordinate patches in which time-translation Killing vectors generate these one-parameter subgroups $SO(2)$, $E(1)$ and $SO(1,1)$.
- In Lorentzian signature, these coordinate patches are given by the global, Poincaré and Rindler coordinates, respectively.

coordinate patch	time-translation group		frequency spectrum	
	Lorentzian	Euclidean	Lorentzian	Euclidean
global	$SO(2)$	$SO(1,1)$	discrete	continuous
Poincaré	$E(1)$	$E(1)$	continuous	continuous
Rindler	$SO(1,1)$	$SO(2)$	continuous	discrete (Matsubara frequency)

1d conformal algebra: $SO(2)$ basis

- 1-dimensional conformal algebra $\mathfrak{so}(2,1)$ is spanned by the three generators $\{J_1, J_2, J_3\}$ that satisfy the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = -iJ_1, \quad [J_3, J_1] = -iJ_2$$

- In terms of the ladder operators $J_\pm := -J_1 \pm iJ_2$ the commutation relations become

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_3$$

- The quadratic Casimir of the Lie algebra $\mathfrak{so}(2,1)$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = J_3(J_3 \pm 1) - J_\mp J_\pm$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of C and J_3 that satisfies

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle \quad \text{and} \quad J_3|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle$$

Then the state $J_\pm|\Delta, \omega\rangle$ satisfies $J_3 J_\pm|\Delta, \omega\rangle = (\omega \pm 1)J_\pm|\Delta, \omega\rangle$, which implies the ladder equations

$$\boxed{J_\pm|\Delta, \omega\rangle \propto |\Delta, \omega \pm 1\rangle}$$

1d conformal algebra: $SO(1,1)$ basis

- Let us next consider the following hermitian linear combinations

$$A_1 = J_1, \quad A_\pm = J_2 \pm J_3$$

which satisfy the commutation relations

$$[A_1, A_\pm] = \pm i A_\pm, \quad [A_+, A_-] = 2iA_1$$

- The quadratic Casimir of the Lie algebra $\mathfrak{so}(2,1)$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = -A_1(A_1 \pm i) - A_\mp A_\pm$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of C and A_1 that satisfies

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle \quad \text{and} \quad A_1|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle$$

Then the state $A_\pm|\Delta, \omega\rangle$ satisfies $A_1 A_\pm|\Delta, \omega\rangle = (\omega \pm i)A_\pm|\Delta, \omega\rangle$, which implies the ladder equations

$$\boxed{A_\pm|\Delta, \omega\rangle \propto |\Delta, \omega \pm i\rangle}$$

1d conformal algebra: holographic realization

- In the Rindler-AdS$_2$ problem, the $SO(2,1)$ generators (**Killing vectors**) are given by the following first-order differential operators:

$$\begin{aligned} A_1 &= i\partial_t \\ A_\pm &= e^{\pm t}\left[\sinh x(i\partial_x) \pm \cosh x(i\partial_t)\right] \end{aligned}$$

- The Casimir gives the d'Alembertian $\Box = (1/\sqrt{|g|})\partial_\mu \sqrt{|g|} g^{\mu\nu}\partial_\nu$ on Rindler-AdS$_2$:

$$C = A_1(A_1 \pm i) - A_\mp A_\pm = \sinh^2 x \left(-\partial_t^2 + \partial_x^2\right)$$

- The eigenvalue equations are

$$\begin{aligned} A_1|\Delta,\omega\rangle = \omega|\Delta,\omega\rangle \quad &\Leftrightarrow \quad i\partial_t \Phi_{\Delta,\omega}(t,x) = \omega\Phi_{\Delta,\omega}(t,x) \\ C|\Delta,\omega\rangle = \Delta(\Delta-1)|\Delta,\omega\rangle \quad &\Leftrightarrow \quad \left(-\partial_x^2 + \frac{\Delta(\Delta-1)}{\sinh^2 x}\right)\Phi_{\Delta,\omega}(t,x) = \omega^2\Phi_{\Delta,\omega}(t,x) \end{aligned}$$

  These are nothing but the Klein-Gordon equation $(\Box - m^2)\Phi_{\Delta,\omega} = 0$ for a scalar field of definite frequency ω, where Δ and m^2 are related as $\Delta(\Delta-1) = m^2$.

- The ladder equations are

$$\boxed{\boldsymbol{A_\pm \Phi_{\Delta,\omega} \propto \Phi_{\Delta,\omega\pm i}}}$$

1d conformal algebra: holographic realization

- Finite-temperature CFT_1 lives on the boundary $x = 0$. To analyze this, let us consider the asymptotic near-boundary limit $x \to 0$ of the Killing vectors

$$A_1^{\text{near}} := \lim_{x\to 0} A_1 = i\partial_t$$

$$A_\pm^{\text{near}} := \lim_{x\to 0} A_\pm = \mathrm{e}^{\pm t}\left(ix\partial_x \pm i\partial_t\right)$$

- The quadratic Casimir near the boundary is

$$C^{\text{near}} = A_1^{\text{near}}(A_1^{\text{near}} \pm i) - A_\mp^{\text{near}} A_\pm^{\text{near}} = x^2\partial_x^2$$

- The eigenvalue equations are

$$i\partial_t \Phi_{\Delta,\omega}^{\text{near}}(t,x) = \omega \Phi_{\Delta,\omega}^{\text{near}}(t,x)$$

$$\left(-\partial_x^2 + \frac{\Delta(\Delta-1)}{x^2}\right)\Phi_{\Delta,\omega}^{\text{near}}(t,x) = 0$$

which are easily solved with the result

$$\Phi_{\Delta,\omega}^{\text{near}}(t,x) = A_\Delta(\omega)x^\Delta \mathrm{e}^{-i\omega t} + B_\Delta(\omega)x^{1-\Delta}\mathrm{e}^{-i\omega t}$$

where $A_\Delta(\omega)$ and $B_\Delta(\omega)$ are integration constants which may depend on Δ and ω.

Correlator recurrence relations

- The ladder equations $A_{\pm}^{\text{near}}\Phi_{\Delta,\omega}^{\text{near}} \propto \Phi_{\Delta,\omega\pm i}^{\text{near}}$ become

$$(i\Delta \pm \omega)A_\Delta(\omega)x^\Delta e^{-i(\omega\pm i)t} + (i(1-\Delta)\pm\omega)B_\Delta(\omega)x^{1-\Delta}e^{-i(\omega\pm i)t}$$
$$\propto \quad A_\Delta(\omega\pm i)x^\Delta e^{-i(\omega\pm i)t} + \quad B_\Delta(\omega\pm i)x^{1-\Delta}e^{-i(\omega\pm i)t}$$

  from which we get

$$(i\Delta \pm \omega)A_\Delta(\omega) \propto A_\Delta(\omega\pm i)$$
$$(i(1-\Delta)\pm\omega)B_\Delta(\omega) \propto B_\Delta(\omega\pm i)$$

- Now we use the holography technique. According to the real-time prescription of the AdS/CFT correspondence, 2-point functions are given by the ratio **[Iqbal-Liu ’09]**

$$\tilde{G}_\Delta(\omega) = (2\Delta-1)\frac{A_\Delta(\omega)}{B_\Delta(\omega)}$$

  which satisfies the **recurrence relations** in the complex ω-plane:

$$\boxed{\tilde{G}_\Delta(\omega) = \frac{-1+\Delta\pm i\omega}{-\Delta\pm i\omega}\tilde{G}_\Delta(\omega\pm i)}$$

Correlator recurrence relations

- Now, all that remains is to solve the recurrence relations exactly:

$$\tilde{G}_\Delta(\omega) = \frac{-1+\Delta \pm i\omega}{-\Delta \pm i\omega}\tilde{G}_\Delta(\omega \pm i)$$

  In fact, there are many nontrivial solutions to these equations. Among them are the following "minimal" solutions:

$$\tilde{G}^{\pm}_\Delta(\omega) = N^{\pm}_\Delta \mathrm{e}^{\pm\pi\omega}\Gamma(\Delta - i\omega)\Gamma(\Delta + i\omega)$$
$$\tilde{G}^{A/R}_\Delta(\omega) = N^{A/R}_\Delta \frac{\Gamma(\Delta \pm i\omega)}{\Gamma(1-\Delta \pm i\omega)}$$

  where $N^{\pm}_\Delta$ and $N^{A/R}_\Delta$ are ω-independent normalization factors.

- **Remark**. Here "minimal" means that the solutions do not contain a factor $f(\omega) = f(\omega \pm i)$. A typical example of such factor is $f = f(\mathrm{e}^{2\pi\omega})$, where f is an arbitrary function.

Correlator recurrence relations

- It can be checked that $\tilde{G}^{\pm}_{\Delta}(\omega)$ correspond to the positive- and negative-frequency 2-point Wightman functions, whereas $\tilde{G}^{A/R}_{\Delta}(\omega)$ the advanced and retarded 2-point functions:

$$\tilde{G}^{+}_{\Delta}(\omega) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t} \langle \mathcal{O}_{\Delta}(t)\mathcal{O}_{\Delta}(0)\rangle$$

$$\tilde{G}^{-}_{\Delta}(\omega) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t} \langle \mathcal{O}_{\Delta}(0)\mathcal{O}_{\Delta}(t)\rangle$$

$$\tilde{G}^{A}_{\Delta}(\omega) = -i\int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t} \theta(-t)\langle [\mathcal{O}_{\Delta}(0), \mathcal{O}_{\Delta}(t)]\rangle$$

$$\tilde{G}^{R}_{\Delta}(\omega) = -i\int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t} \theta(t)\langle [\mathcal{O}_{\Delta}(t), \mathcal{O}_{\Delta}(0)]\rangle$$

where

$$\langle \mathcal{O}_{\Delta}(t_1)\mathcal{O}_{\Delta}(t_2)\rangle = c_{\Delta}\left[\frac{1}{2\sinh\left(\frac{t_1 - t_2 - i\epsilon}{2}\right)}\right]^{2\Delta}$$

- **Remark**. Computations of $\tilde{G}^{A/R}_{\Delta}(\omega)$ are very complicated due to the presence of $\theta(\mp t)$.

CFT Perspective: $SO(2,1)$ Intertwiners

$SO(2,1)$ intertwiners

- It has been long known that CFT 2-point functions are nothing but the kernels of intertwining operators G_Δ of conformal algebra [Koller '75]. In one dimension, the intertwining relations are given by

$$A_1^{\Delta} G_\Delta = G_\Delta A_1^{1-\Delta}$$
$$A_\pm^{\Delta} G_\Delta = G_\Delta A_\pm^{1-\Delta}$$

where A_1^α, $A_\pm^\alpha$ are $SO(2,1)$ generators that act on the representation space $\mathcal{V}_\alpha$ of scaling dimension $\alpha = \Delta, 1-\Delta$.

- I shall show that the recurrence relations $(i\Delta \pm \omega)\tilde{G}_\Delta(\omega) = (i(1-\Delta) \pm \omega)\tilde{G}_\Delta(\omega \pm i)$ can be derived only from the intertwining relations and the $SO(1,1)$ basis $|\alpha, \omega\rangle$ of the representation space $\mathcal{V}_\alpha$ that satisfies the following equations:

$$A_1^\alpha |\alpha, \omega\rangle = \omega |\alpha, \omega\rangle$$
$$A_\pm^\alpha |\alpha, \omega\rangle = (i\alpha \pm \omega)|\alpha, \omega \pm i\rangle \quad (\alpha = \Delta, 1-\Delta)$$

- **Remark**. In coordinate realizations, A_1^α and $A_\pm^\alpha$ are given by

$$A_1^\alpha = i\partial_t, \quad A_\pm^\alpha = \mathrm{e}^{\pm t}(i\alpha \pm i\partial_t)$$

The eigenfunction of A_1^α is just the plane wave $\mathrm{e}^{-i\omega t}$. Hence $|\alpha, \omega\rangle = \mathrm{e}^{-i\omega t}$.

$SO(2,1)$ intertwiners

Intertwining relations:

$$A_1^{\Delta} G_{\Delta} = G_{\Delta} A_1^{1-\Delta} \tag{A}$$

$$A_{\pm}^{\Delta} G_{\Delta} = G_{\Delta} A_{\pm}^{1-\Delta} \tag{B}$$

$SO(1,1)$ basis $|\alpha, \omega\rangle$, $\alpha \in \{\Delta, 1-\Delta\}$:

$$A_1^{\alpha} |\alpha, \omega\rangle = \omega |\alpha, \omega\rangle \tag{C}$$

$$A_{\pm}^{\alpha} |\alpha, \omega\rangle = (i\alpha \pm \omega) |\alpha, \omega \pm i\rangle \tag{D}$$

Step 1. Let us first consider the state $G_{\Delta}|1-\Delta, \omega\rangle$. The intertwining relation (A) gives

$$A_1^{\Delta} G_{\Delta} |1-\Delta, \omega\rangle \overset{(A)}{=} G_{\Delta} A_1^{1-\Delta} |1-\Delta, \omega\rangle \overset{(C)}{=} \omega G_{\Delta} |1-\Delta, \omega\rangle$$

which implies that the state $G_{\Delta}|1-\Delta, \omega\rangle$ is an eigenstate of A_1^{Δ} with an eigenvalue ω. Hence $G_{\Delta}|1-\Delta, \omega\rangle$ must be proportional to $|\Delta, \omega\rangle$. Thus we write

$$G_{\Delta} |1-\Delta, \omega\rangle = \tilde{G}_{\Delta}(\omega) |\Delta, \omega\rangle \tag{E}$$

where the proportional coefficient $\tilde{G}_{\Delta}(\omega)$ is the frequency-space 2-point function.

$SO(2,1)$ intertwiners

Intertwining relations:

$$A_1^{\Delta} G_{\Delta} = G_{\Delta} A_1^{1-\Delta} \tag{A}$$

$$A_{\pm}^{\Delta} G_{\Delta} = G_{\Delta} A_{\pm}^{1-\Delta} \tag{B}$$

$SO(1,1)$ basis $|\alpha, \omega\rangle$, $\alpha \in \{\Delta, 1-\Delta\}$:

$$A_1^{\alpha} |\alpha, \omega\rangle = \omega |\alpha, \omega\rangle \tag{C}$$

$$A_{\pm}^{\alpha} |\alpha, \omega\rangle = (i\alpha \pm \omega) |\alpha, \omega \pm i\rangle \tag{D}$$

Step 2. Let us next consider the equation $A_{\pm}^{\Delta} G_{\Delta} |1-\Delta, \omega\rangle = G_{\Delta} A_{\pm}^{1-\Delta} |1-\Delta, \omega\rangle$. The left-hand side is evaluated as follows:

$$A_{\pm}^{\Delta} G_{\Delta} |1-\Delta, \omega\rangle \overset{\text{(E)}}{=} \tilde{G}_{\Delta}(\omega) A_{\pm}^{\Delta} |\Delta, \omega\rangle \overset{\text{(D)}}{=} (i\Delta \pm \omega) \tilde{G}_{\Delta}(\omega) |\Delta, \omega \pm i\rangle$$

while the right-hand side becomes

$$G_{\Delta} A_{\pm}^{1-\Delta} |1-\Delta, \omega\rangle \overset{\text{(D)}}{=} (i(1-\Delta) \pm \omega) G_{\Delta} |1-\Delta, \omega \pm i\rangle \overset{\text{(E)}}{=} (i(1-\Delta) \pm \omega) \tilde{G}_{\Delta}(\omega \pm i) |\Delta, \omega \pm i\rangle$$

Comparing these we get the recurrence relations $(i\Delta \pm \omega) \tilde{G}_{\Delta}(\omega) = (i(1-\Delta) \pm \omega) \tilde{G}_{\Delta}(\omega \pm i)$.

Summary & Outlook

Summary & outlook

Summary

- $SO(2,1)$ isometry of Rindler-AdS$_2$ induces the recurrence relations for finite-temperature CFT$_1$ 2-point functions in frequency space:

$$G_\Delta(\omega) = \frac{-1+\Delta \pm i\omega}{-\Delta \pm i\omega} G_\Delta(\omega \pm i)$$

- By solving these recurrence relations we can obtain frequency-space 2-point functions without performing complicated Fourier transforms.

Outlook

- Generalizations to finite-temperature CFT$_d$. The simplest approach would be to consider Rindler-AdS$_{d+1}$ described by the metric

$$ds^2_{\text{Rindler-AdS}_{d+1}} = -\left(\frac{r^2}{R^2} - 1\right) dt^2 + \frac{dr^2}{r^2/R^2 - 1} + r^2 dH^2_{d-1}$$

where dH_{d-1} stands for the line element of $(d-1)$-dimensional hyperbolic space $\mathbb{H}^{d-1}$. (The case $d = 2$ has been done in the previous work `arXiv:1312.7348`.)