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P-Branes Dynamics, AdS/CFT and Correlation Functions

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Contents

1 Introduction

The *probe* branes approach for studying issues in the string/M-theory uses an approximation, in which one neglects the back-reaction of the branes on the background. In this sense, the probe branes are multidimensional dynamical systems, evolving in given, variable in general, external fields.

The probe branes method is widely used in the string/M-theory to investigate many different problems at a classical, semiclassical and quantum levels. The literature in this field of research can be *conditionally* divided into several parts. One of them is devoted to the properties of the probe branes themselves, e.g., [31]. The subject of another part of the papers is to probe the geometries of the string/M-theory backgrounds, e.g., [32]. One another part can be described as connected with the investigation of the correspondence between the string/M-theory geometries and their field theory duals, e.g., [33], [34]. Let us also mention the application of the probe branes technique in the 'Mirage cosmology'- an approach to the brane world scenario, e.g., [35].

In view of the wide implementation of the probe branes as a tool for investigation of different problems in the string/M-theory, it will be useful to have a method describing their dynamics, which is general enough to include as many cases of interest as possible, and on the other hand, to give the possibility for obtaining *explicit exact* solutions.

Here, we propose such an approach, which is appropriate for p-branes and Dp-branes, for arbitrary worldvolume and space-time dimensions, for tensile and tensionless branes, for different variable background fields with minimal restrictions on them, and finally, for different space-time and worldvolume gauges (embeddings).

This thesis is based on [1]-[30].

Our notations for the special functions we use are summarized in Appendix A.

2 P-branes dynamics in general backgrounds

Now, we are going to consider probe p-branes and Dp-branes dynamics in D-dimensional string/M-theory backgrounds of general type. Unified description for the tensile and tensionless branes is used. We obtain exact solutions of their equations of motion and constraints in static gauge as well as in more general gauges. Their dynamics in the whole space-time is also analyzed and exact solutions are found [4].

Before considering the problem for obtaining exact brane solutions in general string theory backgrounds, it will be useful first to choose appropriate actions, which will facilitate our task. Generally speaking, there are two types of brane actions - with and without square roots

[1]. The former ones are not well suited to our purposes, because the square root introduces additional nonlinearities in the equations of motion. Nevertheless, they have been used when searching for exact brane solutions in fixed backgrounds, because there are no constraints in the Lagrangian description and one has to solve only the equations of motion. The other type of actions contain additional worldvolume fields (Lagrange multipliers). Varying with respect to them, one obtains constraints, which, in general, are not independent. Starting with an action without square root, one escapes the nonlinearities connected with the square root, but has to solve the equations of motion and the (dependent) constraints.

Independently of their type, all actions proportional to the brane tension cannot describe the tensionless branes. The latter appear in many important cases in the string theory, and it is preferable to have a unified description for tensile and tensionless branes.

Our aim now is to find brane actions, which do not contain square roots, generate only independent constraints and give a unified description for tensile and tensionless branes.

2.1 P-brane actions

The Polyakov type action for the bosonic p-brane in a D-dimensional curved space-time with metric tensor $g_{MN}(x)$, interacting with a background (p+1)-form gauge field b_{p+1} via Wess-Zumino term, can be written as

$$\begin{aligned} S_p^P = &- \int d^{p+1}\xi \Big\{ \frac{T_p}{2}\sqrt{-\gamma}\left[\gamma^{mn}\partial_m X^M \partial_n X^N g_{MN}(X) - (p-1)\right] \\ &- Q_p \frac{\varepsilon^{m_1 \ldots m_{p+1}}}{(p+1)!}\partial_{m_1} X^{M_1} \ldots \partial_{m_{p+1}} X^{M_{p+1}} b_{M_1 \ldots M_{p+1}}(X) \Big\}, \\ &\partial_m = \partial/\partial\xi^m, \quad m,n = 0,1,\ldots,p; \quad M,N = 0,1,\ldots,D-1, \end{aligned} \tag{2.1}$$

where γ is the determinant of the auxiliary worldvolume metric γ_{mn}, and γ^{mn} is its inverse. The position of the brane in the background space-time is given by $x^M = X^M(\xi^m)$, and T_p, Q_p are the p-brane tension and charge, respectively. If we consider the action (2.1) as a bosonic part of a supersymmetric one, we have to set $Q_p = \pm T_p$. In what follows, $Q_p = T_p$.

The requirement that the variation of the action (2.1) with respect to γ_{mn} vanishes, leads to

$$\left(\gamma^{kl}\gamma^{mn} - 2\gamma^{km}\gamma^{ln}\right) G_{mn} = (p-1)\gamma^{kl}, \tag{2.2}$$

where $G_{mn} = \partial_m X^M \partial_n X^N g_{MN}(X)$ is the metric induced on the p-brane worldvolume. Taking the trace of the above equality, one obtains

$$\gamma^{mn} G_{mn} = p+1,$$

[1]Examples of these two type of actions are the Nambu-Goto and Polyakov actions for the string.

i.e., γ^{mn} is the inverse of G_{mn}: $\gamma^{mn} = G^{mn}$. If one inserts this back into (2.1), the result will be the corresponding Nambu-Goto type action ($G \equiv \det(G_{mn})$):

$$\begin{aligned} S_p^{NG} &= \int d^{p+1}\xi \mathcal{L}^{NG} \\ &= -T_p \int d^{p+1}\xi \left[\sqrt{-G} - \frac{\varepsilon^{m_1\ldots m_{p+1}}}{(p+1)!}\partial_{m_1}X^{M_1}\ldots\partial_{m_{p+1}}X^{M_{p+1}}b_{M_1\ldots M_{p+1}}(X)\right]. \end{aligned} \tag{2.3}$$

This means that the two actions, (2.1) and (2.3), are classically equivalent.

As already discussed, the action (2.3) contains a square root, the constraints (2.2), following from (2.1), are not independent and none of these actions is appropriate for description of the *tensionless* branes. To find an action of the type we are looking for, we first compute the explicit expressions for the generalized momenta, following from (2.3):

$$P_M(\xi) = -T_p\left(\sqrt{-G}G^{0n}\partial_n X^N g_{MN} - \partial_1 X^{M_1}\ldots\partial_p X^{M_p} b_{MM_1\ldots M_p}\right).$$

It can be checked that $P_M(\xi)$ satisfy the constraints

$$\begin{aligned} &\mathcal{C}_0 \equiv g^{MN}P_M P_N - 2T_p g^{MN}D_{M1\ldots p}P_N + T_p^2\left[GG^{00} + (-1)^p D_{1\ldots pM}g^{MN}D_{N1\ldots p}\right] = 0, \\ &\mathcal{C}_i \equiv P_M\partial_i X^M = 0, \quad (i = 1,\ldots,p), \end{aligned}$$

where we have introduced the notation

$$D_{M1\ldots p} \equiv b_{MM_1\ldots M_p}\partial_1 X^{M_1}\ldots\partial_p X^{M_p}.$$

Let us now find the canonical Hamiltonian for this dynamical system. The result is:

$$H_{canon} = \int d^p\xi \left(P_M\partial_0 X^M - \mathcal{L}^{NG}\right) = 0.$$

Therefore, according to Dirac [36], we have to take as a Hamiltonian the linear combination of the first class primary constraints $\mathcal{C}_n$: [2]

$$H = \int d^p\xi\mathcal{H} = \int d^p\xi\left(\lambda^0\mathcal{C}_0 + \lambda^i\mathcal{C}_i\right).$$

The corresponding Hamiltonian equations of motion for X^M are

$$\left(\partial_0 - \lambda^i\partial_i\right)X^M = 2\lambda^0 g^{MN}\left(P_N - T_p D_{N1\ldots p}\right),$$

from where one obtains the explicit expressions for P_M

$$P_M = \frac{1}{2\lambda^0}g_{MN}\left(\partial_0 - \lambda^j\partial_j\right)X^N + T_p D_{M1\ldots p}. \tag{2.4}$$

[2]In the case under consideration, secondary constraints do not appear. The first class property of $\mathcal{C}_n$ follows from their Poisson bracket algebra.

With the help of (2.4), one arrives at the following configuration space action

$$\begin{aligned}S_p &= \int d^{p+1}\xi \mathcal{L}_p = \int d^{p+1}\xi \left(P_M \partial_0 X^M - \mathcal{H}\right) \qquad (2.5)\\ &= \int d^{p+1}\xi \Big\{ \frac{1}{4\lambda^0}\Big[g_{MN}(X)\left(\partial_0-\lambda^i\partial_i\right)X^M\left(\partial_0-\lambda^j\partial_j\right)X^N - \left(2\lambda^0 T_p\right)^2 GG^{00}\Big]\\ &+ T_p b_{M_0\dots M_p}(X)\partial_0 X^{M_0}\dots\partial_p X^{M_p}\Big\}\\ &= \int d^{p+1}\xi \Big\{ \frac{1}{4\lambda^0}\Big[G_{00}-2\lambda^j G_{0j}+\lambda^i\lambda^j G_{ij} - \left(2\lambda^0 T_p\right)^2 GG^{00}\Big]\\ &+ T_p b_{M_0\dots M_p}(X)\partial_0 X^{M_0}\dots\partial_p X^{M_p}\Big\},\end{aligned}$$

which does not contain square root, generates the independent $(p+1)$ constraints, as we will show below, and in which the limit $T_p \to 0$ may be taken.

It can be proven that this action is classically equivalent to the previous two actions. It is enough to show that (2.3) and (2.5) are equivalent, because we already saw that this is true for (2.1) and (2.3).

Varying the action S_p with respect to Lagrange multipliers λ^m and requiring these variations to vanish, one obtains the constraints

$$G_{00}-2\lambda^j G_{0j}+\lambda^i\lambda^j G_{ij} + \left(2\lambda^0 T_p\right)^2 GG^{00} = 0, \qquad (2.6)$$

$$G_{0j}-\lambda^i G_{ij} = 0. \qquad (2.7)$$

By using them, the Lagrangian density $\mathcal{L}_p$ from (2.5) can be rewritten in the form

$$\mathcal{L}_p = -T_p\sqrt{-GG^{00}\left[G_{00}-G_{0i}\left(G^{-1}\right)^{ij}G_{j0}\right]} + T_p b_{M_0\dots M_p}(X)\partial_0 X^{M_0}\dots\partial_p X^{M_p}. \qquad (2.8)$$

Now, applying the equalities

$$GG^{00} = \det\left(G_{ij}\right) \equiv \mathbf{G}, \quad G = \left[G_{00}-G_{0i}\left(G^{-1}\right)^{ij}G_{j0}\right]\mathbf{G}, \qquad (2.9)$$

one finds that

$$G^{00}\left[G_{00}-G_{0i}\left(G^{-1}\right)^{ij}G_{j0}\right] = 1.$$

Inserting this in (2.8), one obtains the Nambu-Goto type Lagrangian density $\mathcal{L}^{NG}$ from (2.3). Thus, the classical equivalence of the actions (2.3) and (2.5) is established.

We will work further in the gauge $\lambda^m = constants$, in which the equations of motion for X^M, following from (2.5), are given by

$$\begin{aligned}&g_{LN}\Big[\left(\partial_0-\lambda^i\partial_i\right)\left(\partial_0-\lambda^j\partial_j\right)X^N - \left(2\lambda^0 T_p\right)^2\partial_i\left(\mathbf{G}G^{ij}\partial_j X^N\right)\Big] \qquad (2.10)\\ &+\Gamma_{L,MN}\Big[\left(\partial_0-\lambda^i\partial_i\right)X^M\left(\partial_0-\lambda^j\partial_j\right)X^N - \left(2\lambda^0 T_p\right)^2\mathbf{G}G^{ij}\partial_i X^M\partial_j X^N\Big]\\ &= 2\lambda^0 T_p H^{\mathbf{b}}_{LM_0\dots M_p}\partial_0 X^{M_0}\dots\partial_p X^{M_p},\end{aligned}$$

where $\mathbf{G}$ is defined in (2.9),

$$\Gamma_{L,MN} = g_{LK}\Gamma^K_{MN} = \frac{1}{2}\left(\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}\right)$$

are the components of the symmetric connection compatible with the metric g_{MN} and $H^{\mathbf{b}}_{p+2} = db_{p+1}$ is the field strength of the $(p+1)$-form gauge potential b_{p+1}.

2.2 Dp-brane actions

The Dirac-Born-Infeld type action for the bosonic part of the super- Dp-brane in a D-dimensional space-time with metric tensor $g_{MN}(x)$, interacting with a background (p+1)-form Ramond-Ramond gauge field c_{p+1} via Wess-Zumino term, can be written as

$$\begin{aligned} S^{DBI} &= -T_{D_p}\int d^{p+1}\xi\Big\{e^{-a(p,D)\Phi}\sqrt{-\det\left(G_{mn} + B_{mn} + 2\pi\alpha' F_{mn}\right)} \\ &- \frac{\varepsilon^{m_1\ldots m_{p+1}}}{(p+1)!}\partial_{m_1}X^{M_1}\ldots\partial_{m_{p+1}}X^{M_{p+1}}c_{M_1\ldots M_{p+1}}\Big\}. \end{aligned} \tag{2.11}$$

T_{D_P}=$(2\pi)^{-(p-1)/2}g_s^{-1}T_p$ is the D-brane tension, $g_s = \exp\langle\Phi\rangle$ is the string coupling expressed by the dilaton vacuum expectation value $\langle\Phi\rangle$ and $2\pi\alpha'$ is the inverse string tension. $G_{mn} = \partial_m X^M\partial_n X^N g_{MN}(X)$, $B_{mn} = \partial_m X^M\partial_n X^N b_{MN}(X)$ and $\Phi(X)$ are the pullbacks of the background metric, antisymmetric tensor and dilaton to the Dp-brane worldvolume, while $F_{mn}(\xi)$ is the field strength of the worldvolume $U(1)$ gauge field $A_m(\xi)$: $F_{mn} = 2\partial_{[m}A_{n]}$. The parameter $a(p,D)$ depends on the brane and space-time dimensions p and D, respectively.

A Dp-brane action, which generalizes the Polyakov type p-brane action, has been introduced in [37]. Namely, the action, classically equivalent to (2.11), is given by

$$\begin{aligned} S^{AZH} &= -\frac{T_{D_p}}{2}\int d^{p+1}\xi\Big\{e^{-a(p,D)\Phi}\sqrt{-\mathcal{K}}\left[\mathcal{K}^{mn}\left(G_{mn} + B_{mn} + 2\pi\alpha' F_{mn}\right) - (p-1)\right] \\ &- 2\frac{\varepsilon^{m_1\ldots m_{p+1}}}{(p+1)!}\partial_{m_1}X^{M_1}\ldots\partial_{m_{p+1}}X^{M_{p+1}}c_{M_1\ldots M_{p+1}}\Big\}, \end{aligned}$$

where $\mathcal{K}$ is the determinant of the matrix $\mathcal{K}_{mn}$, $\mathcal{K}^{mn}$ is its inverse, and these matrices have symmetric as well as antisymmetric part

$$\mathcal{K}^{mn} = \mathcal{K}^{(mn)} + \mathcal{K}^{[mn]},$$

where the symmetric part $\mathcal{K}^{(mn)}$ is the analogue of the auxiliary metric γ^{mn} in the p-brane action (2.1).

Again, none of these actions satisfy all our requirements. In the same way as in the

p-brane case, just considered, one can prove that the action

$$\begin{aligned} S_{Dp} &= \int d^{p+1}\xi \mathcal{L}_{Dp} = \int d^{p+1}\xi \frac{e^{-a\Phi}}{4\lambda^0}\Big[G_{00} - 2\lambda^i G_{0i} + \left(\lambda^i\lambda^j - \kappa^i\kappa^j\right) G_{ij} \\ &- \left(2\lambda^0 T_{D_p}\right)^2 \mathbf{G} + 2\kappa^i\left(\mathcal{F}_{0i} - \lambda^j\mathcal{F}_{ji}\right) + 4\lambda^0 T_{D_p} e^{a\Phi} c_{M_0\dots M_p}\partial_0 X^{M_0}\dots\partial_p X^{M_p}\Big], \\ & \mathcal{F}_{mn} = B_{mn} + 2\pi\alpha' F_{mn}, \end{aligned} \tag{2.12}$$

which possesses the necessary properties, is classically equivalent to the action (2.11). Here additional Lagrange multipliers κ^i are introduced, in order to linearize the quadratic term

$$\left(\mathcal{F}_{0i} - \lambda^k\mathcal{F}_{ki}\right)\left(G^{-1}\right)^{ij}\left(\mathcal{F}_{0j} - \lambda^l\mathcal{F}_{lj}\right)$$

arising in the action. For other actions of this type, see [38] - [40].

Varying the action S_{Dp} with respect to Lagrange multipliers λ^m, κ^i, and requiring these variations to vanish, one obtains the constraints

$$G_{00} - 2\lambda^j G_{0j} + \left(\lambda^i\lambda^j - \kappa^i\kappa^j\right) G_{ij} + \left(2\lambda^0 T_{Dp}\right)^2 \mathbf{G} + 2\kappa^i\left(\mathcal{F}_{0i} - \lambda^j\mathcal{F}_{ji}\right) = 0, \tag{2.13}$$

$$G_{0j} - \lambda^i G_{ij} = \kappa^i \mathcal{F}_{ij} \tag{2.14}$$

$$\mathcal{F}_{0j} - \lambda^i \mathcal{F}_{ij} = \kappa^i G_{ij}. \tag{2.15}$$

Instead with the constraint (2.13), we will work with the simpler one

$$G_{00} - 2\lambda^j G_{0j} + \left(\lambda^i\lambda^j + \kappa^i\kappa^j\right) G_{ij} + \left(2\lambda^0 T_{Dp}\right)^2 \mathbf{G} = 0, \tag{2.16}$$

which is obtained by inserting (2.15) into (2.13).

We will use the gauge $(\lambda^m, \kappa^i) =$ *constants* and for simplicity, we will restrict our considerations to constant dilaton $\Phi = \Phi_0$ and constant electro-magnetic field $F_{mn} = F^o_{mn}$ on the Dp-brane worldvolume. In this case, the equations of motion for X^M, following from (2.12), are

$$\begin{aligned} & g_{LN}\Big[\left(\partial_0 - \lambda^i\partial_i\right)\left(\partial_0 - \lambda^j\partial_j\right) X^N - \left(2\lambda^0 T_{Dp}\right)^2 \partial_i\left(\mathbf{G}G^{ij}\partial_j X^N\right) - \kappa^i\kappa^j\partial_i\partial_j X^N\Big] \\ & +\Gamma_{L,MN}\Big[\left(\partial_0 - \lambda^i\partial_i\right) X^M \left(\partial_0 - \lambda^j\partial_j\right) X^N \\ & - \left(2\lambda^0 T_{Dp}\right)^2 \mathbf{G}G^{ij}\partial_i X^M\partial_j X^N - \kappa^i\kappa^j\partial_i X^M\partial_j X^N\Big] \\ & = 2\lambda^0 T_{Dp} e^{a\Phi_0} H^{\mathbf{c}}_{LM_0\dots M_p}\partial_0 X^{M_0}\dots\partial_p X^{M_p} + H_{LMN}\kappa^j\left(\partial_0 - \lambda^i\partial_i\right) X^M\partial_j X^N, \end{aligned} \tag{2.17}$$

where $H^{\mathbf{c}}_{p+2} = dc_{p+1}$ and $H_3 = db_2$ are the corresponding field strengths.

2.3 Exact solutions in general backgrounds

The main idea in the mostly used approach for obtaining exact solutions of the probe branes equations of motion in variable external fields is to reduce the problem to a particle-like one, and even more - to solving one dimensional dynamical problem, if possible. To achieve this, one must get rid of the dependence on the spatial worldvolume coordinates ξ^i. To this end, since the brane actions contain the first derivatives $\partial_i X^M$, the brane coordinates $X^M(\xi^m)$ have to depend on ξ^i at most linearly:

$$X^M(\xi^0, \xi^i) = \Lambda_i^M \xi^i + Y^M(\xi^0), \quad \Lambda_i^M \text{ are arbitrary constants.} \tag{2.18}$$

Besides, the background fields entering the action depend implicitly on ξ^i through their dependence on X^M. If we choose $\Lambda_i^M = 0$ in (2.18), the connection with the p-brane setting will be lost. If we suppose that the background fields do not depend on X^M, the result will be constant background, which is not interesting in the case under consideration. The *compromise* is to accept that the external fields depend only on part of the coordinates, say X^a, and to set namely for this coordinates $\Lambda_i^a = 0$. In other words, we propose the ansatz ($X^M = (X^\mu, X^a)$):

$$X^\mu(\xi^0, \xi^i) = \Lambda_j^\mu \xi^j + Y^\mu(\xi^0), \quad X^a(\xi^0, \xi^i) = Y^a(\xi^0), \tag{2.19}$$

$$\partial_\mu g_{MN} = 0, \quad \partial_\mu b_{MN} = 0, \quad \partial_\mu b_{M_0 \ldots M_p} = 0, \quad \partial_\mu c_{M_0 \ldots M_p} = 0. \tag{2.20}$$

The resulting reduced Lagrangian density will depend only on $\xi^0 = \tau$ if the Lagrange multipliers λ^m, κ^i do not depend on ξ^i. Actually, this property follows from their equations of motion, from where they can be expressed through quantities depending only on the temporal worldvolume parameter τ.

Thus, we have obtained the general conditions, under which the probe branes dynamics reduces to the particle-like one. However, we will not start our considerations relaying on the generic ansatz (2.19). Instead, we will begin in the framework of the commonly used in ten space-time dimensions *static gauge*: $X^m(\xi^n) = \xi^m$. The latter is a particular case of (2.19), obtained under the following restrictions:

$$\begin{aligned} &(1){:}\mu = i = 1, \ldots, p; \quad (2){:}\Lambda_j^\mu = \Lambda_j^i = \delta_j^i; \\ &(3){:}Y^\mu(\tau) = Y^i(\tau) = 0; \quad (4){:}Y^0(\tau) = \tau \in \{Y^a\}. \end{aligned} \tag{2.21}$$

Therefore, the static gauge is appropriate for backgrounds which may depend on $X^0 = Y^0(\tau)$, but must be independent on X^i, $(i = 1, \ldots p)$. Such properties are not satisfactory in the lower dimensions. For instance, in four dimensional black hole backgrounds, the metric depends on X^1, X^2 and the static gauge ansatz does not work. That is why, our next step is to consider the probe branes dynamics in the framework of the ansatz

$$X^\mu(\tau, \xi^i) = \Lambda_m^\mu \xi^m = \Lambda_0^\mu \tau + \Lambda_i^\mu \xi^i, \quad X^a(\tau, \xi^i) = Y^a(\tau), \tag{2.22}$$

which is obtained from (2.19) under the restriction $Y^\mu(\tau) = \Lambda_0^\mu \tau$. Here, for the sake of symmetry between the worldvolume coordinates $\xi^0 = \tau$ and ξ^i, we have included in X^μ a term linear in τ. At any time, one can put $\Lambda_0^\mu = 0$ and the corresponding terms in the formulas will disappear. Further, we will refer to the ansatz (2.22) as *linear gauges*, as far as X^μ are linear combinations of ξ^m with *arbitrary* constant coefficients.

Finally, we will investigate the classical branes dynamics by using the general ansatz (2.19), rewritten in the form

$$X^{\mu}(\tau,\xi^{i}) = \Lambda_{0}^{\mu}\tau + \Lambda_{i}^{\mu}\xi^{i} + Y^{\mu}(\tau), \quad X^{a}(\tau,\xi^{i}) = Y^{a}(\tau). \tag{2.23}$$

Compared with (2.19), here we have separated the linear part of Y^{μ} as in the previous ansatz (2.22). This will allow us to compare the role of the term $\Lambda_{0}^{\mu}\tau$ in these two cases.

2.3.1 Static gauge dynamics

Here we begin our analysis of the probe branes dynamics in the framework of the *static gauge* ansatz. In order not to introduce too many type of indices, we will denote with Y^{a}, Y^{b}, etc., the coordinates, which are *not fixed by the gauge*. However, one have not to forget that *by definition*, Y^{a} are the coordinates on which the background fields can depend. In *static gauge*, according to (2.21), one of this coordinates, the temporal one $Y^{0}(\tau)$, is *fixed* to coincide with τ. Therefore, in this gauge, the remaining coordinates Y^{a} are *spatial* ones in space-times with signature $(-,+,\ldots,+)$.

Let us start with the p-branes case.

In static gauge, and under the conditions (2.20), the action (2.5) reduces to (the over-dot is used for $d/d\tau$)

$$S_{p}^{SG} = \int d\tau L_{p}^{SG}(\tau), \quad V_{p} = \int d^{p}\xi, \tag{2.24}$$

$$L_{p}^{SG}(\tau) = \frac{V_{p}}{4\lambda^{0}}\Big\{g_{ab}(Y^{a})\dot{Y}^{a}\dot{Y}^{b} + 2\left[g_{0a}(Y^{a}) - \lambda^{i}g_{ia}(Y^{a}) + 2\lambda^{0}T_{p}b_{a1\ldots p}(Y^{a})\right]\dot{Y}^{a}$$
$$+g_{00}(Y^{a}) - 2\lambda^{i}g_{0i}(Y^{a}) + \lambda^{i}\lambda^{j}g_{ij}(Y^{a}) - \left(2\lambda^{0}T_{p}\right)^{2}\det(g_{ij}(Y^{a})) + 4\lambda^{0}T_{p}b_{01\ldots p}(Y^{a})\Big\}.$$

To have finite action, we require the fraction V_{p}/λ^{0} to be finite one. For example, in the string case ($p = 1$) and in conformal gauge ($\lambda^{1} = 0, (2\lambda^{0}T_{1})^{2} = 1$), this means that the quantity $V_{1}/\alpha' = 2\pi V_{1}T_{1}$ must be finite.

The constraints derived from the action (2.24) are:

$$g_{ab}\dot{Y}^{a}\dot{Y}^{b} + 2\left(g_{0a} - \lambda^{i}g_{ia}\right)\dot{Y}^{a} + g_{00} - 2\lambda^{i}g_{0i} + \lambda^{i}\lambda^{j}g_{ij} + \left(2\lambda^{0}T_{p}\right)^{2}\det(g_{ij}) = 0, \tag{2.25}$$

$$g_{ia}\dot{Y}^{a} + g_{i0} - g_{ij}\lambda^{j} = 0. \tag{2.26}$$

The Lagrangian L_{p}^{SG} does not depend on τ explicitly, so the energy $E_{p} = p_{a}^{SG}\dot{Y}^{a} - L_{p}^{SG}$ is conserved:

$$g_{ab}\dot{Y}^{a}\dot{Y}^{b} - g_{00} + 2\lambda^{i}g_{0i} - \lambda^{i}\lambda^{j}g_{ij} + \left(2\lambda^{0}T_{p}\right)^{2}\det(g_{ij}) - 4\lambda^{0}T_{p}b_{01\ldots p} = \frac{4\lambda^{0}E_{p}}{V_{p}} = constant.$$

With the help of the constraints, we can replace this equality by the following one

$$g_{0a}\dot{Y}^a + g_{00} - \lambda^i g_{i0} + 2\lambda^0 T_p b_{01\dots p} = -\frac{2\lambda^0 E_p}{V_p}. \tag{2.27}$$

To clarify the physical meaning of the equalities (2.26) and (2.27), we compute the momenta (2.4) in static gauge

$$2\lambda^0 P_M^{SG} = g_{Ma}\dot{Y}^a + g_{M0} - \lambda^i g_{Mi} + 2\lambda^0 T_p b_{M1\dots p}. \tag{2.28}$$

The comparison of (2.28) with (2.27) and (2.26) shows that $P_0^{SG} = -E_p/V_p = const$ and $P_i^{SG} = const = 0$. Inserting these conserved momenta into (2.25), we obtain the *effective* constraint

$$g_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}^S, \tag{2.29}$$

where

$$\mathcal{U}^S = -\left(2\lambda^0 T_p\right)^2 \det(g_{ij}) + g_{00} - 2\lambda^i g_{0i} + \lambda^i\lambda^j g_{ij} + 4\lambda^0\left(T_p b_{01\dots p} + E_p/V_p\right).$$

In the gauge $\lambda^m = constants$, the equations of motion following from S_p^{SG} (or from (2.10) after imposing the static gauge) take the form:

$$g_{ab}\ddot{Y}^b + \Gamma_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U}^S + 2\partial_{[a}\mathcal{A}_{b]}^S\dot{Y}^b, \tag{2.30}$$

where

$$\mathcal{A}_a^S = g_{a0} - \lambda^i g_{ai} + 2\lambda^0 T_p b_{a1\dots p}.$$

Thus, in general, the time evolution of the reduced dynamical system does not correspond to a geodesic motion. The deviation from the geodesic trajectory is due to the appearance of the *effective* scalar potential $\mathcal{U}^S$ and of the field strength $2\partial_{[a}\mathcal{A}_{b]}^S$ of the *effective* $U(1)$-gauge potential $\mathcal{A}_a^S$. In addition, our dynamical system is subject to the *effective* constraint (2.29).

Next, we proceed with the Dp-branes case.

In static gauge, and for background fields independent of the coordinates X^i (conditions(2.20)), the reduced Lagrangian, obtained from (2.12), is given by

$$\begin{aligned}
L_{Dp}^{SG}(\tau) &= \frac{V_{Dp}e^{-a\Phi_0}}{4\lambda^0}\Big[g_{ab}\dot{Y}^a\dot{Y}^b + g_{00} - 2\lambda^i g_{0i} + \left(\lambda^i\lambda^j - \kappa^i\kappa^j\right)g_{ij} \\
&+ 2\left(g_{0a} - \lambda^i g_{ia} + 2\lambda^0 T_{Dp}e^{a\Phi_0}c_{a1\dots p} + \kappa^i b_{ai}\right)\dot{Y}^a \\
&- \left(2\lambda^0 T_{Dp}\right)^2 \det(g_{ij}) + 4\lambda^0 T_{Dp}e^{a\Phi_0}c_{01\dots p} \\
&+ 2\kappa^i\left(b_{0i} - \lambda^j b_{ji}\right) + 4\pi\alpha'\kappa^i\left(F_{0i}^o - \lambda^j F_{ji}^o\right)\Big].
\end{aligned}$$

As we already mentioned at the end of Section 2, we restrict our considerations to the case of constant dilaton $\Phi = \Phi_0$ and constant electro-magnetic field F^o_{mn} on the Dp-brane worldvolume.

Now, the constraints (2.16), (2.14) and (2.15) take the form

$$g_{ab}\dot{Y}^a\dot{Y}^b + 2\left(g_{0a} - \lambda^i g_{ia}\right)\dot{Y}^a + g_{00} - 2\lambda^i g_{0i} \tag{2.31}$$
$$+\left(\lambda^i\lambda^j + \kappa^i\kappa^j\right)g_{ij} + \left(2\lambda^0 T_{Dp}\right)^2\det(g_{ij}) = 0,$$

$$g_{ja}\dot{Y}^a + g_{0j} - \lambda^i g_{ij} - \kappa^i b_{ij} = 2\pi\alpha'\kappa^i F^o_{ij} \tag{2.32}$$
$$b_{aj}\dot{Y}^a + b_{0j} - \lambda^i b_{ij} - \kappa^i g_{ij} = -2\pi\alpha'\left(F^o_{0j} - \lambda^i F^o_{ij}\right).$$

The reduced Lagrangian L^{SG}_{Dp} does not depend on τ explicitly. As a consequence, the energy E_{Dp} is conserved:

$$g_{ab}\dot{Y}^a\dot{Y}^b - g_{00} + 2\lambda^i g_{0i} - \left(\lambda^i\lambda^j - \kappa^i\kappa^j\right)g_{ij} + \left(2\lambda^0 T_{Dp}\right)^2\det(g_{ij}) - 4\lambda^0 T_{Dp}e^{a\Phi_0}c_{01\ldots p}$$
$$-2\kappa^i\left(b_{0i} - \lambda^j b_{ji}\right) - 4\pi\alpha'\kappa^i\left(F^o_{0i} - \lambda^j F^o_{ji}\right) = \frac{4\lambda^0 E_{Dp}}{V_{Dp}}e^{a\Phi_0} = constant.$$

By using the constraints (2.31) and (2.32), the above equality can be replaced by the following one

$$g_{0a}\dot{Y}^a + g_{00} - \lambda^i g_{i0} + 2\lambda^0 T_{Dp}e^{a\Phi_0}c_{01\ldots p} + \kappa^i\left(b_{0i} + 2\pi\alpha' F^o_{0i}\right) = -\frac{2\lambda^0 E_{Dp}}{V_{Dp}}e^{a\Phi_0}. \tag{2.33}$$

Now, we compute the momenta, obtained from the initial action (2.12), in static gauge

$$2\lambda^0 e^{a\Phi_0}P^{SG}_M = g_{Ma}\dot{Y}^a + g_{M0} - \lambda^j g_{Mj} + 2\lambda^0 T_{Dp}e^{a\Phi_0}c_{M1\ldots p} + \kappa^j b_{Mj}. \tag{2.34}$$

Comparing (2.34) with (2.33) and (2.32), one finds that

$$P^{SG}_0 = -\left(\frac{E_{Dp}}{V_{Dp}} + \frac{\pi\alpha'}{\lambda^0}e^{-a\Phi_0}\kappa^j F^o_{0j}\right) = constant,$$
$$P^{SG}_i = -\frac{\pi\alpha'}{\lambda^0}e^{-a\Phi_0}\kappa^j F^o_{ij} = constants.$$

As in the p-brane case, not only the energy, but also the spatial components of the momenta P^{SG}_i, along the X^i coordinates, are conserved. In the Dp-brane case however, P^{SG}_i are not identically zero due the existence of a constant worldvolume magnetic field F^o_{ij}.

Inserting (2.32) and (2.33) into (2.31), one obtains the effective constraint

$$g_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}^{DS}, \tag{2.35}$$

where

$$\mathcal{U}^{DS} = -\left(2\lambda^0 T_{Dp}\right)^2\det(g_{ij}) + g_{00} - 2\lambda^i g_{0i} + \left(\lambda^i\lambda^j - \kappa^i\kappa^j\right)g_{ij}$$
$$+4\lambda^0 e^{a\Phi_0}\left(T_{Dp}c_{01\ldots p} + \frac{E_{Dp}}{V_{Dp}}\right) + 2\kappa^i\left(b_{0i} - \lambda^j b_{ji}\right) + 4\pi\alpha'\kappa^i\left(F^o_{0i} - \lambda^j F^o_{ji}\right).$$

In the gauge $(\lambda^m, \kappa^i) = constants$, the equations of motion following from L^{SG}_{Dp} (or from (2.17) after using the static gauge ansatz) take the form:

$$g_{ab}\ddot{Y}^b + \Gamma_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U}^{DS} + 2\partial_{[a}\mathcal{A}^{DS}_{b]}\dot{Y}^b, \tag{2.36}$$

where

$$\mathcal{A}^{DS}_a = g_{a0} - \lambda^i g_{ai} + 2\lambda^0 T_{Dp} e^{a\Phi_0} c_{a1\ldots p} + \kappa^i b_{ai}.$$

It is obvious that the equations of motion (2.30), (2.36) and the effective constraints (2.29), (2.35) have the *same form* for p-branes and for Dp-branes. The difference is in the explicit expressions for the *effective* scalar and 1-form gauge potentials.

2.3.2 Branes dynamics in linear gauges

Now we will repeat our analysis of the probe branes dynamics in the framework of the more general *linear gauges*, given by the ansatz (2.22). The *static gauge* is a particular case of the *linear gauges*, corresponding to the following restrictions:

$$(1){:}\mu = i = 1,\ldots,p; \quad (2){:}\Lambda^\mu_0 = \Lambda^i_0 = 0;$$
$$(3){:}\Lambda^\mu_j = \Lambda^i_j = \delta^i_j; \quad (4){:}Y^0(\tau) = \tau \in \{Y^a\}.$$

P-branes case

In linear gauges, and under the conditions (2.20), one obtains the following reduced Lagrangian, arising from the action (2.5)

$$L^{LG}_p(\tau) = \frac{V_p}{4\lambda^0}\Big\{g_{ab}\dot{Y}^a\dot{Y}^b + 2\Big[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right) g_{\mu a} + 2\lambda^0 T_p B_{a1\ldots p}\Big]\dot{Y}^a \tag{2.37}$$
$$+ \left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu} - \left(2\lambda^0 T_p\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu})$$
$$+4\lambda^0 T_p \Lambda^\mu_0 B_{\mu 1\ldots p}\Big\}, \qquad B_{M1\ldots p} \equiv b_{M\mu_1\ldots\mu_p}\Lambda^{\mu_1}_1\ldots\Lambda^{\mu_p}_p.$$

The constraints derived from the Lagrangian (2.37) are:

$$g_{ab}\dot{Y}^a\dot{Y}^b + 2\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right) g_{\mu a}\dot{Y}^a + \left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right) \tag{2.38}$$
$$\times\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu} + \left(2\lambda^0 T_p\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) = 0,$$
$$\Lambda^\mu_i\left[g_{\mu a}\dot{Y}^a + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu}\right] = 0. \tag{2.39}$$

The Lagrangian L^{LG}_p does not depend on τ explicitly, so the energy $E_p = p^{LG}_a\dot{Y}^a - L^{LG}_p$ is conserved:

$$g_{ab}\dot{Y}^a\dot{Y}^b - \left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu} + \left(2\lambda^0 T_p\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu})$$
$$-4\lambda^0 T_p\Lambda^\mu_0 B_{\mu1\ldots p} = \frac{4\lambda^0 E_p}{V_p} = constant.$$

With the help of the constraints (2.38) and (2.39), one can replace this equality by the following one

$$\Lambda_0^\mu \left[g_{\mu a}\dot{Y}^a + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} + 2\lambda^0 T_p B_{\mu 1\dots p}\right] = -\frac{2\lambda^0 E_p}{V_p}. \tag{2.40}$$

In linear gauges, the momenta (2.4) take the form

$$2\lambda^0 P_M^{LG} = g_{Ma}\dot{Y}^a + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{M\nu} + 2\lambda^0 T_p B_{M1\dots p}. \tag{2.41}$$

The comparison of (2.41) with (2.40) and (2.39) gives

$$\Lambda_0^\mu P_\mu^{LG} = -\frac{E_p}{V_p} = constant, \quad \Lambda_i^\mu P_\mu^{LG} = constants = 0.$$

Therefore, in the linear gauges, the projections of the momenta P_μ^{LG} onto Λ_n^μ are conserved. Moreover, as far as the Lagrangian (2.37) does not depend on the coordinates X^μ, the corresponding conjugated momenta P_μ^{LG} are also conserved.

Inserting (2.40) and (2.39) into (2.38), we obtain the *effective* constraint

$$g_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}^L,$$

where the *effective* scalar potential is given by

$$\mathcal{U}^L = -\left(2\lambda^0 T_p\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) + \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu}$$
$$+4\lambda^0\left(T_p\Lambda_0^\mu B_{\mu 1\dots p} + \frac{E_p}{V_p}\right).$$

In the gauge $\lambda^m = constants$, the equations of motion following from L_p^{LG} take the form:

$$g_{ab}\ddot{Y}^b + \Gamma_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U}^L + 2\partial_{[a}\mathcal{A}_{b]}^L\dot{Y}^b,$$

where

$$\mathcal{A}_a^L = \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) g_{a\mu} + 2\lambda^0 T_p B_{a1\dots p},$$

is the *effective* 1-form gauge potential, generated by the non-diagonal components $g_{a\mu}$ of the background metric and by the components $b_{a\mu_1\dots\mu_p}$ of the background $(p+1)$-form gauge field.

Dp-branes case

In linear gauges, and for background fields independent of the coordinates X^μ (conditions(2.20)), the reduced Lagrangian, obtained from (2.12), is given by

$$\begin{aligned} L^{LG}_{Dp}(\tau) &= \frac{V_{Dp}e^{-a\Phi_0}}{4\lambda^0}\Big\{g_{ab}\dot{Y}^a\dot{Y}^b + \left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) - \kappa^i\kappa^j\Lambda^\mu_i\Lambda^\nu_j\right] g_{\mu\nu} \\ &+ 2\left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right) g_{\mu a} + 2\lambda^0 T_{Dp}e^{a\Phi_0}C_{a1\dots p} + \kappa^i\Lambda^\mu_i b_{a\mu}\right]\dot{Y}^a \\ &- \left(2\lambda^0 T_{Dp}\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) + 4\lambda^0 T_{Dp}e^{a\Phi_0}\Lambda^\mu_0 C_{\mu 1\dots p} \\ &- 2\kappa^i\Lambda^\mu_i\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) b_{\mu\nu} + 4\pi\alpha'\kappa^i\left(F^o_{0i} - \lambda^j F^o_{ji}\right)\Big\}, \end{aligned}$$

where the following shorthand notation has been introduced

$$C_{M1\dots p} \equiv c_{M\mu_1\dots\mu_p}\Lambda^{\mu_1}_1\dots\Lambda^{\mu_p}_p.$$

Now, the constraints (2.16), (2.14), and (2.15) take the form

$$\begin{aligned} &g_{ab}\dot{Y}^a\dot{Y}^b + 2\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right) g_{\mu a}\dot{Y}^a + \left(2\lambda^0 T_{Dp}\right)^2\det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) \\ &+ \left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) + \kappa^i\kappa^j\Lambda^\mu_i\Lambda^\nu_j\right] g_{\mu\nu} = 0, \end{aligned} \tag{2.42}$$

$$\Lambda^\mu_i\left[g_{\mu a}\dot{Y}^a + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu} + \kappa^j\Lambda^\nu_j b_{\mu\nu}\right] = 2\pi\alpha'\kappa^j F^o_{ji} \tag{2.43}$$

$$\Lambda^\mu_i\left[b_{\mu a}\dot{Y}^a + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) b_{\mu\nu} + \kappa^j\Lambda^\nu_j g_{\mu\nu}\right] = 2\pi\alpha'\left(F^o_{0i} - \lambda^j F^o_{ji}\right).$$

The reduced Lagrangian L^{LG}_{Dp} does not depend on τ explicitly. As a consequence, the energy E_{Dp} is conserved:

$$\begin{aligned} &g_{ab}\dot{Y}^a\dot{Y}^b - \left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) - \kappa^i\kappa^j\Lambda^\mu_i\Lambda^\nu_j\right] g_{\mu\nu} \\ &+ \left(2\lambda^0 T_{Dp}\right)^2\det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) - 4\lambda^0 T_{Dp}e^{a\Phi_0}\Lambda^\mu_0 C_{\mu 1\dots p} \\ &-2\kappa^i\Lambda^\mu_i\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) b_{\mu\nu} - 4\pi\alpha'\kappa^i\left(F^o_{0i} - \lambda^j F^o_{ji}\right) = \frac{4\lambda^0 E_{Dp}}{V_{Dp}}e^{a\Phi_0} = constant. \end{aligned}$$

By using the constraints (2.42) and (2.43), the above equality can be replaced by the following one

$$\begin{aligned} &\Lambda^\mu_0\left[g_{\mu a}\dot{Y}^a + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu} + 2\lambda^0 T_{Dp}e^{a\Phi_0}C_{\mu 1\dots p} + \kappa^j\Lambda^\nu_j b_{\mu\nu}\right] \\ &+2\pi\alpha'\kappa^i F^o_{0i} = -\frac{2\lambda^0 E_{Dp}}{V_{Dp}}e^{a\Phi_0}. \end{aligned} \tag{2.44}$$

In linear gauges, the momenta obtained from the initial action (2.12), are

$$2\lambda^0 e^{a\Phi_0}P^{LG}_M = g_{Ma}\dot{Y}^a + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{M\nu} + 2\lambda^0 T_{Dp}e^{a\Phi_0}C_{M1\dots p} + \kappa^j\Lambda^\nu_j b_{M\nu}. \tag{2.45}$$

Comparing (2.45) with (2.44) and (2.43), one finds that the following equalities hold

$$\Lambda_0^\mu P_\mu^{LG} = -\left(\frac{E_{Dp}}{V_{Dp}} + \frac{\pi\alpha'}{\lambda^0} e^{-a\Phi_0}\kappa^j F_{0j}^o\right) = constant,$$
$$\Lambda_i^\mu P_\mu^{LG} = -\frac{\pi\alpha'}{\lambda^0} e^{-a\Phi_0}\kappa^j F_{ij}^o = constants.$$

They may be viewed as restrictions on the number of the arbitrary parameters, presented in the theory.

As in the p-brane case, the momenta P_μ^{LG} are conserved quantities, due to the independence of the Lagrangian on the coordinates X^μ.

Inserting (2.43) and (2.44) into (2.42), one obtains the effective constraint

$$g_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}^{DL},$$

where

$$\mathcal{U}^{DL} = \left[\left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) - \kappa^i\kappa^j\Lambda_i^\mu\Lambda_j^\nu\right] g_{\mu\nu}$$
$$- \left(2\lambda^0 T_{Dp}\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) + 4\lambda^0 e^{a\Phi_0}\left(T_{Dp}\Lambda_0^\mu C_{\mu 1\dots p} + \frac{E_{Dp}}{V_{Dp}}\right)$$
$$+2\kappa^i\Lambda_i^\mu\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) b_{\mu\nu} + 4\pi\alpha'\kappa^i\left(F_{0i}^o - \lambda^j F_{ji}^o\right).$$

In the gauge $(\lambda^m, \kappa^i) = constants$, the equations of motion following from L_{Dp}^{LG} take the form:

$$g_{ab}\ddot{Y}^b + \Gamma_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U}^{DL} + 2\partial_{[a}\mathcal{A}_{b]}^{DL}\dot{Y}^b,$$

where

$$\mathcal{A}_a^{DL} = \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) g_{a\mu} + 2\lambda^0 T_{Dp} e^{a\Phi_0} C_{a1\dots p} + \kappa^i\Lambda_i^\mu b_{a\mu}.$$

It is clear that the equations of motion and the effective constraints have the *same form* for p-branes and for Dp-branes in linear gauges, as well as in static gauge. The only difference is in the explicit expressions for the *effective* scalar and 1-form gauge potentials.

2.3.3 Branes dynamics in the whole space-time

Working in *static gauge* $X^m(\xi^n) = \xi^m$, we actually imply that the probe branes have no dynamics along the background coordinates x^m. The (proper) time evolution is possible only in the transverse directions, described by the coordinates x^a.

Using the *linear gauges*, we have the possibility to place the probe branes in general position with respect to the coordinates x^μ, on which the background fields do not depend. However, the real dynamics is again in the transverse directions only.

Actually, in the framework of our approach, the probe branes can have 'full' dynamical freedom only when the ansatz (2.23) is used, because only then *all* of the brane coordinates X^M are allowed to vary *nonlinearly* with the proper time τ. Therefore, with the help of (2.23), we can probe the *whole* space-time.

We will use the superscript A to denote that the corresponding quantity is taken on the ansatz (2.23). It is understood that the conditions (2.20) are also fulfilled.

P-branes

Now, the reduced Lagrangian obtained from the action (2.5) is given by

$$\begin{aligned} L_p^A(\tau) = &\frac{V_p}{4\lambda^0}\Big\{g_{MN}\dot{Y^M}\dot{Y^N} + 2\Big[\left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) g_{\mu N} + 2\lambda^0 T_p B_{N1\ldots p}\Big]\dot{Y^N} \\ &+ \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} - \left(2\lambda^0 T_p\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) \\ &+4\lambda^0 T_p \Lambda_0^\mu B_{\mu 1\ldots p}\Big\}. \end{aligned}$$

The constraints, derived from the above Lagrangian, are:

$$\begin{aligned} &g_{MN}\dot{Y^M}\dot{Y^N} + 2\left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) g_{\mu N}\dot{Y^N} + \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) \\ &\times\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} + \left(2\lambda^0 T_p\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) = 0, \end{aligned} \tag{2.46}$$

$$\Lambda_i^\mu\left[g_{\mu N}\dot{Y^N} + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu}\right] = 0. \tag{2.47}$$

The corresponding momenta are ($P_M = P_M^A/V_p$)

$$2\lambda^0 P_M = g_{MN}\dot{Y}^N + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{M\nu} + 2\lambda^0 T_p B_{M1\ldots p},$$

and part of them, P_μ, are conserved

$$g_{\mu N}\dot{Y}^N + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} + 2\lambda^0 T_p B_{\mu 1\ldots p} = 2\lambda^0 P_\mu = constants, \tag{2.48}$$

because L_p^A does not depend on X^μ. From (2.47) and (2.48), the compatibility conditions follow

$$\Lambda_i^\mu P_\mu = 0. \tag{2.49}$$

We will regard on (2.49) as a solution of the constraints (2.47), which restricts the number of the arbitrary parameters Λ_i^μ and P_μ. That is why from now on, we will deal only with the constraint (2.46).

In the gauge $\lambda^m = constants$, the equations of motion for Y^N, following from L_p^A, have the form

$$g_{LN}\ddot{Y}^N + \Gamma_{L,MN}\dot{Y}^M\dot{Y}^N = \frac{1}{2}\partial_L\mathcal{U}^{in} + 2\partial_{[L}\mathcal{A}^{in}_{N]}\dot{Y}^N, \qquad (2.50)$$

where

$$\begin{aligned}
\mathcal{U}^{in} &= -\left(2\lambda^0T_p\right)^2\det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) + \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right)g_{\mu\nu}\\
&+4\lambda^0T_p\Lambda_0^\mu B_{\mu1\ldots p},\\
\mathcal{A}^{in}_N &= \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)g_{N\mu} + 2\lambda^0T_pB_{N1\ldots p}.
\end{aligned}$$

Let us first consider this part of the equations of motion (2.50), which corresponds to $L = \lambda$. It follows from (2.20) that the connection coefficients $\Gamma_{\lambda,MN}$, involved in these equations, are

$$\Gamma_{\lambda,ab} = \frac{1}{2}\left(\partial_a g_{b\lambda} + \partial_b g_{a\lambda}\right), \quad \Gamma_{\lambda,\mu a} = \frac{1}{2}\partial_a g_{\mu\lambda}, \quad \Gamma_{\lambda,\mu\nu} = 0.$$

Inserting these expressions in the part of the differential equations (2.50) corresponding to $L = \lambda$ and using that $\dot{g}_{MN} = \dot{Y}^a\partial_a g_{MN}$, $\dot{B}_{M1\ldots p} = \dot{Y}^a\partial_a B_{M1\ldots p}$, one receives

$$\frac{d}{d\tau}\left[g_{\mu N}\dot{Y}^N + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right)g_{\mu\nu} + 2\lambda^0T_pB_{\mu1\ldots p}\right] = 0.$$

These equalities express the fact that the momenta P_μ are conserved (compare with (2.48)). Therefore, we have to deal only with the other part of the equations of motion, corresponding to $L = a$

$$g_{aN}\ddot{Y}^N + \Gamma_{a,MN}\dot{Y}^M\dot{Y}^N = \frac{1}{2}\partial_a\mathcal{U}^{in} + 2\partial_{[a}\mathcal{A}^{in}_{N]}\dot{Y}^N. \qquad (2.51)$$

Our next task is to *separate the variables* $\dot{Y}^\mu$ and $\dot{Y}^a$ in these equations and in the constraint (2.46). To this end, we will use the conservation laws (2.48) to express $\dot{Y}^\mu$ through $\dot{Y}^a$. The result is

$$\dot{Y}^\mu = \left(g^{-1}\right)^{\mu\nu}\left[2\lambda^0(P_\nu - T_pB_{\nu1\ldots p}) - g_{\nu a}\dot{Y}^a\right] - (\Lambda_0^\mu - \lambda^i\Lambda_i^\mu). \qquad (2.52)$$

We will need also the explicit expressions for the connection coefficients $\Gamma_{a,\mu b}$ and $\Gamma_{a,\mu\nu}$, which under the conditions (2.20) reduce to

$$\Gamma_{a,\mu b} = -\frac{1}{2}\left(\partial_a g_{b\mu} - \partial_b g_{a\mu}\right) = -\partial_{[a}g_{b]\mu}, \quad \Gamma_{a,\mu\nu} = -\frac{1}{2}\partial_a g_{\mu\nu}. \qquad (2.53)$$

By using (2.52) and (2.53), after some calculations, one rewrites the equations of motion (2.51) and the constraint (2.46) in the form

$$h_{ab}\ddot{Y}^b + \Gamma^{\mathbf{h}}_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U}^A + 2\partial_{[a}\mathcal{A}^A_{b]}\dot{Y}^b, \qquad (2.54)$$

$$h_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}^A, \qquad (2.55)$$

where a new, *effective metric* appeared

$$h_{ab} = g_{ab} - g_{a\mu}(g^{-1})^{\mu\nu} g_{\nu b}.$$

$\Gamma^{\mathbf{h}}_{a,bc}$ is the connection compatible with this metric

$$\Gamma^{\mathbf{h}}_{a,bc} = \frac{1}{2}\left(\partial_b h_{ca} + \partial_c h_{ba} - \partial_a h_{bc}\right).$$

The new, *effective* scalar and gauge potentials are given by

$$\begin{aligned}
\mathcal{U}^A &= -\left(2\lambda^0 T_p\right)^2 \det(\Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu}) - (2\lambda^0)^2 \left(P_\mu - T_p B_{\mu 1\ldots p}\right)\left(g^{-1}\right)^{\mu\nu}\left(P_\nu - T_p B_{\nu 1\ldots p}\right),\\
\mathcal{A}^A_a &= 2\lambda^0\left[g_{a\mu}\left(g^{-1}\right)^{\mu\nu}\left(P_\nu - T_p B_{\nu 1\ldots p}\right) + T_p B_{a1\ldots p}\right].
\end{aligned}$$

We note that Eqs. (2.54), (2.55), and therefore their solutions, do not depend on the parameters Λ^μ_0 and λ^i in contrast to the previously considered cases. However, they have the same form as before.

Dp-branes

The reduced Lagrangian, obtained from (2.12), is given by

$$\begin{aligned}
L^A_{Dp}(\tau) &= \frac{V_{Dp} e^{-a\Phi_0}}{4\lambda^0}\Big\{ g_{MN}\dot{Y^M}\dot{Y^N} + \left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) - \kappa^i\kappa^j\Lambda^\mu_i\Lambda^\nu_j\right] g_{\mu\nu}\\
&+ 2\left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right) g_{\mu N} + 2\lambda^0 T_{Dp} e^{a\Phi_0} C_{N1\ldots p} + \kappa^i\Lambda^\mu_i b_{N\mu}\right]\dot{Y^N}\\
&- \left(2\lambda^0 T_{Dp}\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) + 4\lambda^0 T_{Dp} e^{a\Phi_0}\Lambda^\mu_0 C_{\mu 1\ldots p}\\
&- 2\kappa^i\Lambda^\mu_i\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) b_{\mu\nu} + 4\pi\alpha'\kappa^i\left(F^o_{0i} - \lambda^j F^o_{ji}\right)\Big\},
\end{aligned}$$

The constraints (2.16), (2.14), and (2.15) take the form

$$g_{MN}\dot{Y^M}\dot{Y^N} + 2\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right) g_{\mu N}\dot{Y^N} + \left(2\lambda^0 T_{Dp}\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) \tag{2.56}$$
$$+ \left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) + \kappa^i\kappa^j\Lambda^\mu_i\Lambda^\nu_j\right] g_{\mu\nu} = 0,$$

$$\Lambda^\mu_i\left[g_{\mu N}\dot{Y^N} + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu} + \kappa^j\Lambda^\nu_j b_{\mu\nu}\right] = 2\pi\alpha'\kappa^j F^o_{ji} \tag{2.57}$$

$$\Lambda^\mu_i\left[b_{\mu N}\dot{Y^N} + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) b_{\mu\nu} + \kappa^j\Lambda^\nu_j g_{\mu\nu}\right] = 2\pi\alpha'\left(F^o_{0i} - \lambda^j F^o_{ji}\right). \tag{2.58}$$

Because of the independence of L^A_{Dp} on X^μ, the momenta $P^D_\mu = P^{DA}_\mu / V_{Dp}$ are conserved

$$2\lambda^0 e^{a\Phi_0} P^D_\mu = g_{\mu N}\dot{Y^N} + \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{\mu\nu} + 2\lambda^0 T_{Dp} e^{a\Phi_0} C_{\mu 1\ldots p} + \kappa^j\Lambda^\nu_j b_{\mu\nu} = constants. \tag{2.59}$$

From (2.57) and (2.59), one obtains the following compatibility conditions

$$\Lambda^\mu_j P^D_\mu = \frac{\pi\alpha'}{\lambda^0} e^{-a\Phi_0}\kappa^i F^o_{ij},$$

which we interpret as a solution of the constraints (2.57).

In the gauge $(\lambda^m, \kappa^i) = constants$, the equations of motion for Y^N, following from L^A_{Dp}, take the form

$$g_{LN}\ddot{Y}^N + \Gamma_{L,MN}\dot{Y}^M\dot{Y}^N = \frac{1}{2}\partial_L\mathcal{U}^{Din} + 2\partial_{[L}\mathcal{A}^{Din}_{N]}\dot{Y}^N, \tag{2.60}$$

where

$$\begin{aligned}
\mathcal{U}^{Din} &= -\left(2\lambda^0 T_p\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) + \left[\left(\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i\right)\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) - \kappa^i\kappa^j\Lambda^\mu_i\Lambda^\nu_j\right] g_{\mu\nu} \\
&+4\lambda^0 T_{Dp}e^{a\Phi_0}\Lambda^\mu_0 C_{\mu 1\dots p} - 2\kappa^i\Lambda^\mu_i\left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) b_{\mu\nu}, \\
\mathcal{A}^{Din}_N &= \left(\Lambda^\nu_0 - \lambda^j\Lambda^\nu_j\right) g_{N\nu} + 2\lambda^0 T_{Dp}e^{a\Phi_0} C_{N1\dots p} + \kappa^j\Lambda^\nu_j b_{N\nu}.
\end{aligned}$$

As in the p-brane case, this part of the equations of motion (2.60), which corresponds to $L = \lambda$, expresses the conservation of the momenta P^D_μ, in accordance with (2.59). The remaining equations of motion, which we have to deal with, are

$$g_{aN}\ddot{Y}^N + \Gamma_{a,MN}\dot{Y}^M\dot{Y}^N = \frac{1}{2}\partial_a\mathcal{U}^{Din} + 2\partial_{[a}\mathcal{A}^{Din}_{N]}\dot{Y}^N. \tag{2.61}$$

To exclude the dependence on $\dot{Y}^\mu$ in the Eqs. (2.61) and in the constraints (2.56), (2.58), we use the conservation laws (2.59) to express $\dot{Y}^\mu$ through $\dot{Y}^a$:

$$\dot{Y}^\mu = \left(g^{-1}\right)^{\mu\nu}\left[2\lambda^0 e^{a\Phi_0}(P^D_\nu - T_{Dp}C_{\nu 1\dots p}) - g_{\nu a}\dot{Y}^a - \kappa^j\Lambda^\rho_j b_{\nu\rho}\right] - (\Lambda^\mu_0 - \lambda^i\Lambda^\mu_i). \tag{2.62}$$

By using (2.62) and (2.53), one can rewrite the equations of motion (2.61) and the constraint (2.56) as

$$h_{ab}\ddot{Y}^b + \Gamma^{\mathbf{h}}_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U}^{DA} + 2\partial_{[a}\mathcal{A}^{DA}_{b]}\dot{Y}^b, \tag{2.63}$$

$$h_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}^{DA}. \tag{2.64}$$

Now, the *effective* scalar and 1-form gauge potentials are given by

$$\begin{aligned}
\mathcal{U}^{DA} &= -\left(2\lambda^0 T_p\right)^2 \det(\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu}) - \kappa^i\kappa^j\Lambda^\mu_i\Lambda^\nu_j g_{\mu\nu} \\
&- \left[2\lambda^0 e^{a\Phi_0}(P^D_\mu - T_{Dp}C_{\mu 1\dots p}) - \kappa^i\Lambda^\lambda_i b_{\mu\lambda}\right]\left(g^{-1}\right)^{\mu\nu} \\
&\times \left[2\lambda^0 e^{a\Phi_0}(P^D_\nu - T_{Dp}C_{\nu 1\dots p}) - \kappa^j\Lambda^\rho_j b_{\nu\rho}\right], \\
\mathcal{A}^{DA}_a &= g_{a\mu}\left(g^{-1}\right)^{\mu\nu}\left[2\lambda^0 e^{a\Phi_0}(P^D_\nu - T_{Dp}C_{\nu 1\dots p}) - \kappa^j\Lambda^\rho_j b_{\nu\rho}\right] \\
&+ 2\lambda^0 T_{Dp}e^{a\Phi_0}C_{a1\dots p} + \kappa^i\Lambda^\mu_i b_{a\mu}.
\end{aligned}$$

Eqs. (2.63), (2.64), have the same form as in static and linear gauges, but now they do not depend on the parameters Λ^μ_0 and λ^i. Another difference is the appearance of a new, *effective* background metric h_{ab} and the corresponding connection $\Gamma^{\mathbf{h}}_{a,bc}$.

In the D-brane case, we have another set of constraints (2.58), generated by the Lagrange multipliers κ^i. With the help of (2.62), they acquire the form

$$\Big\{\left[b_{a\nu} - g_{a\rho}\left(g^{-1}\right)^{\rho\mu} b_{\mu\nu}\right]\dot{Y}^a + b_{\mu\nu}\left(g^{-1}\right)^{\mu\rho}\left[2\lambda^0 e^{a\Phi_0}(P^D_\rho - T_{Dp}C_{\rho 1\ldots p}) - \kappa^j\Lambda^\lambda_j b_{\rho\lambda}\right]$$
$$-\kappa^i\Lambda^\mu_i g_{\mu\nu}\Big\}\Lambda^\nu_j = -2\pi\alpha'\left(F^o_{0j} - \lambda^i F^o_{ij}\right).$$

2.3.4 Explicit solutions of the equations of motion

All cases considered so far, have one common feature. The dynamics of the corresponding reduced particle-like system is described by *effective* equations of motion and one *effective* constraint, which have the *same form*, independently of the ansatz used to reduce the p-branes or Dp-branes dynamics. Our aim here is to find *explicit exact solutions* to them. [3] To be able to describe all cases simultaneously, let us first introduce some general notations.

We will search for solutions of the following system of *nonlinear* differential equations

$$\mathcal{G}_{ab}\ddot{Y}^b + \Gamma^{\mathcal{G}}_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U} + 2\partial_{[a}\mathcal{A}_{b]}\dot{Y}^b, \tag{2.65}$$
$$\mathcal{G}_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}, \tag{2.66}$$

where $\mathcal{G}_{ab}$, $\Gamma^{\mathcal{G}}_{a,bc}$, $\mathcal{U}$, and $\mathcal{A}_a$ can be as follows

$$\mathcal{G}_{ab} = (g_{ab}, h_{ab}), \quad \Gamma^{\mathcal{G}}_{a,bc} = \left(\Gamma_{a,bc}, \Gamma^{\mathbf{h}}_{a,bc}\right),$$
$$\mathcal{U} = \left(\mathcal{U}^S, \mathcal{U}^{DS}, \mathcal{U}^L, \mathcal{U}^{DL}, \mathcal{U}^A, \mathcal{U}^{DA}\right),$$
$$\mathcal{A}_a = \left(\mathcal{A}^S_a, \mathcal{A}^{DS}_a, \mathcal{A}^L_a, \mathcal{A}^{DL}_a, \mathcal{A}^A_a, \mathcal{A}^{DA}_a\right),$$

depending on the ansatz and on the type of the brane (p-brane or Dp-brane).

Let us start with the simplest case, when the background fields depend on only one coordinate $X^a = Y^a(\tau)$. [4] In this case the Eqs. (2.65), (2.66) simplify to ($d_a \equiv d/dY^a$)

$$\frac{d}{d\tau}\left(\mathcal{G}_{aa}\dot{Y}^a\right) - \frac{1}{2}d_a\mathcal{G}_{aa}\left(\dot{Y}^a\right)^2 = \frac{1}{2}d_a\mathcal{U}, \tag{2.67}$$
$$\mathcal{G}_{aa}\left(\dot{Y}^a\right)^2 = \mathcal{U}, \tag{2.68}$$

where we have used that

$$\mathcal{G}_{ab}\ddot{Y}^b + \Gamma^{\mathcal{G}}_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{d}{d\tau}\left(\mathcal{G}_{ab}\dot{Y}^b\right) - \frac{1}{2}\partial_a\mathcal{G}_{bc}\dot{Y}^b\dot{Y}^c.$$

[3] The additional restrictions on the solutions, depending on the ansatz and on the type of the branes, will be discussed in the next section.

[4] An example of such background is the generalized Kasner type metric, arising in the superstring cosmology [41] (see also [42], [43]).

After multiplying with $2\mathcal{G}_{aa}\dot{Y}^a$ and after using the constraint (2.68), the Eq. (2.67) reduces to

$$\frac{d}{d\tau}\left[\left(\mathcal{G}_{aa}\dot{Y}^a\right)^2 - \mathcal{G}_{aa}\mathcal{U}\right] = 0. \tag{2.69}$$

The solution of (2.69), compatible with (2.68), is just the constraint (2.68). In other words, (2.68) is first integral of the equation of motion for the coordinate Y^a. By integrating (2.68), one obtains the following exact probe branes solution

$$\tau\left(X^a\right) = \tau_0 \pm \int_{X_0^a}^{X^a}\left(\frac{\mathcal{U}}{\mathcal{G}_{aa}}\right)^{-1/2} dx, \tag{2.70}$$

where τ_0 and X_0^a are arbitrary constants.

When one works in the framework of the general ansatz (2.23), one has to also write down the solution for the remaining coordinates X^μ. It can be obtained as follows. One represents $\dot{Y}^\mu$ as

$$\dot{Y}^\mu = \frac{dY^\mu}{dY^a}\dot{Y}^a,$$

and use this and (2.68) in (2.52) for the p-brane, and in (2.62) for the Dp-brane. The result is a system of ordinary differential equations of first order with separated variables, which integration is straightforward. Replacing the obtained solution for $Y^\mu(X^a)$ in the ansatz (2.23), one finally arrives at

$$\begin{aligned} X^\mu(X^a,\xi^i) &= X_0^\mu + \Lambda_i^\mu\left[\lambda^i\tau(X^a)+\xi^i\right] \\ &- \int_{X_0^a}^{X^a}\left(g^{-1}\right)^{\mu\nu}\left[g_{\nu a} \mp 2\lambda^0(P_\nu - T_p B_{\nu 1\ldots p})\left(\frac{\mathcal{U}^A}{h_{aa}}\right)^{-1/2}\right] dx \end{aligned} \tag{2.71}$$

for the p-brane case, and at

$$\begin{aligned} X^\mu(X^a,\xi^i) &= X_0^\mu + \Lambda_i^\mu\left[\lambda^i\tau(X^a)+\xi^i\right] \\ &- \int_{X_0^a}^{X^a}\left(g^{-1}\right)^{\mu\nu}\left\{g_{\nu a} \mp \left[2\lambda^0 e^{a\Phi_0}(P_\nu^D - T_{Dp}C_{\nu 1\ldots p}) - \kappa^j\Lambda_j^\rho b_{\nu\rho}\right]\left(\frac{\mathcal{U}^{DA}}{h_{aa}}\right)^{-1/2}\right\} dx \end{aligned} \tag{2.72}$$

for the Dp-brane case correspondingly. In the above two exact branes solutions, X_0^μ are arbitrary constants, and $\tau(X^a)$ is given in (2.70). We note that the comparison of the solutions $X^\mu(X^a,\xi^i)$ with the initial ansatz (2.23) for X^μ shows, that the dependence on Λ_0^μ has disappeared. We will comment on this later on.

Let us turn to the more complicated case, when the background fields depend on more than one coordinate $X^a = Y^a(\tau)$. We would like to apply the same procedure for solving the system of differential equations (2.65), (2.66), as in the simplest case just considered. To be able to do this, we need to suppose that the metric $\mathcal{G}_{ab}$ is a diagonal one. Then one can rewrite the effective equations of motion (2.65) and the effective constraint (2.66) in the form

$$\begin{aligned}
&\frac{d}{d\tau}\left(\mathcal{G}_{aa}\dot{Y}^a\right)^2 - \dot{Y}^a\partial_a\left(\mathcal{G}_{aa}\mathcal{U}\right) \qquad (2.73)\\
&+\dot{Y}^a\sum_{b\neq a}\left[\partial_a\left(\frac{\mathcal{G}_{aa}}{\mathcal{G}_{bb}}\right)\left(\mathcal{G}_{bb}\dot{Y}^b\right)^2 - 4\partial_{[a}\mathcal{A}_{b]}\mathcal{G}_{aa}\dot{Y}^b\right] = 0,\\
&\mathcal{G}_{aa}\left(\dot{Y}^a\right)^2 + \sum_{b\neq a}\mathcal{G}_{bb}\left(\dot{Y}^b\right)^2 = \mathcal{U}. \qquad (2.74)
\end{aligned}$$

To find solutions of the above equations without choosing particular background, we fix all coordinates X^a except one. Then the exact probe brane solution of the equations of motion is given again by the same expression (2.70) for $\tau\left(X^a\right)$. In the case when one is using the general ansatz (2.23), the solutions (2.71) and (2.72) still also hold.

To find solutions depending on more than one coordinate, we have to impose further conditions on the background fields. Let us show, how a number of *sufficient* conditions, which allow us to reduce the order of the equations of motion by one, can be obtained.

First of all, we split the index a in such a way that Y^r is one of the coordinates Y^a, and Y^α are the others. Then we assume that the effective 1-form gauge field $\mathcal{A}_a$ can be represented in the form

$$\mathcal{A}_a = (\mathcal{A}_r, \mathcal{A}_\alpha) = (\mathcal{A}_r, \partial_\alpha f), \qquad (2.75)$$

i.e., it is oriented along the coordinate Y^r, and the remaining components $\mathcal{A}_\alpha$ are pure gauges. Now, the Eq.(2.73) read

$$\begin{aligned}
&\frac{d}{d\tau}\left(\mathcal{G}_{\alpha\alpha}\dot{Y}^\alpha\right)^2 - \dot{Y}^\alpha\partial_\alpha\left(\mathcal{G}_{\alpha\alpha}\mathcal{U}\right) \qquad (2.76)\\
&+\dot{Y}^\alpha\left[\partial_\alpha\left(\frac{\mathcal{G}_{\alpha\alpha}}{\mathcal{G}_{rr}}\right)\left(\mathcal{G}_{rr}\dot{Y}^r\right)^2 - 2\mathcal{G}_{\alpha\alpha}\partial_\alpha\left(\mathcal{A}_r - \partial_r f\right)\dot{Y}^r\right]\\
&+\dot{Y}^\alpha\sum_{\beta\neq\alpha}\partial_\alpha\left(\frac{\mathcal{G}_{\alpha\alpha}}{\mathcal{G}_{\beta\beta}}\right)\left(\mathcal{G}_{\beta\beta}\dot{Y}^\beta\right)^2 = 0,\\
&\frac{d}{d\tau}\left(\mathcal{G}_{rr}\dot{Y}^r\right)^2 - \dot{Y}^r\partial_r\left(\mathcal{G}_{rr}\mathcal{U}\right) \qquad (2.77)\\
&+\dot{Y}^r\sum_{\alpha}\left[\partial_r\left(\frac{\mathcal{G}_{rr}}{\mathcal{G}_{\alpha\alpha}}\right)\left(\mathcal{G}_{\alpha\alpha}\dot{Y}^\alpha\right)^2 + 2\mathcal{G}_{rr}\partial_\alpha\left(\mathcal{A}_r - \partial_r f\right)\dot{Y}^\alpha\right] = 0.
\end{aligned}$$

After imposing the conditions

$$\partial_\alpha\left(\frac{\mathcal{G}_{\alpha\alpha}}{\mathcal{G}_{aa}}\right) = 0, \quad \partial_\alpha\left(\mathcal{G}_{rr}\dot{Y}^r\right)^2 = 0, \qquad (2.78)$$

the Eq.(2.76) reduce to

$$\frac{d}{d\tau}\left(\mathcal{G}_{\alpha\alpha}\dot{Y}^{\alpha}\right)^{2}-\dot{Y}^{\alpha}\partial_{\alpha}\left\{\mathcal{G}_{\alpha\alpha}\left[\mathcal{U}+2\left(\mathcal{A}_{r}-\partial_{r}f\right)\dot{Y}^{r}\right]\right\}=0,$$

which are solved by

$$\left(\mathcal{G}_{\alpha\alpha}\dot{Y}^{\alpha}\right)^{2}=D_{\alpha}\left(Y^{a\neq\alpha}\right)+\mathcal{G}_{\alpha\alpha}\left[\mathcal{U}+2\left(\mathcal{A}_{r}-\partial_{r}f\right)\dot{Y}^{r}\right]=E_{\alpha}\left(Y^{\beta}\right)\geq 0, \tag{2.79}$$

where D_α, E_α are arbitrary functions of their arguments. ($E_\alpha = E_\alpha\left(Y^\beta\right)$ follows from (2.80)).

To integrate the Eq. (2.77), we impose the condition

$$\partial_{r}\left(\mathcal{G}_{\alpha\alpha}\dot{Y}^{\alpha}\right)^{2}=0. \tag{2.80}$$

After using the second of the conditions (2.78), the condition (2.80), and the already obtained solution (2.79), the Eq. (2.77) can be recast in the form

$$\begin{aligned}&\frac{d}{d\tau}\left[\left(\mathcal{G}_{rr}\dot{Y}^{r}\right)^{2}+2\mathcal{G}_{rr}\left(\mathcal{A}_{r}-\partial_{r}f\right)\dot{Y}^{r}\right]\\&=\dot{Y}^{r}\partial_{r}\left\{\mathcal{G}_{rr}\left[\left(1-n_{\alpha}\right)\left(\mathcal{U}+2\left(\mathcal{A}_{r}-\partial_{r}f\right)\dot{Y}^{r}\right)-\sum_{\alpha}\frac{D_{\alpha}\left(Y^{a\neq\alpha}\right)}{\mathcal{G}_{\alpha\alpha}}\right]\right\},\end{aligned}$$

where n_α is the number of the coordinates Y^α. The solution of this equation, compatible with (2.79) and with the effective constraint (2.74), is

$$\left(\mathcal{G}_{rr}\dot{Y}^{r}\right)^{2}=\mathcal{G}_{rr}\left[(1-n_{\alpha})\mathcal{U}-2n_{\alpha}\left(\mathcal{A}_{r}-\partial_{r}f\right)\dot{Y}^{r}-\sum_{\alpha}\frac{D_{\alpha}\left(Y^{a\neq\alpha}\right)}{\mathcal{G}_{\alpha\alpha}}\right]=E_{r}\left(Y^{r}\right)\geq 0, \tag{2.81}$$

where E_r is again an arbitrary function.

Thus, we succeeded to separate the variables $\dot{Y}^a$ and to obtain the first integrals (2.79), (2.81) for the equations of motion (2.73), when the conditions (2.75), (2.78), (2.80) on the background are fulfilled. [5] Further progress is possible, when working with particular background configurations, having additional symmetries (see for instance, [35]).

To summarize, we addressed here the problem of obtaining *explicit exact* solutions for probe branes moving in general string theory backgrounds. We concentrated our attention to the *common* properties of the p-branes and Dp-branes dynamics and tried to formulate an approach, which is effective for different embeddings, for arbitrary worldvolume and space-time dimensions, for different variable background fields, for tensile and tensionless branes.

[5] An example, when the obtained sufficient conditions are satisfied, is given by the evolution of a tensionless brane in Kerr space-time. Moreover, in this case, one is able to find the orbit $r = r(\theta)$ [44].

To achieve this, we first performed an analysis with the aim to choose brane actions, which are most appropriate for our purposes.

Next, we formulated the frameworks in which to search for exact probe branes solutions. The guiding idea is the reduction of the brane dynamics to a particle-like one. In view of the existing practice, we first consider the case of *static gauge* embedding, which is the mostly used one in higher dimensions. Then we turn to the more general case of *linear embeddings*, which are appropriate for lower dimensions too. After that, we consider the branes dynamics by using a more general ansatz, allowing for its reduction to particle-like one. The obtained results reveal one common property in all the cases considered. The *effective* equations of motion and one of the constraints, the *effective* constraint, have the *same form* independently of the ansatz used to reduce the p-branes or Dp-branes dynamics. In general, the effective equations of motion do not coincide with the geodesic ones. The deviation from the geodesic motion is due to the appearance of *effective* scalar and 1-form gauge potentials. The same scalar potential arises in the effective constraint.

Also, we considered the problem of obtaining *explicit exact* solutions of the effective equations of motion and the effective constraint, without using the explicit structure of the effective potentials.

In the case when the background fields depend on only one coordinate $x^a = X^a(\tau)$, we showed that these equations can always be integrated and give the probe brane solution in the form $\tau = \tau(X^a)$, where τ is the worldvolume temporal parameter. We also give the explicit solutions for the brane coordinates X^μ in the form $X^\mu = X^\mu(X^a, \xi^i)$. They are nontrivial when one uses the more general ansatz (2.23). Let us remind that x^μ are the coordinates, on which the background fields do not depend.

In the case when the background fields depend on more than one coordinate, and we fix all brane coordinates X^a except one, the exact solutions are given by the same expressions as in the case considered before, if the metric $\mathcal{G}_{ab}$ is a diagonal one. In this way, we have realized the possibility to obtain probe brane solutions as functions of *every* single one coordinate, on which the background depends. In the case when none of the brane coordinates is kept fixed, we were able to find *sufficient* conditions, which ensure the separation of the variables $\dot{X}^a = \dot{Y}^a(\tau)$. As a result, we have found the manifest expressions for n_a first integrals of the equations of motion, where n_a is the number of the brane coordinates Y^a.

In obtaining the solutions described above, it was not taken into account that some restrictions on them can arise, depending on the ansatz used and on the type of the branes considered. As far as we are interested here in the *common* properties of the probe branes dynamics, we will not make an exhaustive investigation of all possible peculiarities, which can arise in different particular cases. Nevertheless, we will point out some specific properties, characterizing the dynamics of the different type of branes for different embeddings.

We note that in static gauge, the brane coordinates X^a figuring in our solutions, are spatial ones. This is so, because in this gauge the background temporal coordinate, on which the background fields can depend, is identified with the worldvolume time τ.

The solutions $X^\mu(X^a,\xi^i)$, given by (2.71) for the p-brane and by (2.72) for the Dp-brane, depend on the worldvolume parameters (τ,ξ^i) through the specific combination $\Lambda_i^\mu\left(\lambda^i\tau+\xi^i\right)$. It is interesting to understand if its origin has some physical meaning. To this end, let us consider the p-branes equations of motion (2.10) and constraints (2.6), (2.7) in the tensionless limit $T_p\to 0$, when they take the form [1, 2]

$$\begin{aligned}
&g_{LN}\left(\partial_0-\lambda^i\partial_i\right)\left(\partial_0-\lambda^j\partial_j\right)X^N+\Gamma_{L,MN}\left(\partial_0-\lambda^i\partial_i\right)X^M\left(\partial_0-\lambda^j\partial_j\right)X^N=0,\\
&g_{MN}\left(\partial_0-\lambda^i\partial_i\right)X^M\left(\partial_0-\lambda^j\partial_j\right)X^N=0, \qquad (2.82)\\
&g_{MN}\left(\partial_0-\lambda^i\partial_i\right)X^M\partial_jX^N=0.
\end{aligned}$$

It is easy to check that in D-dimensional space-time, any D arbitrary functions of the type $F^M=F^M\left(\lambda^i\tau+\xi^i\right)$ solve this system of partial differential equations. Hence, the linear part of the *tensile* p-brane and Dp-brane solutions (2.71) and (2.72), is a background independent solution of the *tensionless* p-brane equations of motion and constraints.

Let us also point out here that by construction, the actions used in our considerations allow for taking the tensionless limit $T_p\to 0$ ($T_{Dp}\to 0$). Moreover, from the explicit form of the obtained exact probe branes solutions it is clear that the opposite limit $T_p\to\infty$ ($T_{Dp}\to\infty$) can be also taken.

We have obtained solutions of the probe branes equations of motion and one of the constrains, which have the same form for all of the considered cases. Now, let us see how we can satisfy the other constraints present in the theory. These are p constraints, obtained by varying the corresponding actions with respect to the Lagrange multipliers λ^i. For the Dp-brane, we have p additional constraints, obtained by varying the action with respect to the Lagrange multipliers κ^i. Actually, the constraints generated by the λ^i-multipliers are satisfied. Due to the conservation of the corresponding momenta, they just restrict the number of the *independent* parameters present in the solutions. The only exception is the p-brane in static gauge case, where the momenta P_i^{SG} must be zero. Let us give an example how the problem can be resolved in a particular situation, which is nevertheless general enough. Let the background metric along the probe p-brane be a diagonal one. Then from (2.26) and (2.28) it follows that the momenta P_i^{SG} will be identically zero, if we work in the gauge $\lambda^i=0$. In the general case, and this is also valid for the κ^i- generated constraints, we have to insert the obtained solution of the equations of motion into the unresolved constraints. The result will be a number of algebraic relations between the background fields. If they are not satisfied (on the solution) at least for some particular values of the free parameters in the solution, it would be fair to say that our approach does not work properly in this case, and some modification is needed.

Finally, let us say a few words about some possible generalizations of the obtained results.

As is known, the branes charges are restricted up to a sign to be equal to the branes tensions from the condition for space-time supersymmetry of the corresponding actions. In our computations, however, the coefficients in front of the background antisymmetric fields do not play any special role. That is why, to account for nonsupersymmetric probe branes, it is enough to make the replacements

$$T_p b_{p+1} \to Q_p b_{p+1}, \quad T_{Dp} c_{p+1} \to Q_{Dp} c_{p+1}.$$

In our Dp-brane action (2.12), we have included only the leading Wess-Zumino term of the possible Dp-brane couplings. It is easy to see that our results can be generalized to include other interaction terms just by the replacement

$$c_{p+1} \to c_{p+1} + c_{p-1} \wedge b_2 + \dots .$$

This is a consequence of the fact that we do not used the explicit form of the background field $c_{M_0 \dots M_p}$. We have used only its antisymmetry and its independence on part of the background coordinates.

2.4 Particular cases

2.4.1 Tensionless string in Demianski-Newman background

The metric for Demianski-Newman space-time is of the following type:

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01}dx^0dx^1 + 2g_{03}dx^0dx^3 + 2g_{13}dx^1dx^3 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2.$$

It can be put in a form in which the manifest expressions for $g_{\mu\nu}$ are given by the equalities (all other components are zero):

$$\begin{aligned} g_{00} &= -\exp(+2U) \quad , \quad g_{11} = \left(1 - \frac{a^2 \sin^2\theta}{\mathcal{R}^2}\right)\exp(-2U), \\ g_{22} &= \left(\mathcal{R}^2 - a^2\sin^2\theta\right)\exp(-2U), \\ g_{33} &= \mathcal{R}^2\sin^2\theta\exp(-2U) - \left[\frac{2(Mr+l)a\sin^2\theta + 2\mathcal{R}^2 l\cos\theta}{\mathcal{R}^2 - a^2\sin^2\theta}\right]^2 \exp(+2U), \\ g_{03} &= -\frac{2(Mr+l)a\sin^2\theta + 2\mathcal{R}^2 l\cos\theta}{\mathcal{R}^2 - a^2\sin^2\theta}\exp(+2U), \end{aligned} \tag{2.83}$$

where

$$\exp(\pm 2U) = \left[1 - 2\frac{Mr + l\,(a\cos\theta + l)}{r^2 + (a\cos\theta + l)^2}\right]^{\pm 1}, \quad \mathcal{R}^2 = r^2 - 2Mr + a^2 - l^2,$$

M is a mass parameter, a is an angular momentum per unit mass and l is the NUT parameter. The metric (2.83) is a stationary axisymmetric metric. It belongs to the vacuum solutions of the Einstein field equations, which are of type D under Petrov's classification. The Kerr and NUT space-times are particular cases of the considered metric and can be obtained by putting $l = 0$ or $a = 0$ in (2.83). The case $l = 0, a = 0$ obviously corresponds to the Schwarzschild solution.

Solving the equations of motion and constraints (2.82) for the string case ($p = 1$) in Demianski-Newman background, one can find the following solution [1]

$$\begin{aligned}
t - t_0 &= \pm \int_{r_0}^{r} dr \left[\frac{(C^0 + C^3 \mathcal{A}_0)\, \mathcal{A}_0}{\mathcal{R}^2 \sin^2\theta_0} \exp(+2U_0) - C^3 \exp(-2U_0) \right] W^{-1/2}, \\
\varphi - \varphi_0 &= \mp \int_{r_0}^{r} dr \frac{C^0 + C^3 \mathcal{A}_0}{\mathcal{R}^2 \sin^2\theta_0} \exp(+2U_0) W^{-1/2}, \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^{r} dr W^{-1/2}, \\
W &= \left(\mathcal{R}^2 - a^2 \sin^2\theta_0\right)^{-1} \left[\left(C^3\right)^2 \mathcal{R}^2 - \frac{\left(C^0 + C^3 \mathcal{A}_0\right)^2}{\sin^2\theta_0} \exp(+4U_0) \right], \\
\mathcal{A}_0 &= \frac{2(Mr + l) a \sin^2\theta_0 + 2\mathcal{R}^2 l \cos\theta_0}{\mathcal{R}^2 - a^2 \sin^2\theta_0}, \quad U_0 = U_{|\theta=\theta_0}, \\
t_0, r_0, \varphi_0, \tau_0 &- \mathit{constants}.
\end{aligned} \tag{2.84}$$

2.4.2 Tensionless p-branes in a solitonic background

The solitonic $(\tilde{d} - 1)$-brane background is given by

$$\begin{gathered}
ds^2 = g_{MN} dx^M dx^N = \exp(2A)\, \eta_{\mu\nu} dx^\mu dx^\nu + \exp(2B) \left(dr^2 + r^2 d\Omega^2_{D-\tilde{d}-1} \right), \\
\exp(2A) = \left(1 + \frac{k_{\tilde{d}}}{r^d}\right)^{-\frac{d}{d+\tilde{d}}}, \quad \exp(2B) = \left(1 + \frac{k_{\tilde{d}}}{r^d}\right)^{+\frac{\tilde{d}}{d+\tilde{d}}}, \quad k_{\tilde{d}} = const, \\
d + \tilde{d} = D - 2, \quad \eta_{\mu\nu} = \mathrm{diag}(-, +, ..., +), \quad \mu, \nu = 0, 1, ..., \tilde{d} - 1.
\end{gathered}$$

The $(D - \tilde{d} - 1)$-dimensional sphere $\mathbf{S}^{D-\tilde{d}-1}$ is supposed to be parameterized so that

$$\begin{aligned}
g_{kk} &= \exp(2B)\, r^2 \prod_{n=1}^{D-k-1} \sin^2\theta_n, \quad D - k - 1 = 1, 2, ..., D - \tilde{d} - 2, \\
g_{D-1,D-1} &= \exp(2B)\, r^2.
\end{aligned}$$

In this background the following tensionless p-brane solutions of the equations of motion and constraints (2.82) do exist [2]

$$
\begin{aligned}
y^{\mu} = y^{\mu}_0 \pm E^{\mu}\int_{r_0}^{r} du\left(1+\frac{k_{\tilde{d}}}{u^d}\right)\left(\mathcal{E}-\frac{\left(C^{D-1}\right)^2}{u^2}+\frac{\mathcal{E}k_{\tilde{d}}}{u^d}\right)^{-1/2}, \quad r\equiv y^{\tilde{d}},\\
E^{\mu} = \left(E^0, E^1,\dots,E^{\tilde{d}-1}\right) = \left(CC^{\tilde{d}-1}, C^1,\dots, CC^0\right);\\
\varphi = \varphi_0 \pm C^{D-1}\int_{r_0}^{r}\frac{du}{u^2}\left(\mathcal{E}-\frac{\left(C^{D-1}\right)^2}{u^2}+\frac{\mathcal{E}k_{\tilde{d}}}{u^d}\right)^{-1/2}, \quad \varphi\equiv y^{D-1};\\
\tau = \tau_0 \pm \int_{r_0}^{r} du\left(1+\frac{k_{\tilde{d}}}{u^d}\right)^{\frac{\tilde{d}}{d+\tilde{d}}}\left(\mathcal{E}-\frac{\left(C^{D-1}\right)^2}{u^2}+\frac{\mathcal{E}k_{\tilde{d}}}{u^d}\right)^{-1/2};\\
\mathcal{E} \equiv -E^{\mu}E^{\nu}\eta_{\mu\nu} = \left(CC^{\tilde{d}-1}\right)^2-\left(CC^0\right)^2-\sum_{\alpha=1}^{\tilde{d}-2}\left(C^{\alpha}\right)^2 \geq 0.
\end{aligned}
\tag{2.85}
$$

Let us restrict ourselves to the particular case of ten dimensional solitonic 5-brane background. The corresponding values of the parameters D, $\tilde{d}$ and d are $D=10$, $\tilde{d}=6$, $d=2$. Taking this into account and performing the integration in (2.85), one obtains the following explicit exact solution of the equations of motion and constraints for a tensionless p-brane living in such curved space-time ($\mathcal{C}=k_6-\left(C^9\right)^2/\mathcal{E}>0$)

$$
\begin{aligned}
y^{\mu} = y^{\mu}_0 \mp \frac{k_6 E^{\mu}}{\left(\mathcal{C}\mathcal{E}\right)^{1/2}}\ln\left(\frac{\frac{\mathcal{C}^{1/2}}{r}+\left(1+\frac{\mathcal{C}}{r^2}\right)^{1/2}}{\frac{\mathcal{C}^{1/2}}{r_0}+\left(1+\frac{\mathcal{C}}{r_0^2}\right)^{1/2}}\right)\\
\pm\frac{E^{\mu}}{\mathcal{E}^{1/2}}\left[\left(\mathcal{C}+r^2\right)^{1/2}-\left(\mathcal{C}+r_0^2\right)^{1/2}\right],\\
\varphi = \varphi_0 \mp \frac{C^9}{\left(\mathcal{C}\mathcal{E}\right)^{1/2}}\ln\left(\frac{\frac{\mathcal{C}^{1/2}}{r}+\left(1+\frac{\mathcal{C}}{r^2}\right)^{1/2}}{\frac{\mathcal{C}^{1/2}}{r_0}+\left(1+\frac{\mathcal{C}}{r_0^2}\right)^{1/2}}\right),\\
\tau = \tau_0 \mp \left(\frac{k_6^3}{\mathcal{C}^4\mathcal{E}^2}\right)^{1/4} r^{3/2}\left(1+\frac{\mathcal{C}}{r^2}\right)^{1/2} F_2\left(3/4, 1, -3/4; 3/2, 3/4; 1+\frac{r^2}{\mathcal{C}}, -\frac{r^2}{k_6}\right)\\
\mp 2\left(\frac{k_6^3 r_0^2}{\mathcal{C}^2\mathcal{E}^2}\right)^{1/4} F_1\left(1/4, 1/2, -3/4; 5/4; -\frac{r_0^2}{\mathcal{C}}, -\frac{r_0^2}{k_6}\right)\\
\pm\frac{\Gamma(1/4)k_6}{4\Gamma(3/4)}\sqrt{\frac{\pi}{\mathcal{C}\mathcal{E}}}\,{}_2F_1\left(1/4, 1/2; -1/4; 1-\frac{k_6}{\mathcal{C}}\right)\\
\mp\frac{\Gamma(1/4)k_6}{2\Gamma(3/4)}\sqrt{\frac{\pi}{\mathcal{C}\mathcal{E}}}\left(1-\frac{k_6}{\mathcal{C}}\right)^{-1/4}{}_2F_1\left(1/4, 3/2; 3/4; \left(1-\frac{k_6}{\mathcal{C}}\right)^{-1}\right).
\end{aligned}
\tag{2.86}
$$

2.4.3 String and D-string solutions in other backgrounds

Let us first give an explicit example of exact solution for a string moving in four dimensional cosmological Kasner type background. Namely, the line element is ($x^0 \equiv t$)

$$ds^2 = g_{MN}dx^M dx^N = -(dt)^2 + \sum_{\mu=1}^{3} t^{2q_\mu}(dx^\mu)^2, \tag{2.87}$$

$$\sum_{\mu=1}^{3} q_\mu = 1, \quad \sum_{\mu=1}^{3} q_\mu^2 = 1. \tag{2.88}$$

For definiteness, we choose $q_\mu = (2/3, 2/3, -1/3)$. Then, one can obtain the following exact solution of the equations of motion and constraints in the considered particular metric [3]

$$\begin{aligned} X^\mu(t,\sigma) &= X_0^\mu + C_\pm^\mu\left(\lambda^1\tau + \sigma\right) \pm A_\mu^\pm I^\mu(t), \quad \tau(t) = \tau_0 \pm I^0(t), \\ I^M(t) &\equiv \int_{t_0}^{t} du u^{-2q_M}\left(V^\pm\right)^{-1/2}, \quad q_M = (0, 2/3, 2/3, -1/3), \end{aligned} \tag{2.89}$$

and $V^\pm$ is the corresponding effective scalar potential.

Although we have chosen relatively simple background metric, the expressions for I^M are too complicated. Because of that, we shall write down here only the formulas for the two limiting cases $T = 0$ and $T \to \infty$ for $t > t_0 \geq 0$. The former corresponds to considering tensionless strings (high energy string limit).

When $T = 0$, I^M reads

$$\begin{aligned} I^M = & \frac{1}{2\left[\left(A_1^\pm\right)^2 + \left(A_2^\pm\right)^2\right]^{1/2}}\Bigg[\frac{t^{2/3-2q_M}}{(q_M - 1/3)A}\, {}_2F_1\left(1/2, q_M - 1/3; q_M + 2/3; -\frac{1}{A^2t^2}\right) \\ & + \frac{t_0^{5/3-2q_M}}{q_M - 5/6}\, {}_2F_1\left(1/2, 5/6 - q_M; 11/6 - q_M; -A^2t_0^2\right) + \frac{\Gamma(q_M - 1/3)\Gamma(5/6 - q_M)}{\sqrt{\pi}A^{5/3-2q_M}}\Bigg], \end{aligned}$$

where

$$A^2 \equiv \frac{\left(A_3^\pm\right)^2}{\left(A_1^\pm\right)^2 + \left(A_2^\pm\right)^2}.$$

When $T \to \infty$, I^M are given by the equalities

$$\begin{aligned}
I^0 &= \pm\frac{1}{4\lambda^0 T C^3_\pm}\left[\frac{6}{C}t^{1/3}{}_2F_1\left(1/2,-1/6;5/6;-\frac{1}{C^2t^2}\right)\right.\\
&- \left.\frac{3}{2}t_0^{4/3}{}_2F_1\left(1/2,2/3;5/3;-C^2t_0^2\right)+\frac{\Gamma\left(-1/6\right)\Gamma\left(2/3\right)}{\sqrt{\pi}C^{4/3}}\right],\\
I^{1,2} &= \pm\frac{1}{4\lambda^0 T C^3_\pm}\left[\ln\left|\frac{\left(1+C^2t^2\right)^{1/2}-1}{\left(1+C^2t^2\right)^{1/2}+1}\right|-\ln\left|\frac{\left(1+C^2t_0^2\right)^{1/2}-1}{\left(1+C^2t_0^2\right)^{1/2}+1}\right|\right],\\
I^3 &= \pm\frac{1}{2\lambda^0 T C^3_\pm C^2}\left[\left(1+C^2t^2\right)^{1/2}-\left(1+C^2t_0^2\right)^{1/2}\right],
\end{aligned}$$

where

$$C^2 \equiv \frac{\left(C^1_\pm\right)^2+\left(C^2_\pm\right)^2}{\left(C^3_\pm\right)^2}.$$

Our choice of the scale factors $\left(t^{2/3}, t^{2/3}, t^{-1/3}\right)$ was dictated only by the simplicity of the solution. However, this is a very special case of a Kasner type metric. Actually, this is one of the two solutions of the constraints (2.88) (up to renaming of the coordinates x^μ) for which two of the exponents q_μ are equal. The other such solution is $q_\mu = (0,0,1)$ and it corresponds to flat space-time. Now, we will write down the exact tensionless string solution ($T = 0$) for a gravity background with arbitrary, but different q_μ. It is given by (2.89), where

$$\begin{aligned}
&I^M(t) = constant - \frac{\sqrt{\pi}}{A_2^\pm}\sum_{k=0}^{\infty}\frac{\left(A_3^\pm/A_2^\pm\right)^{2k}}{k!\Gamma\left(1/2-k\right)}\frac{t^{\mathcal{P}}}{\mathcal{P}}\times\\
&{}_2F_1\left(1/2+k,\frac{\mathcal{P}}{2(q_2-q_1)};\frac{2(q_2-q_3)k+3q_2-2q_1+1-2q_M}{2(q_2-q_1)};-\left(\frac{A_1^\pm}{A_2^\pm}\right)^2 t^{2(q_2-q_1)}\right),\\
&\mathcal{P}\equiv 2(q_2-q_3)k+q_2+1-2q_M, \quad q_M=(0,q_1,q_2,q_3), \quad \text{for } q_1>q_2,
\end{aligned}$$

and

$$\begin{aligned}
&I^M(t) = constant + \frac{\sqrt{\pi}}{A_1^\pm}\sum_{k=0}^{\infty}\frac{\left(A_3^\pm/A_1^\pm\right)^{2k}}{k!\Gamma\left(1/2-k\right)}\frac{t^{\mathcal{Q}}}{\mathcal{Q}}\times\\
&{}_2F_1\left(1/2+k,\frac{\mathcal{Q}}{2(q_1-q_2)};\frac{2(q_1-q_3)k+3q_1-2q_2+1-2q_M}{2(q_1-q_2)};-\left(\frac{A_2^\pm}{A_1^\pm}\right)^2 t^{2(q_1-q_2)}\right),\\
&\mathcal{Q}\equiv 2(q_1-q_3)k+q_1+1-2q_M, \quad \text{for } q_1<q_2.
\end{aligned}$$

Because there are no restrictions on q_μ, except $q_1 \neq q_2 \neq q_3$, the above probe string solution is also valid in generalized Kasner type backgrounds arising in superstring cosmology. In string frame, the effective Kasner constraints for the four dimensional dilaton-moduli-vacuum solution are

$$\sum_{\mu=1}^{3} q_\mu = 1 + \mathcal{K}, \quad \sum_{\mu=1}^{3} q_\mu^2 = 1 - \mathcal{B}^2,$$
$$-1 - \sqrt{3\left(1-\mathcal{B}^2\right)} \leq \mathcal{K} \leq -1 + \sqrt{3\left(1-\mathcal{B}^2\right)}, \quad \mathcal{B}^2 \in [0,1]\,.$$

In Einstein frame, the metric has the same form, but in new, rescaled coordinates and with new powers $\tilde{q}_\mu$ of the scale factors. The generalized Kasner constraints are also modified as follows

$$\sum_{\mu=1}^{3} \tilde{q}_\mu = 1, \quad \sum_{\mu=1}^{3} \tilde{q}_\mu^2 = 1 - \tilde{\mathcal{B}}^2 - \frac{1}{2}\tilde{\mathcal{K}}^2, \quad \tilde{\mathcal{B}}^2 + \frac{1}{2}\tilde{\mathcal{K}}^2 \in [0,1]\,.$$

Actually, the obtained tensionless string solution is also relevant to considerations within a *pre-big bang* context, because there exist a class of models for pre-big bang cosmology, which is a particular case of the given generalized Kasner backgrounds.

Our next example is for a string moving in the following ten dimensional supergravity background given in Einstein frame

$$ds^2 = g_{MN}^E dx^M dx^N = \exp(2A)\eta_{mn}dx^m dx^n + \exp(2B)\left(dr^2 + r^2 d\Omega_7^2\right),$$
$$\exp\left[-2(\phi-\phi_0)\right] = 1 + \frac{k}{r^6}, \quad \phi_0, k = constants,$$
$$A = \frac{3}{4}(\phi-\phi_0), \quad B = -\frac{1}{4}(\phi-\phi_0), \quad B_{01} = -\exp\left[-2\left(\phi - \frac{3}{4}\phi_0\right)\right].$$

All other components of B_{MN} as well as all components of the gravitino ψ_M and dilatino λ are zero. If we parameterize the sphere S^7 so that

$$g_{10-j,10-j}^E = \exp(2B)r^2 \prod_{l=1}^{j-1} \sin^2 x^{10-l}, \quad j = 2,3,...,7, \quad g_{99}^E = \exp(2B)r^2,$$

the metric g_{MN}^E does not depend on x^0, x^1 and x^3, i.e. $\mu = 0,1,3$. Then we set [3] $y^\alpha = y_0^\alpha = constants$ for $\alpha = 4,...,9$ and obtain a solution of the equations of motion and constraints as a function of the radial coordinate r:

$$X^\mu(r,\sigma) = X_0^\mu + C_\pm^\mu\left(\lambda^1\tau + \sigma\right) \pm I^\mu(r), \quad \tau(r) = \tau_0 \pm I(r),$$
$$I^m(r) = \eta^{mn}\int_{r_0}^{r} du\left[B_n^\pm + 2\lambda^0 T\varepsilon_{nk}C_\pm^k \exp\left(-\phi_0/2\right)\left(1+\frac{k}{u^6}\right)\right]\left(1+\frac{k}{u^6}\right)W_\pm^{-1/2},$$
$$I^3(r) = \frac{B_3^\pm}{s^2}\int_{r_0}^{r}\frac{du}{u^2}W_\pm^{-1/2}, \quad s \equiv \prod_{l=1}^{6}\sin y_0^{10-l}, \quad I(r) = \exp\left(\phi_0/2\right)\int_{r_0}^{r} du W_\pm^{-1/2},$$

where

$$W_{\pm} = \left\{ \left[B_0^{\pm} + 2\lambda^0 T C_{\pm}^1 \exp\left(-\phi_0/2\right) \left(1 + \frac{k}{u^6}\right)\right]^2 \right.$$
$$- \left. \left[B_1^{\pm} - 2\lambda^0 T C_{\pm}^0 \exp\left(-\phi_0/2\right) \left(1 + \frac{k}{u^6}\right)\right]^2 \right\} \left(1 + \frac{k}{u^6}\right) - \left(\frac{B_3^{\pm}}{s}\right)^2 \frac{1}{u^2}$$
$$+ \left(2\lambda^0 T\right)^2 \exp\left(\phi_0\right) \left\{ \left[\left(C_{\pm}^0\right)^2 - \left(C_{\pm}^1\right)^2 \right] \left(1 + \frac{k}{u^6}\right)^{-1} - \left(C_{\pm}^3 s\right)^2 u^2 \right\}.$$

This is the solution also in the string frame, because we have one and the same metric in the action expressed in two different ways.

The above solution extremely simplifies in the tensionless limit $T \to 0$. Let us give the manifest expressions for this case. For $r_0 < r$, they are:

$$\lim_{T\to 0} I^m(r) = \eta^{mn} B_m^{\pm} \left(J^0 + kJ^6\right), \quad \lim_{T\to 0} I^3(r) = \frac{B_3^{\pm}}{s^2} J^2, \quad \lim_{T\to 0} I(r) = J^0,$$

where

$$J^{\beta}(r) = -\sqrt{\frac{\pi}{\left(B_0^{\pm}\right)^2 - \left(B_1^{\pm}\right)^2}}$$
$$\times \left\{ \frac{1}{r^{\beta-1}} \sum_{n=0}^{\infty} \frac{\Gamma\left(-\frac{6n+\beta-5}{4}\right) \left(k/r^6\right)^n}{(6n+\beta-1)\,\Gamma\left(\frac{1-2n}{2}\right)\Gamma\left(-\frac{2n+\beta-5}{4}\right)} P_n^{\left(-\frac{6n+\beta-1}{4}, -n-1\right)}\left(1 - 2\frac{\delta}{k} r^4\right) \right.$$
$$\left. - \frac{1}{r_0^{\beta-1}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+\beta+3}{4}\right)\left(-\delta/r_0^2\right)^n}{(2n+\beta-1)\,\Gamma\left(\frac{1-2n}{2}\right)\Gamma\left(\frac{6n+\beta+3}{4}\right)} P_n^{\left(\frac{2n+\beta-1}{4}, -n-1\right)}\left(1 - 2\frac{k}{\delta} r_0^{-4}\right) \right\},$$
$$\delta \equiv \frac{\left(B_3^{\pm}/s\right)^2}{\left(B_0^{\pm}\right)^2 - \left(B_1^{\pm}\right)^2},$$

and $P_n^{(\alpha,\beta)}(z)$ are the Jacobi polynomials. To obtain the solution for $r_0 > r$, one has to exchange r and r_0 in the expression for J^{β}.

Now let us turn to the case of a D-string living in five dimensional *anti de Sitter* space-time. The corresponding metric may be written as

$$g_{00} = -\left(1 + \frac{r^2}{R^2}\right), \quad g_{11} = \left(1 + \frac{r^2}{R^2}\right)^{-1},$$
$$g_{22} = r^2 \sin^2 x^3 \sin^2 x^4, \quad g_{33} = r^2 \sin^2 x^4, \quad g_{44} = r^2,$$

where $K = -1/R^2$ is the constant curvature.

The exact string solution as a function of r found in [3] is

$$X^0(r,\sigma) = X_0^0 + C_\pm^0\left(\lambda^1\tau+\sigma\right) \mp B_0^\pm \int_{r_0}^r du \left(1+\frac{u^2}{R^2}\right)^{-1}\left(g_{00}V_D^{\pm 0}\right)^{-1/2},$$
$$X^2(r,\sigma) = X_0^2 + C_\pm^2\left(\lambda^1\tau+\sigma\right) \pm \frac{B_2^\pm}{c^2}\int_{r_0}^r \frac{du}{u^2}\left(g_{00}V_D^{\pm 0}\right)^{-1/2},$$
$$\tau(r) = \tau_0 \pm \int_{r_0}^r du\left(g_{00}V_D^{\pm 0}\right)^{-1/2}, \qquad c \equiv \sin x_0^3 \sin x_0^4,$$
$$F_{01}(r) = -g_s T_D \lambda^2 \left\{\left[\left(\frac{C_\pm^0}{R}\right)^2 - \left(cC_\pm^2\right)^2\right] r^2 + \left(C_\pm^0\right)^2\right\},$$

where

$$g_{00}V_D^{\pm 0} = \left[\left(B_0^\pm\right)^2 - \left(\frac{B_2^\pm}{cR}\right)^2 + \left(A_\pm C_\pm^0\right)^2\right] - \left(\frac{B_2^\pm}{c}\right)^2 \frac{1}{u^2}$$
$$+A_\pm^2\left[2\left(\frac{C_\pm^0}{R}\right)^2 - \left(cC_\pm^2\right)^2\right] u^2 + \left(\frac{A_\pm}{R}\right)^2\left[\left(\frac{C_\pm^0}{R}\right)^2 - \left(cC_\pm^2\right)^2\right] u^4.$$

This solution describes a D-string evolving in the subspace (x^0, x^1, x^2).

Alternatively, we could fix the coordinates $r = r_0$, $x^4 = x_0^4 \equiv \psi_0$ and obtain a solution as a function of the coordinate $x^3 \equiv \theta$. In this case, the result is the following [3]

$$X^0(\theta,\sigma) = X_0^0 + C_\pm^0\left(\lambda^1\tau+\sigma\right) \mp \frac{B_0^\pm \varrho}{g_{00}^0}\int_{\theta_0}^\theta du\left(-V_D^{\pm 0}\right)^{-1/2},$$
$$X^2(\theta,\sigma) = X_0^2 + C_\pm^2\left(\lambda^1\tau+\sigma\right) \pm \frac{B_2^\pm}{\varrho}\int_{\theta_0}^\theta \frac{du}{\sin^2 u}\left(-V_D^{\pm 0}\right)^{-1/2},$$
$$\tau(\theta) = \tau_0 \pm \varrho \int_{\theta_0}^\theta du\left(-V_D^{\pm 0}\right)^{-1/2}, \qquad \varrho \equiv r_0 \sin\psi_0,$$
$$F_{01}(\theta) = g_s T_D \lambda^2\left[\left(C_\pm^0\right)^2 g_{00}^0 + \left(\varrho C_\pm^2\right)^2 \sin^2\theta\right],$$

where

$$-V_D^{\pm 0} = \left[\left(B_0^\pm\right)^2 g_{11}^0 - \left(A_\pm C_\pm^0\right)^2 g_{00}^0\right] - \left(A_\pm C_\pm^2 \varrho\right)^2 \sin^2 u - \frac{\left(B_2^\pm/\varrho\right)^2}{\sin^2 u}.$$

This is a solution for D-string placed in the subspace described by the coordinates (x^0, x^2, x^3).

Finally, we will give an example of exact solution for a D-string moving in a non-diagonal metric. To this end, let us consider the ten dimensional black hole solution of [45]. In string frame metric, it can be written as [46]

$$
\begin{aligned}
ds^2 = & \left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)^{-1/2}\left(1+\frac{r_0^2\sinh^2\gamma}{r^2}\right)^{-1/2} \\
\times & \left\{-dt^2+(dx^9)^2+\frac{r_0^2}{r^2}\left(\cosh\chi dt+\sinh\chi dx^9\right)^2\right. \\
+ & \left.\left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)\left[(dx^5)^2+(dx^6)^2+(dx^7)^2+(dx^8)^2\right]\right\} \\
+ & \left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)^{1/2}\left(1+\frac{r_0^2\sinh^2\gamma}{r^2}\right)^{1/2}\left[\left(1-\frac{r_0^2}{r^2}\right)^{-1}dr^2+r^2d\Omega_3^2\right], \\
\exp\left[-2(\phi-\phi_\infty)\right] = & \left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)^{-1}\left(1+\frac{r_0^2\sinh^2\gamma}{r^2}\right). \qquad (2.90)
\end{aligned}
$$

The equalities (2.90) define a solution of type IIB string theory, which low energy action in Einstein frame contains the terms

$$\int d^{10}x\sqrt{-g}\left[R-\frac{1}{2}\left(\nabla\phi\right)^2-\frac{1}{12}\exp(\phi)H'^2\right], \qquad (2.91)$$

where H' is the Ramond-Ramond three-form field strength. The Neveu-Schwarz 3-form field strength, the selfdual 5-form field strength and the second scalar are set to zero. After some simplifications [3], the exact D-string solution as a function of the radial coordinate r reads

$$
\begin{aligned}
& X^\mu\left(r,\sigma\right) = X_0^\mu + C_\pm^\mu\left(\lambda^1\tau+\sigma\right) \pm I^\mu(r), \qquad \mu = 0,2,5,6,7,8,9, \\
& X^{3,4} = X_0^{3,4} = constants, \qquad \tau(r) = \tau_0 \pm I(r),
\end{aligned}
$$

where

$$
\begin{aligned}
I^0 = & -\int_{r_0}^{r} du\left[B_0^\pm g_{99} - B_9^\pm g_{09}\right]g_{11}\mathcal{W}^{-1/2}, \\
I^9 = & \int_{r_0}^{r} du\left[B_0^\pm g_{09} - B_9^\pm g_{00}\right]g_{11}\mathcal{W}^{-1/2}, \\
I^l = & B_l^\pm\int_{r_0}^{r}\frac{du}{g_{ll}^0}\mathcal{W}^{-1/2}, \qquad l=2,5,6,7,8, \qquad I = \int_{r_0}^{r} du\mathcal{W}^{-1/2}, \\
F_{01} = & g_s T_D \lambda^2 C_\pm^\mu C_\pm^\nu g_{\mu\nu}^0(r), \\
\mathcal{W} = & \left[\left(B_9^\pm\right)^2 - \frac{\left(A_\pm C_\pm^0\right)^2}{g_{11}}\right]g_{00} + \left[\left(B_0^\pm\right)^2 - \frac{\left(A_\pm C_\pm^9\right)^2}{g_{11}}\right]g_{99} \\
& -2\left(B_0^\pm B_9^\pm + \frac{A_\pm^2 C_\pm^0 C_\pm^9}{g_{11}}\right)g_{09} \\
& -\frac{1}{g_{11}}\left\{\sum_{l=5}^{8}\left[\left(A_\pm C_\pm^l\right)^2 + \left(B_\pm^l\right)^2\right] + \left(\frac{B_\pm^2}{c}\right)^2\frac{1}{u^2} + \left(A_\pm C_\pm^2 c\right)^2 u^2\right\}.
\end{aligned}
$$

All previous considerations are based on the ansatz (2.19). However, there exist other embeddings which allow for simplification of the dynamics and as a consequence for obtaining exact solutions of the equations of motion and constraints. An example is the following one

$$X^{\mu}(\xi^{m}) = \Lambda^{\mu}_{m}\xi^{m} + Z^{\mu}(\alpha\sigma + \beta\tau), \quad X^{a}(\xi^{m}) = Z^{a}(\alpha\sigma + \beta\tau), \quad \alpha, \beta = constants, \tag{2.92}$$

where $\tau = \xi^0$, σ is one of the worldvolume spatial coordinates ξ^i, and Z^μ, Z^a are arbitrary functions. This ansatz will be used further on.

3 AdS/CFT

The AdS/CFT duality [33] between string/M-theory on curved space-times with Anti-de Sitter subspaces and conformal field theories in different dimensions has been actively investigated in the last years. A lot of impressive progresses have been made in this field of research based mainly on the integrability structures discovered on both sides of the correspondence. The most studied example is the duality between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ target space and the $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four space-time dimensions. However, many other cases are also of interest, and have been investigated intensively (for recent review on the AdS/CFT duality, see [47]).

Different classical string/M-theory solutions play important role in checking and understanding the AdS/CFT correspondence [48]. To establish relations with the dual gauge theory, one has to take the semiclassical limit of *large* conserved charges [49].

An interesting issue to solve is to find the *finite-size effects*, related to the wrapping interactions in the dual field theory [50].

3.1 Classical string solutions and string/field theory duality

In [5] and [6] the string dynamics in general string theory target space-times is considered by using the Polyakov action (2.1) (for strings $p = 1$). Exact solutions of the equations of motion and Virasoro constraints are found. This is done in the covariant worldsheet gauge $\gamma^{mn} = constants$. The considerations in [5] are based on the ansatz (2.19), while in [6] a particular case of the ansatz (2.92) is used, corresponding to $\alpha = 1$, $\beta = 0$. Then, the general results are applied for several string backgrounds having dual field theory description. Namely, $AdS_5 \times S^5$ with field theory dual $\mathcal{N} = 4$ SYM, AdS_5 black hole with field theory dual *finite temperature* $\mathcal{N} = 4$ SYM, and $AdS_3 \times S^3 \times \mathcal{M}$, with NS-NS 2-form gauge field. In accordance with the AdS/CFT duality, the string theory on $AdS_3 \times S^3 \times \mathcal{M}$ is dual to a superconformal field theory on a cylinder, which is the boundary of AdS_3 in global coordinates.

Analogous considerations have been made in [27]. The difference is that the most general form of the embedding (2.92) is applied for strings. Let us describe the general results obtained there.

In what follow we will use *conformal gauge* $\gamma^{mn} = \eta^{mn} = diag(-1, 1)$ in which the string Lagrangian, the Virasoro constraints and the equations of motion take the following form:

$$\mathcal{L} = \frac{T}{2}\left(G_{00} - G_{11} + 2B_{01}\right), \tag{3.1}$$
$$G_{00} + G_{11} = 0, \quad G_{01} = 0,$$
$$g_{LK}\left[\left(\partial_0^2 - \partial_1^2\right) X^K + \Gamma^K_{MN}\left(\partial_0 X^M \partial_0 X^N - \partial_1 X^M \partial_1 X^N\right)\right] = H_{LMN}\partial_0 X^M \partial_1 X^N.$$

Now, let us *suppose* that there exist some number of commuting Killing vector fields along part of X^M coordinates and split X^M into two parts

$$X^M = (X^\mu, X^a),$$

where X^μ are the isometric coordinates, while X^a are the non-isometric ones. The existence of isometric coordinates leads to the following conditions on the background fields:

$$\partial_\mu g_{MN} = 0, \quad \partial_\mu b_{MN} = 0. \tag{3.2}$$

Then from the string action, we can compute the conserved charges

$$Q_\mu = \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_0 X^\mu)} \tag{3.3}$$

under the above conditions.

Next, we introduce the following ansatz for the string embedding

$$X^\mu(\tau, \sigma) = \Lambda^\mu \tau + \tilde{X}^\mu(\alpha\sigma + \beta\tau), \quad X^a(\tau, \sigma) = \tilde{X}^a(\alpha\sigma + \beta\tau), \tag{3.4}$$

where Λ^μ, α, β are arbitrary parameters. Further on, we will use the notation $\xi = \alpha\sigma + \beta\tau$. Applying this ansatz, one can find that the equalities (3.1), (3.3) become

$$\mathcal{L} = \frac{T}{2}\Big[-(\alpha^2 - \beta^2) g_{MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi} + 2\Lambda^\mu\left(\beta g_{\mu N} + \alpha b_{\mu N}\right)\frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu\Lambda^\nu g_{\mu\nu}\Big], \tag{3.5}$$

$$G_{00} + G_{11} = (\alpha^2 + \beta^2) g_{MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi} + 2\beta\Lambda^\mu g_{\mu N}\frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu\Lambda^\nu g_{\mu\nu} = 0, \tag{3.6}$$

$$G_{01} = \alpha\beta g_{MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi} + \alpha\Lambda^\mu g_{\mu N}\frac{d\tilde{X}^N}{d\xi} = 0, \tag{3.7}$$

$$-(\alpha^2-\beta^2)\left[g_{LK}\frac{d^2\tilde{X}^K}{d\xi^2}+\Gamma_{L,MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi}\right]+2\beta\Lambda^\mu\Gamma_{L,\mu N}\frac{d\tilde{X}^N}{d\xi}+\Lambda^\mu\Lambda^\nu\Gamma_{L,\mu\nu}$$
$$=\alpha\Lambda^\mu H_{L\mu N}\frac{d\tilde{X}^N}{d\xi}, \tag{3.8}$$

$$Q_\mu=\frac{T}{\alpha}\int d\xi\left[(\beta g_{\mu N}+\alpha b_{\mu N})\frac{d\tilde{X}^N}{d\xi}+\Lambda^\nu g_{\mu\nu}\right]. \tag{3.9}$$

Our next task is to try to solve the equations of motion (3.8) for the isometric coordinates, i.e. for $L=\lambda$. Due to the conditions (3.2) imposed on the background fields, we obtain that

$$\Gamma_{\lambda,ab}=\frac{1}{2}\left(\partial_a g_{b\lambda}+\partial_b g_{a\lambda}\right),\quad \Gamma_{\lambda,\mu a}=\frac{1}{2}\partial_a g_{\mu\lambda}\quad \Gamma_{\lambda,\mu\nu}=0,$$
$$H_{\lambda ab}=\partial_a b_{b\lambda}+\partial_b b_{\lambda a},\quad H_{\lambda\mu a}=\partial_a b_{\lambda\mu},\quad H_{\lambda\mu\nu}=0.$$

By using this, one can find the following first integrals for $\tilde{X}^\mu$:

$$\frac{d\tilde{X}^\mu}{d\xi}=\frac{1}{\alpha^2-\beta^2}\left[g^{\mu\nu}\left(C_\nu-\alpha\Lambda^\rho b_{\nu\rho}\right)+\beta\Lambda^\mu\right]-g^{\mu\nu}g_{\nu a}\frac{d\tilde{X}^a}{d\xi}, \tag{3.10}$$

where C_ν are arbitrary integration constants. Therefore, according to our ansatz (3.4), the solutions for the string coordinates X^μ can be written as

$$X^\mu(\tau,\sigma)=\Lambda^\mu\tau+\frac{1}{\alpha^2-\beta^2}\int d\xi\left[g^{\mu\nu}\left(C_\nu-\alpha\Lambda^\rho b_{\nu\rho}\right)+\beta\Lambda^\mu\right]-\int g^{\mu\nu}g_{\nu a}d\tilde{X}^a(\xi). \tag{3.11}$$

Now, let us turn to the remaining equations of motion corresponding to $L=a$, where

$$\Gamma_{a,\mu b}=-\frac{1}{2}(\partial_a g_{b\mu}-\partial_b g_{a\mu}),\quad \Gamma_{a,\mu\nu}=-\frac{1}{2}\partial_a g_{\mu\nu},$$
$$H_{a\mu\nu}=\partial_a b_{\mu\nu},\quad H_{a\mu b}=-\partial_a b_{b\mu}+\partial_b b_{a\mu}.$$

Taking this into account and replacing the first integrals for $\tilde{X}^\mu$ already found, one can write these equations in the form (prime is used for $d/d\xi$)

$$(\alpha^2-\beta^2)\left[h_{ab}\tilde{X}^{b''}+\Gamma^h_{a,bc}\tilde{X}^{b'}\tilde{X}^{c'}\right]=2\partial_{[a}A_{b]}\tilde{X}^{b'}-\partial_a U, \tag{3.12}$$

where

$$h_{ab}=g_{ab}-g_{a\mu}g^{\mu\nu}g_{\nu b},\quad \Gamma^h_{a,bc}=\frac{1}{2}\left(\partial_b h_{ca}+\partial_c h_{ba}-\partial_a h_{bc}\right) \tag{3.13}$$
$$A_a=g_{a\mu}g^{\mu\nu}\left(C_\nu-\alpha\Lambda^\rho b_{\nu\rho}\right)+\alpha\Lambda^\mu b_{a\mu}, \tag{3.14}$$
$$U=\frac{1/2}{\alpha^2-\beta^2}\left[\left(C_\mu-\alpha\Lambda^\rho b_{\mu\rho}\right)g^{\mu\nu}\left(C_\nu-\alpha\Lambda^\lambda b_{\nu\lambda}\right)+\alpha^2\Lambda^\mu\Lambda^\nu g_{\mu\nu}\right]. \tag{3.15}$$

One can show that the above equations for $\tilde{X}^a$ can be derived from the effective Lagrangian

$$\mathcal{L}^{eff}(\xi) = \frac{1}{2}(\alpha^2 - \beta^2) h_{ab} \tilde{X}^{a'} \tilde{X}^{b'} + A_a \tilde{X}^{a'} - U.$$

The corresponding effective Hamiltonian is

$$\mathcal{H}^{eff}(\xi) = \frac{1}{2}(\alpha^2 - \beta^2) h_{ab} \tilde{X}^{a'} \tilde{X}^{b'} + U,$$

or in terms of the momenta p_a conjugated to $\tilde{X}^a$

$$\mathcal{H}^{eff}(\xi) = \frac{1}{2}(\alpha^2 - \beta^2) h^{ab} (p_a - A_a)(p_b - A_b) + U.$$

The Virasoro constraints (3.6), (3.7) become:

$$\frac{1}{2}(\alpha^2 - \beta^2) h_{ab} \tilde{X}^{a'} \tilde{X}^{b'} + U = 0, \quad \alpha \Lambda^\mu C_\mu = 0. \tag{3.16}$$

Finally, let us write down the expressions for the conserved charges (3.9)

$$\begin{aligned} Q_\mu &= \frac{T}{\alpha^2 - \beta^2} \int d\xi \left[\frac{\beta}{\alpha} C_\mu + \alpha \Lambda^\nu g_{\mu\nu} + b_{\mu\nu} g^{\nu\rho} \left(C_\rho - \alpha \Lambda^\lambda b_{\rho\lambda} \right) \right. \\ &+ \left. (\alpha^2 - \beta^2)(b_{\mu a} - b_{\mu\nu} g^{\nu\rho} g_{\rho a}) \tilde{X}^{a'} \right]. \end{aligned} \tag{3.17}$$

3.1.1 Rotating strings in type IIA reduction of M-theory on G_2 manifold and their semiclassical limits

The type IIA background, in which we will search for rotating string solutions, has the form [52]

$$\begin{aligned} ds^2_{10} &= r_0^{1/2} C \left\{ -(dx^0)^2 + \delta_{IJ} dx^I dx^J + A^2 \left[(g^1)^2 + (g^2)^2 \right] \right. \\ &+ \left. B^2 \left[(g^3)^2 + (g^4)^2 \right] + D^2 (g^5)^2 \right\} + r_0^{1/2} \frac{dr^2}{C}, \quad (I, J = 1, 2, 3), \quad r_0 = const, \\ & e^\Phi = r_0^{3/4} C^{3/2}, \quad F_2 = \sin\theta_1 d\phi_1 \wedge d\theta_1 - \sin\theta_2 d\phi_2 \wedge d\theta_2. \end{aligned} \tag{3.18}$$

Here, g^1,...,g^5 are given by

$$\begin{aligned} g^1 &= -\sin\theta_1 d\phi_1 - \cos\psi_1 \sin\theta_2 d\phi_2 + \sin\psi_1 d\theta_2, \\ g^2 &= d\theta_1 - \sin\psi_1 \sin\theta_2 d\phi_2 - \cos\psi_1 d\theta_2, \\ g^3 &= -\sin\theta_1 d\phi_1 + \cos\psi_1 \sin\theta_2 d\phi_2 - \sin\psi_1 d\theta_2, \\ g^4 &= d\theta_1 + \sin\psi_1 \sin\theta_2 d\phi_2 + \cos\psi_1 d\theta_2, \\ g^5 &= d\psi_1 + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2, \end{aligned}$$

and the functions A, B, C and D depend on the radial coordinate r only:

$$\begin{aligned} A &= \frac{1}{\sqrt{12}}\sqrt{(r-3r_0/2)(r+9r_0/2)}, \quad B = \frac{1}{\sqrt{12}}\sqrt{(r+3r_0/2)(r-9r_0/2)}, \\ C &= \sqrt{\frac{(r-9r_0/2)(r+9r_0/2)}{(r-3r_0/2)(r+3r_0/2)}}, \quad D = r/3. \end{aligned} \tag{3.19}$$

In (3.18), Φ and F_2 are the Type IIA dilaton and the field strength of the Ramond-Ramond one-form gauge field respectively.

The above ten dimensional background arises as dimensional reduction of M-theory on a G_2 manifold with field theory dual four dimensional $\mathcal{N}=1$ SYM.

The type IIA solution (3.18) describes a D6-brane wrapping the $\mathbf{S}^3$ in the deformed conifold geometry. For $r\to\infty$, the metric becomes that of a singular conifold, the dilaton is constant, and the flux is through the $\mathbf{S}^2$ surrounding the wrapped D6-brane. For $r-9r_0/2=\epsilon\to 0$, the string coupling e^{Φ} goes to zero like $\epsilon^{3/4}$, whereas the curvature blows up as $\epsilon^{-3/2}$ just like in the near horizon region of a flat D6-brane. This means that classical supergravity is valid for sufficiently large radius. However, the singularity in the interior is the same as the one of flat D6 branes, as expected. On the other hand, the dilaton continuously decreases from a finite value at infinity to zero, so that for small r_0 classical string theory is valid everywhere. As explained in [53], the global geometry is that of a warped product of flat Minkowski space and a non-compact space, Y_6, which for large radius is simply the conifold since the backreaction of the wrapped D6 brane becomes less and less important. However, in the interior, the backreaction induces changes on Y_6 away from the conifold geometry. For $r\to 9r_0/2$, the $\mathbf{S}^2$ shrinks to zero size, whereas an $\mathbf{S}^3$ of finite size remains. This behavior is similar to that of the deformed conifold but the two metrics are different.

Three types of rotating string solutions have been found in [10].

The first one is given by ($\Delta r = r-3l$, $\Delta r_1 = r_1-3l$)

$$\begin{aligned} \xi^1(r) &= \frac{8}{(\Lambda_+^2+\Lambda_-^2)^{1/2}}\left[\frac{l\Delta r}{(3l-r_2)\Delta r_1}\right]^{1/2}\times \\ &F_D^{(5)}\left(1/2;-1/2,-1/2,1/2,1/2,1/2;3/2;-\frac{\Delta r}{2l},-\frac{\Delta r}{4l},-\frac{\Delta r}{6l},-\frac{\Delta r}{3l-r_2},\frac{\Delta r}{\Delta r_1}\right). \end{aligned} \tag{3.20}$$

Now, we can compute the conserved momenta on the obtained solution. They are:

$$\begin{aligned} \frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} &= T\left[\frac{2^7 l\Delta r_1}{(\Lambda_+^2+\Lambda_-^2)(3l-r_2)}\right]^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2} \\ &\times F_D^{(1)}\left(1/2;1/2;3/2;\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right), \end{aligned} \tag{3.21}$$

$$P_{\theta_1} = \left(\Lambda_0^{\theta_1} - \Lambda_0^{\theta_2}\cos\psi_1^0\right) I_A + \left(\Lambda_0^{\theta_1} + \Lambda_0^{\theta_2}\cos\psi_1^0\right) I_B, \qquad (3.22)$$
$$P_{\theta_2} = \left(\Lambda_0^{\theta_2} - \Lambda_0^{\theta_1}\cos\psi_1^0\right) I_A + \left(\Lambda_0^{\theta_2} + \Lambda_0^{\theta_1}\cos\psi_1^0\right) I_B,$$

where

$$I_A = T\left[\frac{2^7 l^5 \Delta r_1}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)}\right]^{1/2} \left(1 + \frac{\Delta r_1}{2l}\right)\left(1 + \frac{\Delta r_1}{6l}\right)\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \qquad (3.23)$$
$$\times F_D^{(3)}\left(1/2; -1, -1, 1/2; 3/2; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right),$$

$$I_B = \frac{T}{9}\left[\frac{2^9 \left(l\Delta r_1\right)^3}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)}\right]^{1/2} \left(1 + \frac{\Delta r_1}{4l}\right)\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \qquad (3.24)$$
$$\times F_D^{(2)}\left(1/2; -1, 1/2; 5/2; \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right).$$

Our next task is to find the relation between the energy E and the other conserved quantities P_I, P_{θ_1}, P_{θ_2}, in the semiclassical limit (large conserved charges), which corresponds to $r_1 \to \infty$. In this limit,

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = \frac{\pi T \left(2^3 l\right)^{1/2}}{\left(\Lambda_+^2 + \Lambda_-^2\right)^{1/2}}, \qquad I_A = I_B = \frac{\pi T \left(2l\right)^{1/2} v_0^2}{\left(\Lambda_+^2 + \Lambda_-^2\right)^{3/2}},$$

which leads to the following energy-charge relation

$$E^2 = \mathbf{P}^2 + 2\pi T \left(6r_0\right)^{1/2} \left(P_{\theta_1}^2 + P_{\theta_2}^2\right)^{1/2}, \qquad \mathbf{P}^2 = \delta_{IJ} P_I P_J. \qquad (3.25)$$

The second type of rotating string solution can be written as

$$\xi^1(r) = \frac{8}{\left(\bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3\right)^{1/2}} \left[\frac{l\Delta r}{(3l - r_2)\Delta r_1}\right]^{1/2} \times \qquad (3.26)$$
$$F_D^{(5)}\left(1/2; -1/2, -1/2, 1/2, 1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{\Delta r_1}\right).$$

For E and P_I we have

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = 8T\left[\frac{2l\Delta r_1}{\left(\bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3\right)(3l - r_2)}\right]^{1/2} \left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \qquad (3.27)$$
$$\times F_D^{(1)}\left(1/2; 1/2; 3/2; \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right).$$

For the conserved angular momenta P_θ and P_ϕ one finds

$$\begin{aligned} P_\theta &= \left(\Lambda_0^\theta - \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right) J_A + \left(\Lambda_0^\theta + \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right) J_B, \\ P_\phi &= \left(\Lambda_0^\phi \sin\theta_2^0 - \Lambda_0^\theta \sin\psi_1^0\right) \sin\theta_2^0 J_A + \left(\Lambda_0^\phi \sin\theta_2^0 + \Lambda_0^\theta \sin\psi_1^0\right) \sin\theta_2^0 J_B \\ &\quad + \Lambda_0^\phi \cos^2\theta_2^0 J_D, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} J_A &= 8T \left[\frac{2l^5 \Delta r_1}{\left(\bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3\right)(3l - r_2)}\right]^{1/2} \left(1 + \frac{\Delta r_1}{2l}\right)\left(1 + \frac{\Delta r_1}{6l}\right) \\ &\times \left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} F_D^{(3)}\left(1/2; -1, -1, 1/2; 3/2; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right), \end{aligned} \tag{3.29}$$

$$\begin{aligned} J_B &= \frac{16}{9} T \left[\frac{2(l\Delta r_1)^3}{\left(\bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3\right)(3l - r_2)}\right]^{1/2} \left(1 + \frac{\Delta r_1}{4l}\right)\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \\ &\times F_D^{(2)}\left(1/2; -1, 1/2; 5/2; \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right), \end{aligned} \tag{3.30}$$

$$\begin{aligned} J_D &= 8T \left[\frac{2l^5 \Delta r_1}{\left(\bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3\right)(3l - r_2)}\right]^{1/2} \left(1 + \frac{\Delta r_1}{3l}\right)\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \\ &\times F_D^{(2)}\left(1/2; -2, 1/2; 3/2; \frac{1}{1 + \frac{3l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right). \end{aligned} \tag{3.31}$$

In the semiclassical limit $r_1 \to \infty$, one gets the following dependence of the energy on the charges P_I, P_θ and P_ϕ

$$E^2 = \mathbf{P}^2 + 2\pi T (6r_0)^{1/2} \left(P_\theta^2 + \frac{3P_\phi^2}{3 - \cos^2\theta_2^0}\right)^{1/2}. \tag{3.32}$$

The third solution and the conserved charges can be computed in the same way. These computations lead to the following energy-charge relation after taking the semiclassical limit

$$\begin{aligned} E^2 &= \mathbf{P}^2 + 2\pi T \left(\frac{6r_0}{\Delta}\right)^{1/2} \times \\ &\quad \left[\left(3 - \cos^2\theta_2^0\right) P_{\phi_1}^2 + \left(3 - \cos^2\theta_1^0\right) P_{\phi_2}^2 - 4P_{\phi_1} P_{\phi_2} \cos\theta_1^0 \cos\theta_2^0\right]^{1/2}, \end{aligned} \tag{3.33}$$

where

$$\Delta = 3 - \cos^2\theta_1^0 - \cos^2\theta_2^0 - \cos^2\theta_1^0 \cos^2\theta_2^0.$$

3.1.2 Strings in $AdS_5 \times S^5$, integrable systems and finite-size effects for single spikes

We use the reduction of the string dynamics on $R_t \times S^3$ subspace of $AdS_5 \times S^5$ to the Neumann-Rosochatius (NR) integrable system to map *all* string solutions described by this dynamical system onto solutions of the complex sine-Gordon (CSG) integrable model. This mapping relates the parameters in the solutions on both sides of the correspondence. Then, we find finite-size string solutions, their images in the (complex) sine-Gordon (SG) system, and the leading finite-size effects of the single spike "$E - \Delta\varphi$" relation for both $R_t \times S^2$ and $R_t \times S^3$ cases [14].

Strings on $R_t \times S^3$ and the NR integrable system

We choose to work in *conformal gauge* in which the string Lagrangian and the Virasoro constraints take the form (3.1) (in $AdS_5 \times S^5$ the 2-form B-field is zero)

$$\mathcal{L}_s = \frac{T}{2}\left(G_{00} - G_{11}\right) \tag{3.34}$$

$$G_{00} + G_{11} = 0, \qquad G_{01} = 0. \tag{3.35}$$

We embed the string in $R_t \times S^3$ subspace of $AdS_5 \times S^5$ as follows

$$Z_0 = Re^{it(\tau,\sigma)}, \quad W_j = Rr_j(\tau,\sigma)e^{i\varphi_j(\tau,\sigma)}, \quad \sum_{j=1}^{2} W_j\bar{W}_j = R^2,$$

where R is the common radius of AdS_5 and S^5, and t is the AdS time. For this embedding, the metric induced on the string worldsheet is given by

$$G_{ab} = -\partial_{(a}Z_0\partial_{b)}\bar{Z}_0 + \sum_{j=1}^{2}\partial_{(a}W_j\partial_{b)}\bar{W}_j = R^2\left[-\partial_a t\partial_b t + \sum_{j=1}^{2}\left(\partial_a r_j\partial_b r_j + r_j^2\partial_a\varphi_j\partial_b\varphi_j\right)\right].$$

The corresponding string Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_s + \Lambda_s\left(\sum_{j=1}^{2} r_j^2 - 1\right),$$

where Λ_s is a Lagrange multiplier. In the case at hand, the background metric does not depend on t and φ_j. Therefore, the conserved quantities are the string energy E_s and two angular momenta J_j, given by

$$E_s = -\int d\sigma \frac{\partial\mathcal{L}_s}{\partial(\partial_0 t)}, \quad J_j = \int d\sigma \frac{\partial\mathcal{L}_s}{\partial(\partial_0\varphi_j)}. \tag{3.36}$$

It is known that restricting ourselves to the case

$$t(\tau,\sigma) = \kappa\tau, \quad r_j(\tau,\sigma) = r_j(\xi), \quad \varphi_j(\tau,\sigma) = \omega_j\tau + f_j(\xi), \tag{3.37}$$

$$\xi = \alpha\sigma + \beta\tau, \quad \kappa, \omega_j, \alpha, \beta = constants,$$

reduces the problem to solving the NR integrable system [56]. For the case under consideration, the NR Lagrangian reads (prime is used for $d/d\xi$)

$$L_{NR} = (\alpha^2 - \beta^2) \sum_{j=1}^{2} \left[r_j'^2 - \frac{1}{(\alpha^2-\beta^2)^2} \left(\frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] + \Lambda_s \left(\sum_{j=1}^{2} r_j^2 - 1 \right), \quad (3.38)$$

where the parameters C_j are integration constants after single time integration of the equations of motion for $f_j(\xi)$:

$$f_j' = \frac{1}{\alpha^2 - \beta^2} \left(\frac{C_j}{r_j^2} + \beta \omega_j \right). \quad (3.39)$$

The constraints (3.35) give the conserved Hamiltonian H_{NR} and a relation between the embedding parameters and the arbitrary constants C_j:

$$H_{NR} = (\alpha^2 - \beta^2) \sum_{j=1}^{2} \left[r_j'^2 + \frac{1}{(\alpha^2-\beta^2)^2} \left(\frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] = \frac{\alpha^2+\beta^2}{\alpha^2-\beta^2} \kappa^2, \quad (3.40)$$

$$\sum_{j=1}^{2} C_j \omega_j + \beta \kappa^2 = 0. \quad (3.41)$$

For closed strings, r_j and f_j satisfy the following periodicity conditions

$$r_j(\xi + 2\pi\alpha) = r_j(\xi), \qquad f_j(\xi + 2\pi\alpha) = f_j(\xi) + 2\pi n_\alpha, \quad (3.42)$$

where n_α are integer winding numbers. On the ansatz (3.37), E_s and J_j introduced in (3.36) take the form

$$E_s = \frac{\sqrt{\lambda}}{2\pi} \frac{\kappa}{\alpha} \int d\xi, \qquad J_j = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{\alpha^2 - \beta^2} \int d\xi \left(\frac{\beta}{\alpha} C_j + \alpha \omega_j r_j^2 \right), \quad (3.43)$$

where we have used that the string tension and the 't Hooft coupling constant λ are related by $TR^2 = \frac{\sqrt{\lambda}}{2\pi}$.

In order to identically satisfy the embedding condition

$$\sum_{j=1}^{2} r_j^2 - 1 = 0,$$

we introduce a new variable $\theta(\xi)$ by

$$r_1(\xi) = \sin\theta(\xi), \qquad r_2(\xi) = \cos\theta(\xi). \quad (3.44)$$

Then, Eq.(3.40) leads to

$$\begin{aligned} \theta'(\xi) &= \pm \frac{1}{\alpha^2-\beta^2} \left[(\alpha^2+\beta^2)\kappa^2 - \frac{C_1^2}{\sin^2\theta} - \frac{C_2^2}{\cos^2\theta} - \alpha^2 \left(\omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta \right) \right]^{1/2} \quad (3.45) \\ &\equiv \pm \frac{1}{\alpha^2-\beta^2} \, \Theta(\theta), \end{aligned}$$

which can be integrated to give

$$\xi(\theta) = \pm(\alpha^2 - \beta^2) \int \frac{d\theta}{\Theta(\theta)}. \tag{3.46}$$

From Eqs.(3.39) and (3.44), we can obtain

$$f_1 = \frac{\beta\omega_1\xi}{\alpha^2 - \beta^2} \pm C_1 \int \frac{d\theta}{\sin^2\theta\ \Theta(\theta)}, \tag{3.47}$$

$$f_2 = \frac{\beta\omega_2\xi}{\alpha^2 - \beta^2} \pm C_2 \int \frac{d\theta}{\cos^2\theta\ \Theta(\theta)}. \tag{3.48}$$

Let us also point out that the solutions for $\xi(\theta)$ and f_j must satisfy the conditions (3.41) and (3.42). All these solve formally the NR system for the present case.

Relationship between the NR and CSG integrable systems

Due to Pohlmeyer [57], we know that the string dynamics on $R_t \times S^3$ can be described by the CSG equation. Here, we derive the relation between the solutions of the two integrable systems - NR and CSG.

The CSG system is defined by the Lagrangian

$$\mathcal{L}(\psi) = \frac{\eta^{ab}\partial_a\bar{\psi}\partial_b\psi}{1 - \bar{\psi}\psi} + M^2\bar{\psi}\psi$$

which give the equation of motion

$$\partial_a\partial^a\psi + \bar{\psi}\frac{\partial_a\psi\partial^a\psi}{1 - \bar{\psi}\psi} - M^2(1 - \bar{\psi}\psi)\psi = 0.$$

If we represent ψ in the form

$$\psi = \sin(\phi/2)\exp(i\chi/2),$$

the Lagrangian can be expressed as

$$\mathcal{L}(\phi, \chi) = \frac{1}{4}\left[\partial_a\phi\partial^a\phi + \tan^2(\phi/2)\partial_a\chi\partial^a\chi + (2M)^2\sin^2(\phi/2)\right],$$

along with the equations of motion

$$\partial_a\partial^a\phi - \frac{1}{2}\frac{\sin(\phi/2)}{\cos^3(\phi/2)}\partial_a\chi\partial^a\chi - M^2\sin\phi = 0, \tag{3.49}$$

$$\partial_a\partial^a\chi + \frac{2}{\sin\phi}\partial_a\phi\partial^a\chi = 0. \tag{3.50}$$

The SG system corresponds to a particular case of $\chi = 0$.

To relate the NR system with the CSG integrable system, we consider the case

$$\phi = \phi(\xi), \qquad \chi = A\sigma + B\tau + \tilde{\chi}(\xi),$$

where ϕ and $\tilde{\chi}$ depend on only one variable $\xi = \alpha\sigma + \beta\tau$ in the same way as in our NR ansatz (3.37). Then the equations of motion (3.49), (3.50) reduce to

$$\phi'' - \frac{1}{2}\frac{\sin(\phi/2)}{\cos^3(\phi/2)}\left[\tilde{\chi}'^2 + 2\frac{A\alpha - B\beta}{\alpha^2-\beta^2}\tilde{\chi}' + \frac{A^2-B^2}{\alpha^2-\beta^2}\right] - \frac{M^2\sin\phi}{\alpha^2-\beta^2} = 0, \tag{3.51}$$

$$\tilde{\chi}'' + \frac{2\phi'}{\sin\phi}\left(\tilde{\chi}' + \frac{A\alpha - B\beta}{\alpha^2-\beta^2}\right) = 0. \tag{3.52}$$

We further restrict ourselves to the case of $A\alpha = B\beta$. A trivial solution of Eq.(3.52) is $\tilde{\chi} = constant$, which corresponds to the solutions of the CSG equations considered in [58, 59] for a GM string on $R_t \times S^3$. More nontrivial solution of (3.52) is

$$\tilde{\chi} = C_\chi \int \frac{d\xi}{\tan^2(\phi/2)}. \tag{3.53}$$

The replacement of the above into (3.51) gives

$$\phi'' = \frac{M^2\sin\phi}{\alpha^2-\beta^2} + \frac{1}{2}\left[C_\chi^2\frac{\cos(\phi/2)}{\sin^3(\phi/2)} - \frac{A^2}{\beta^2}\frac{\sin(\phi/2)}{\cos^3(\phi/2)}\right]. \tag{3.54}$$

Integrating once, we obtain

$$\begin{aligned} \phi' &= \pm\left[\left(C_\phi - \frac{2M^2}{\alpha^2-\beta^2}\right) + \frac{4M^2}{\alpha^2-\beta^2}\sin^2(\phi/2) - \frac{A^2/\beta^2}{1-\sin^2(\phi/2)} - \frac{C_\chi^2}{\sin^2(\phi/2)}\right]^{1/2} \\ &\equiv \pm\Phi(\phi), \end{aligned} \tag{3.55}$$

from which we get

$$\xi(\phi) = \pm\int\frac{d\phi}{\Phi(\phi)}, \qquad \chi(\phi) = \frac{A}{\beta}(\beta\sigma + \alpha\tau) \pm C_\chi\int\frac{d\phi}{\tan^2(\phi/2)\Phi(\phi)}.$$

All these solve the CSG system for the considered particular case. It is clear from (3.55) that the expression inside the square root must be positive.

Now we are ready to establish a correspondence between the NR and CSG integrable systems described above. To this end, we make the following identification

$$\sin^2(\phi/2) \equiv \frac{\sqrt{-G}}{K^2} \tag{3.56}$$

where G is the determinant of the induced metric G_{ab} computed on the constraints (3.35) and K^2 is a parameter which will be fixed later[6]. For our NR system, $\sqrt{-G}$ is given by

$$\sqrt{-G} = \frac{R^2\alpha^2}{\alpha^2-\beta^2}\left[(\kappa^2-\omega_1^2) + (\omega_1^2-\omega_2^2)\cos^2\theta\right]. \tag{3.57}$$

[6]For $K^2 = \kappa^2$, this definition of the angle ϕ coincides with the one used in [58], which is based on the Pohlmeyer's reduction procedure [57].

We want the field ϕ, defined in (3.56) through NR quantities, to *identically* satisfy (3.55) derived from the CSG equations. This imposes relations between the parameters involved, which are given in appendix A. In this way, we mapped *all* string solutions on $R_t \times S^3$ (in particular on $R_t \times S^2$) described by the NR integrable system onto solutions of the CSG (in particular SG) equations. From (A.1) one can see that the parameters A and C_χ are nonzero in general on $R_t \times S^2$ where $\omega_2 = C_2 = 0$. This means that there exist string solutions on $R_t \times S^2$ which correspond to solutions of the CSG system. Only when $M^2 = \kappa^2$, all string solutions on $R_t \times S^2$ are represented by solutions of the SG equation.

For the GM and SS solutions, which we are interested in, the relations between the NR and CSG parameters simplify a lot. Let us write them explicitly. The GM solutions correspond to $C_2 = 0$, $\kappa^2 = \omega_1^2$. This leads to

$$\begin{aligned} C_\phi &= \frac{2}{\alpha^2 - \beta^2}\left[3M^2 - 2\left(\omega_1^2 - \frac{\omega_2^2}{1-\beta^2/\alpha^2}\right)\right], \qquad K^2 = R^2 M^2, \\ A^2 &= \frac{4}{\alpha^2/\beta^2 - 1}\left(M^2 - \omega_1^2 + \frac{\omega_2^2}{1-\beta^2/\alpha^2}\right), \qquad C_\chi = 0. \end{aligned} \tag{3.58}$$

Therefore, for all GM strings the field χ is linear function at most. Since $A = C_\chi = 0$ implies $\chi = 0$, it follows from here that there exist GM string solutions on $R_t \times S^3$, which are mapped not on CSG solutions but on SG solutions instead. This happens exactly when

$$M^2 = \omega_1^2 - \frac{\omega_2^2}{1-\beta^2/\alpha^2}.$$

In that case the nonzero parameters are

$$K^2 = R^2\left(\omega_1^2 - \frac{\omega_2^2}{1-\beta^2/\alpha^2}\right), \qquad C_\phi = \frac{2}{\alpha^2-\beta^2}\left(\omega_1^2 - \frac{\omega_2^2}{1-\beta^2/\alpha^2}\right),$$

and the corresponding solution of the SG equation can be found from (3.55) to be

$$\sin(\phi/2) = \frac{1}{\cosh\left[\sqrt{\frac{\omega_1^2 - \omega_2^2/(1-\beta^2/\alpha^2)}{1-\beta^2/\alpha^2}}\left(\sigma + \frac{\beta}{\alpha}\tau\right) - \eta_0\right]}, \qquad \eta_0 = const. \tag{3.59}$$

Replacing (3.59) in (3.56), (3.57), one obtains the GM solution (A.3) as it should be.

For the SS solutions $C_2 = 0$, $\kappa^2 = \omega_1^2\alpha^2/\beta^2$. This results in

$$\begin{aligned} C_\phi &= \frac{2}{\beta^2-\alpha^2}\left[2\left(2\omega_1^2\alpha^2/\beta^2 + \frac{\omega_2^2}{\beta^2/\alpha^2 - 1}\right) - 3M^2\right], \\ A^2 &= \frac{4}{M^4(1-\alpha^2/\beta^2)}\left(\omega_1^2\alpha^2/\beta^2 - M^2\right)^2\left(\frac{\omega_2^2}{\beta^2/\alpha^2-1} - M^2\right), \\ C_\chi &= \frac{2\omega_1^2\omega_2\alpha^3}{M^2(\beta^2-\alpha^2)\beta^2}, \qquad K^2 = R^2M^2. \end{aligned} \tag{3.60}$$

We want to point out that C_χ is always nonzero on S^3 contrary to the GM case, which makes χ also non-vanishing. To our knowledge, the CSG solutions corresponding to the SS on $R_t \times S^3$ are not given in the literature. To study this problem, we will consider the case when $A = 0$. A can be zero when

$$M^2 = \kappa^2 = \omega_1^2 \alpha^2/\beta^2 \quad \text{or} \quad M^2 = \frac{\omega_2^2}{\beta^2/\alpha^2 - 1}, \tag{3.61}$$

As is seen from (3.61), we have two options, and we restrict ourselves to the first one[7]. Replacing $M^2 = \omega_1^2\alpha^2/\beta^2$ in (3.60) and using the resulting expressions for C_ϕ and C_χ in (3.55), one obtains the simplified equation

$$\phi'^2 = \frac{4}{\beta^2 - \alpha^2}\left[\omega_1^2 \frac{\alpha^2}{\beta^2}\cos^2(\phi/2) - \frac{\omega_2^2}{\beta^2/\alpha^2 - 1}\cot^2(\phi/2)\right]$$

with solution

$$\sin^2(\phi/2) = \tanh^2(C\xi) + \frac{\omega_2^2}{\omega_1^2(1 - \alpha^2/\beta^2)\cosh^2(C\xi)}, \tag{3.62}$$

where

$$C = \frac{\alpha\omega_1\sqrt{1 - \alpha^2/\beta^2 - \omega_2^2/\omega_1^2}}{\beta^2(1 - \alpha^2/\beta^2)}.$$

This agrees with Eqs. (3.56) and (3.57). By inserting (3.62) into (3.53) one can find

$$\chi = \tilde{\chi} = 2\arctan\left[\frac{\omega_1}{\omega_2}\sqrt{1 - \alpha^2/\beta^2 - \omega_2^2/\omega_1^2}\tanh(C\xi)\right].$$

Hence, the CSG field ψ for the case at hand is given by

$$\begin{aligned}\psi \;=\; & \sqrt{\tanh^2(C\xi) + \frac{\omega_2^2}{\omega_1^2(1 - \alpha^2/\beta^2)\cosh^2(C\xi)}} \\ & \times \exp\left\{i \arctan\left[\frac{\omega_1}{\omega_2}\sqrt{1 - \alpha^2/\beta^2 - \omega_2^2/\omega_1^2}\tanh(C\xi)\right]\right\}.\end{aligned} \tag{3.63}$$

Here we have set the integration constants ϕ_0, χ_0 equal to zero. Several examples, which illustrate the established NR - CSG correspondence, are considered in an Appendix.

Finite-size effects

Here, we will obtain finite-size string solutions, their images in the (complex) sine-Gordon system, and the leading corrections to the SS "$E - \Delta\varphi$" relation: first for the $R_t \times S^2$ case, then for the SS string with two angular momenta.

[7] It turns out that the second option does not allow real solutions.

3.1.3 Finite-size effect of the dyonic giant magnons in $\mathcal{N}=6$ super Chern-Simons-matter theory

The AdS/CFT correspondence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ SYM theory led to many exciting developments and to understanding non-perturbative sturctures of the string and gauge theories. Another exciting possibility is that the same type of duality does exist. The promising candidate for the three-dimensional conformal field theory is $\mathcal{N}=6$ super Chern-Simons (CS) theory with $SU(N) \times SU(N)$ gauge symmetry and level k [66]. In the planar limit of $N, k \to \infty$ with a fixed value of 't Hooft coupling $\lambda = N/k$, the $\mathcal{N}=6$ CS is believed to be dual to type IIA superstring theory on $AdS_4 \times \mathbb{CP}^3$.

In [17] we consider finite-size effects for the dyonic giant magnon of the type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ by applying Lüscher μ-term formula which is derived from a proposed S-matrix for the $\mathcal{N}=6$ super Chern-Simons theory. We compute explicitly the effect for the case of a symmetric configuration where the two external bound states, each of A and B particles, have the same momentum p and spin J_2. We compare this with the classical string theory result which we computed by reducing it to the Neumann-Rosochatius integrable system. The two results match perfectly.

Classical string analysis

Let us consider a classical string moving in $R_t \times \mathbb{CP}^3$. Using the complex coordinates

$$z = y^0 + iy^4, \quad w_1 = x^1 + ix^2, \quad w_2 = x^3 + ix^4, \quad w_3 = x^5 + ix^6, \quad w_4 = x^7 + ix^8,$$

we embed the string as follows [67]

$$z = Z(\tau,\sigma) = \frac{R}{2} e^{it(\tau,\sigma)}, \quad w_a = W_a(\tau,\sigma) = R r_a(\tau,\sigma) e^{i\varphi_a(\tau,\sigma)}.$$

Here t is the AdS time. These complex coordinates should satisfy

$$\sum_{a=1}^{4} W_a \bar{W}_a = R^2, \quad \sum_{a=1}^{4} \left(W_a \partial_m \bar{W}_a - \bar{W}_a \partial_m W_a \right) = 0,$$

or

$$\sum_{a=1}^{4} r_a^2 = 1, \quad \sum_{a=1}^{4} r_a^2 \partial_m \varphi_a = 0, \quad m = 0, 1. \tag{3.64}$$

NR reduction

In order to reduce the string dynamics on $R_t \times \mathbb{CP}^3$ to the NR integrable system, we use the ansatz

$$\begin{aligned} &t(\tau,\sigma) = \kappa\tau, \quad r_a(\tau,\sigma) = r_a(\xi), \quad \varphi_a(\tau,\sigma) = \omega_a \tau + f_a(\xi), \\ &\xi = \alpha\sigma + \beta\tau, \quad \kappa, \omega_a, \alpha, \beta = \text{constants}. \end{aligned} \tag{3.65}$$

It can be shown [67] that after integration of the equations of motion for f_a, which gives

$$f_a' = \frac{1}{\alpha^2 - \beta^2}\left(\frac{C_a}{r_a^2} + \beta\omega_a\right), \qquad C_a = constants, \tag{3.66}$$

one ends up with the following effective Lagrangian for the coordinates r_a

$$\begin{aligned} L_{NR} &= (\alpha^2-\beta^2)\sum_{a=1}^{4}\left[r_a'^2 - \frac{1}{(\alpha^2-\beta^2)^2}\left(\frac{C_a^2}{r_a^2} + \alpha^2\omega_a^2 r_a^2\right)\right] \\ &- \Lambda\left(\sum_{a=1}^{4} r_a^2 - 1\right). \end{aligned} \tag{3.67}$$

This is the Lagrangian for the NR integrable system [56]. In addition, the $\mathbb{CP}^3$ embedding conditions in (3.64) lead to

$$\sum_{a=1}^{4}\omega_a r_a^2 = 0, \qquad \sum_{a=1}^{4} C_a = 0. \tag{3.68}$$

The Virasoro constraints give the conserved Hamiltonian H_{NR} and a relation between the embedding parameters and the arbitrary constants C_a:

$$H_{NR} = (\alpha^2-\beta^2)\sum_{a=1}^{4}\left[r_a'^2 + \frac{1}{(\alpha^2-\beta^2)^2}\left(\frac{C_a^2}{r_a^2} + \alpha^2\omega_a^2 r_a^2\right)\right] = \frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\frac{\kappa^2}{4}, \tag{3.69}$$

$$\sum_{a=1}^{4} C_a\omega_a + \beta(\kappa/2)^2 = 0. \tag{3.70}$$

The conserved charges can be defined by

$$E_s = -\int d\sigma \frac{\partial\mathcal{L}}{\partial(\partial_0 t)}, \qquad J_a = \int d\sigma \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_a)}, \qquad a = 1,2,3,4,$$

where $\mathcal{L}$ is the Polyakov string Lagrangian taken in conformal gauge. Using the ansatz (3.65) and (3.66), we can find

$$E_s = \frac{\kappa\sqrt{2\lambda}}{2\alpha}\int d\xi, \qquad J_a = \frac{2\sqrt{2\lambda}}{\alpha^2-\beta^2}\int d\xi\left(\frac{\beta}{\alpha}C_a + \alpha\omega_a r_a^2\right). \tag{3.71}$$

In view of (3.68), one obtains

$$\sum_{a=1}^{4} J_a = 0. \tag{3.72}$$

Dyonic giant magnon solution

We are interested in finding string configurations corresponding to the following particular solution of (3.68)

$$r_1 = r_3 = \frac{1}{\sqrt{2}}\sin\theta, \quad r_2 = r_4 = \frac{1}{\sqrt{2}}\cos\theta, \quad \omega_1 = -\omega_3, \quad \omega_2 = -\omega_4.$$

The two frequencies ω_1, ω_2 are independent and lead to strings moving in $\mathbb{CP}^3$ with two angular momenta. From the NR Hamiltonian (3.40) one finds

$$\begin{aligned} \theta'^2(\xi) &= \frac{1}{(\alpha^2-\beta^2)^2}\left[\frac{\kappa^2}{4}(\alpha^2+\beta^2) - 2\left(\frac{C_1^2+C_3^2}{\sin^2\theta} + \frac{C_2^2+C_4^2}{\cos^2\theta}\right)\right. \\ &- \left.\alpha^2\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right)\right]. \end{aligned}$$

We further restrict ourselves to $C_2 = C_4 = 0$ to search for GM string configurations. Eqs. (3.68) and (3.70) give

$$C_1 = -C_3 = -\frac{\beta\kappa^2}{8\omega_1}.$$

In this case, the above equation for θ' can be rewritten in the form

$$(\cos\theta)' = \mp\frac{\alpha\sqrt{\omega_1^2-\omega_2^2}}{\alpha^2-\beta^2}\sqrt{(z_+^2-\cos^2\theta)(\cos^2\theta - z_-^2)}, \tag{3.73}$$

where

$$\begin{aligned} z_\pm^2 &= \frac{1}{2(1-\frac{\omega_2^2}{\omega_1^2})}\left\{y_1+y_2-\frac{\omega_2^2}{\omega_1^2} \pm \sqrt{(y_1-y_2)^2 - \left[2\left(y_1+y_2-2y_1y_2\right) - \frac{\omega_2^2}{\omega_1^2}\right]\frac{\omega_2^2}{\omega_1^2}}\right\}, \\ y_1 &= 1-\frac{\kappa^2}{4\omega_1^2}, \quad y_2 = 1-\frac{\beta^2}{\alpha^2}\frac{\kappa^2}{4\omega_1^2}. \end{aligned}$$

The solution of (3.73) is given by

$$\cos\theta = z_+\mathbf{dn}\left(C\xi|m\right), \quad C = \mp\frac{\alpha\sqrt{\omega_1^2-\omega_2^2}}{\alpha^2-\beta^2}z_+, \quad m \equiv 1 - z_-^2/z_+^2. \tag{3.74}$$

To find the full string solution, we also need to obtain the explicit expressions for the functions f_a from (3.66)

$$f_a = \frac{1}{\alpha^2-\beta^2}\int d\xi\left(\frac{C_a}{r_a^2} + \beta\omega_a\right).$$

Using the solution (3.74) for $\theta(\xi)$, we can find

$$\begin{aligned} f_1 = -f_3 &= \frac{\beta/\alpha}{z_+\sqrt{1-\omega_2^2/\omega_1^2}}\left[C\xi - \frac{2(\kappa/2)^2/\omega_1^2}{1-z_+^2}\Pi\left(am(C\xi), -\frac{z_+^2-z_-^2}{1-z_+^2}|m\right)\right], \\ f_2 = -f_4 &= \frac{\beta\omega_2}{\alpha^2-\beta^2}\xi. \end{aligned}$$

As a consequence, the string solution can be written as

$$
\begin{aligned}
W_1 &= \frac{R}{\sqrt{2}}\sqrt{1 - z_+^2 \mathbf{dn}^2\left(C\xi|m\right)}\; e^{i(\omega_1\tau + f_1)},\\
W_2 &= \frac{R}{\sqrt{2}} z_+ \mathbf{dn}\left(C\xi|m\right)\; e^{i(\omega_2\tau + f_2)},\\
W_3 &= \frac{R}{\sqrt{2}}\sqrt{1 - z_+^2 \mathbf{dn}^2\left(C\xi|m\right)}\; e^{-i(\omega_1\tau + f_1)},\\
W_4 &= \frac{R}{\sqrt{2}} z_+ \mathbf{dn}\left(C\xi|m\right)\; e^{-i(\omega_2\tau + f_2)}.
\end{aligned}
\tag{3.75}
$$

The GM in infinite volume can be obtained by taking $z_- \to 0$. In this limit, the solution for θ reduces to

$$
\cos\theta = \frac{\sin\frac{p}{2}}{\cosh(C\xi)},
$$

where the constant $z_+ \equiv \sin p/2$ is given by

$$
z_+^2 = \frac{y_2 - \omega_2^2/\omega_1^2}{1 - \omega_2^2/\omega_1^2}.
$$

One spin solution corresponds $\omega_2 = 0$. Inserting this into (3.71), one can find the energy-charge dispersion relation. For the *single* DGM, the energy and angular momentum J_1 become infinite but their difference remains finite:

$$
E_s - J_1 = \sqrt{\frac{J_2^2}{4} + 2\lambda\sin^2\frac{p}{2}}. \tag{3.76}
$$

Finite-size effects

Using the most general solutions (3.75), we can calculate the finite-size corrections to the energy-charge relation (3.76) in the limit when the string energy $E_s \to \infty$. Here we consider the case of $\alpha^2 > \beta^2$ only since it corresponds to the GM case. We obtain from (3.71) the following expressions for the conserved string energy E_s and the angular momenta J_a

$$
\begin{aligned}
\mathcal{E} &= \frac{2\kappa(1-\beta^2/\alpha^2)}{\omega_1 z_+\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{K}\left(1 - z_-^2/z_+^2\right),\\
\mathcal{J}_1 &= \frac{2z_+}{\sqrt{1-\omega_2^2/\omega_1^2}}\left[\frac{1-\beta^2(\kappa/2)^2/\alpha^2\omega_1^2}{z_+^2}\mathbf{K}\left(1 - z_-^2/z_+^2\right) - \mathbf{E}\left(1 - z_-^2/z_+^2\right)\right],\\
\mathcal{J}_2 &= \frac{2z_+\omega_2/\omega_1}{\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{E}\left(1 - z_-^2/z_+^2\right), \qquad \mathcal{J}_3 = -\mathcal{J}_1, \qquad \mathcal{J}_4 = -\mathcal{J}_2.
\end{aligned}
\tag{3.77}
$$

As a result, the condition (3.72) is identically satisfied. Here, we introduced the notations

$$
\mathcal{E} = \frac{E_s}{\sqrt{2\lambda}}, \qquad \mathcal{J}_a = \frac{J_a}{\sqrt{2\lambda}}. \tag{3.78}
$$

The computation of $\Delta\varphi_1$ gives

$$
\begin{aligned}
p \equiv \Delta\varphi_1 &= 2\int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\theta'} f_1' = \qquad (3.79)\\
&- \frac{2\beta/\alpha}{z_+\sqrt{1-\omega_2^2/\omega_1^2}}\left[\frac{(\kappa/2)^2/\omega_1^2}{1-z_+^2}\mathbf{\Pi}\left(-\frac{z_+^2-z_-^2}{1-z_+^2}\middle|1-z_-^2/z_+^2\right) - \mathbf{K}\left(1-z_-^2/z_+^2\right)\right].
\end{aligned}
$$

Expanding the elliptic integrals, we obtain

$$
\begin{aligned}
E - J_1 &= 2\sqrt{\frac{J_2^2}{4} + 2\lambda\sin^2\frac{p}{2}} \qquad (3.80)\\
&- \frac{32\lambda\sin^4\frac{p}{2}}{\sqrt{J_2^2+8\lambda\sin^2\frac{p}{2}}}\exp\left[-\frac{2\sin^2\frac{p}{2}\left(J_1+\sqrt{J_2^2+8\lambda\sin^2\frac{p}{2}}\right)\sqrt{J_2^2+8\lambda\sin^2\frac{p}{2}}}{J_2^2+8\lambda\sin^4\frac{p}{2}}\right].
\end{aligned}
$$

This also gives the finite-size effect for ordinary GM by taking $J_2 \to 0$

$$
E - J_1 = 2\sqrt{2\lambda}\sin\frac{p}{2} - 16\sqrt{\frac{\lambda}{2}}\sin^3\frac{p}{2}\exp\left[-\frac{J_1}{\sqrt{2\lambda}\sin\frac{p}{2}} - 2\right]. \qquad (3.81)
$$

Finite-size effects from the S-matrix

The $\mathcal{N}=6$ CS theory has two sets of excitations, namely A-particles and B-particles, each of which form a four-dimensional representation of $SU(2|2)$ [68, 69]. We propose an S-matrix with the following structure:

$$
\begin{aligned}
S^{AA}(p_1,p_2) &= S^{BB}(p_1,p_2) = S_0(p_1,p_2)\widehat{S}(p_1,p_2)\\
S^{AB}(p_1,p_2) &= S^{BA}(p_1,p_2) = \tilde{S}_0(p_1,p_2)\widehat{S}(p_1,p_2),
\end{aligned}
$$

where $\widehat{S}$ is the matrix part determined by the $SU(2|2)$ symmetry, and is essentially the same as that found for $\mathcal{N}=4$ SYM in [70, 71]. An important difference arises in the dressing phases $S_0, \tilde{S}_0$ due to the fact that the A- and B-particles are related by complex conjugation.

Lüscher μ-term formula

Here we want to generalize multi-particle Lüscher formula [72, 73] to the case of the bound states. Consider M_A number of A-type DGMs, $|Q_1,\dots Q_{M_A}\rangle$, and M_B number of B-type DGMs, $|\tilde{Q}_1,\dots\tilde{Q}_{M_B}\rangle$. We use α_k for the $SU(2|2)$ quantum numbers carried by the DGMs and C_k for A or B, the two types of particles. Then we propose the multi-particle Lüscher formula for generic DGM states as follows:

$$\delta E_\mu = -i\sum_{b=1}^{4}\left\{\sum_{l=1}^{M_A}(-1)^{F_b}\left(1-\frac{\epsilon'_{Q_l}(p_l)}{\epsilon'_1(\tilde{q}^*)}\right)e^{-i\tilde{q}^*L}\left[\operatorname*{Res}_{q^*=\tilde{q}^*} S^{AA}{}_{b\alpha_l}^{b\alpha_l}(q^*,p_l)\right]\right. \tag{3.82}$$
$$\times\prod_{k\neq l}^{M_A+M_B} S^{AC_k}{}_{b\alpha_k}^{b\alpha_k}(q^*,p_k)$$
$$\left.+\sum_{l=1}^{M_B}(-1)^{F_b}\left(1-\frac{\epsilon'_{\tilde{Q}_l}(p_l)}{\epsilon'_1(\tilde{q}^*)}\right)e^{-i\tilde{q}^*L}\left[\operatorname*{Res}_{q^*=\tilde{q}^*} S^{BB}{}_{b\alpha_l}^{b\alpha_l}(q^*,p_l)\right]\prod_{k\neq l}^{M_A+M_B} S^{BC_k}{}_{b\alpha_k}^{b\alpha_k}(q^*,p_k)\right\}.$$

Here, the energy dispersion relation for the DGM is given by

$$\epsilon_Q(p) = \sqrt{\frac{Q^2}{4}+4g^2\sin^2\frac{p}{2}}. \tag{3.83}$$

The coupling constant $g = h(\lambda)$ is still unknown function of λ which behaves as $h(\lambda) \sim \lambda$ for small λ, and $h(\lambda) \sim \sqrt{\lambda/2}$ for large λ.

S-matrix elements for the dyonic GM

The S-matrix elements for the DGM are in general complicated. However, we can consider a simplest case of the DGMs composed of only A-type ϕ_1's which are the first bosonic particle in the fundamental representation of $SU(2|2)$. It is obvious that these bound states do exist since the elementary S-matrix element $S^{AA}{}_{11}^{11}$ does have a pole. The same holds for the B-type DGMs. However, the hybrid type DGMs are not possible because the S^{AB} S-matrix does not have any bound-state pole.

The Lüscher correction needs only those S-matrix elements which have the same incoming and outgoing $SU(2|2)$ quantum numbers after scattering with a virtual particle. In particular, we can easily compute the matrix elements between an elementary magnon and a the bound-state made of only ϕ_1's (Q of them) denoted by $\mathbf{1}_Q$ [74]

$$S^{AA}{}_{b\mathbf{1}_Q}^{b\mathbf{1}_Q}(y,X^{(Q)}) = \prod_{k=1}^{Q} S^{AA}{}_{b1}^{b1}(y,x_k) = \prod_{k=1}^{Q}\left[\frac{1-\frac{1}{y^+x_k^-}}{1-\frac{1}{y^-x_k^+}}\sigma_{\rm BES}(y,x_k)\tilde{a}_b(y,x_k)\right], \tag{3.84}$$

where $\tilde{a}_b$ are given by [70, 71]

$$\begin{aligned}
\tilde{a}_1(y,x) &= a_1(y,x), \quad \tilde{a}_2(y,x) = a_1(y,x)+a_2(y,x), \\
&\quad \tilde{a}_3(y,x) = \tilde{a}_4(y,x) = a_6(y,x) \\
a_1(y,x) &= \frac{x^- - y^+}{x^+ - y^-}\frac{\eta(x)\eta(y)}{\tilde{\eta}(x)\tilde{\eta}(y)} \\
a_2(y,x) &= \frac{(y^- - y^+)(x^- - x^+)(x^- - y^+)}{(y^- - x^+)(x^-y^- - x^+y^+)}\frac{\eta(x)\eta(y)}{\tilde{\eta}(x)\tilde{\eta}(y)} \\
a_6(y,x) &= \frac{y^+ - x^+}{y^- - x^+}\frac{\eta(y)}{\tilde{\eta}(y)}.
\end{aligned} \tag{3.85}$$

As noticed in [74], a_2/a_1 and a_6/a_1 are negligible $\mathcal{O}(1/g)$ corrections in the classical limit $g >> 1$. Therefore, the S-matrix with $b=1$ is a most important factor for our computation which can be written as

$$S^{AA\mathbf{1}\mathbf{1}_Q}_{\mathbf{1}\mathbf{1}_Q}(y, X^{(Q)}) = \sigma_{\text{BES}}(y, X^{(Q)}) \prod_{k=1}^{Q} \left[\frac{1-\frac{1}{y^+x_k^-}}{1-\frac{1}{y^-x_k^+}} \cdot \frac{x_k^- - y^+}{x_k^+ - y^-} \frac{\eta(x_k)\eta(y)}{\tilde{\eta}(x_k)\tilde{\eta}(y)} \right] \tag{3.86}$$

$$= \sigma_{\text{BES}}(y, X^{(Q)}) S_{\text{BDS}}(y, X^{(Q)}) \frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})} \left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^Q,$$

$$S^{AB\mathbf{1}\mathbf{1}_Q}_{\mathbf{1}\mathbf{1}_Q}(y, X^{(Q)}) = \sigma_{\text{BES}}(y, X^{(Q)}) \frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})} \left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^Q, \tag{3.87}$$

where the BDS S-matrix is defined by

$$S_{\text{BDS}}(y, x) \equiv \frac{1-\frac{1}{y^+x^-}}{1-\frac{1}{y^-x^+}} \cdot \frac{x^- - y^+}{x^+ - y^-}. \tag{3.88}$$

The spectral parameter $X^{(Q)}$ for the DGM is defined by

$$X^{(Q)\pm} = \frac{e^{\pm ip/2}}{4g\sin\frac{p}{2}} \left(Q + \sqrt{Q^2 + 16g^2\sin^2\frac{p}{2}}\right) \equiv e^{(\theta \pm ip)/2}, \tag{3.89}$$

where we introduce θ defined by

$$\sinh\frac{\theta}{2} \equiv \frac{Q}{4g\sin\frac{p}{2}}. \tag{3.90}$$

The frame factors η and $\tilde{\eta}$ are given by [71]

$$\frac{\eta(x_1)}{\tilde{\eta}(x_1)} = \frac{\eta(x_2)}{\tilde{\eta}(x_2)} = 1 \tag{3.91}$$

for the spin-chain frame and

$$\frac{\eta(x_1)}{\tilde{\eta}(x_1)} = \sqrt{\frac{x_2^+}{x_2^-}}, \quad \frac{\eta(x_2)}{\tilde{\eta}(x_2)} = \sqrt{\frac{x_1^-}{x_1^+}} \tag{3.92}$$

for the string frame.

Symmetric DGM state

The classical two spins solution is a symmetric DGM configuration for both of S^2 subspaces. Corresponding Lüscher formula is given by Eq.(3.83) with $M_A = M_B = 1$, which can be much simplified as

$$\delta E_\mu = -i\sum_{b=1}^{4}(-1)^{F_b} e^{-i\tilde{q}^* L} \left\{ \left(1 - \frac{\epsilon_Q'(p_1)}{\epsilon_1'(\tilde{q}^*)}\right) \left[\operatorname*{Res}_{q^*=\tilde{q}^*} S^{AA\mathbf{b}\mathbf{1}_Q}_{\mathbf{b}\mathbf{1}_Q}(q^*, p_1)\right] S^{AB\mathbf{b}\mathbf{1}_{\tilde{Q}}}_{\mathbf{b}\mathbf{1}_{\tilde{Q}}}(q^*, p_2) \right.$$

$$\left. + \left(1 - \frac{\epsilon_{\tilde{Q}}'(p_2)}{\epsilon_1'(\tilde{q}^*)}\right) \left[\operatorname*{Res}_{q^*=\tilde{q}^*} S^{AA\mathbf{b}\mathbf{1}_{\tilde{Q}}}_{\mathbf{b}\mathbf{1}_{\tilde{Q}}}(q^*, p_2)\right] S^{AB\mathbf{b}\mathbf{1}_Q}_{\mathbf{b}\mathbf{1}_Q}(q^*, p_1) \right\}. \tag{3.93}$$

As mentioned earlier, only the two cases of $b = 1, 2$ contributes equally in the sum of Eq.(3.93) since these elements contain a_1. Instead of the summation, we can multiply a factor 2 for the case of $b = 1$. In that case, we can compute easily each term using the S-matrix elements (3.86) and (3.87). Furthermore, we restrict ourselves for the case where the two DGMs are symmetric in both spheres, namely, $p_1 = p_2$ and $Q = \tilde{Q}$. This leads to

$$\delta E_\mu = -4ie^{-i\tilde{q}^* L}\left(1 - \frac{\epsilon'_Q(p)}{\epsilon'_1(\tilde{q}^*)}\right)\left[\operatorname*{Res}_{q^*=\tilde{q}^*} S^{AA}{}^{\mathbf{1}\mathbf{1}_Q}_{\mathbf{1}\mathbf{1}_Q}(q^*,p)\right] S^{AB}{}^{\mathbf{1}\mathbf{1}_Q}_{\mathbf{1}\mathbf{1}_Q}(q^*,p). \tag{3.94}$$

Explicit computations of each factor in (3.94) are exactly the same as those in [74]. There are two types of poles of $S_{\rm BDS}(y, X^{(Q)})$. The s-channel pole which describe $(Q+1)$-DGM arises at $y^- = X^{(Q)+}$ while the t-channel pole for $(Q-1)$-DGM (for $Q \geq 2$) at $y^+ = X^{(Q)+}$. We consider the s-channel pole first. Using the location of the pole, we can find

$$\tilde{q}^* = -\frac{i}{2g\sin\left(\frac{p-i\theta}{2}\right)} \quad \rightarrow \quad e^{-i\tilde{q}^* L} \approx \exp\left[-\frac{L}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right]. \tag{3.95}$$

From Eq.(3.83), one can also obtain

$$1 - \frac{\epsilon'_Q(p)}{\epsilon'_1(\tilde{q}^*)} \approx \frac{\sin\frac{p}{2}\sin\frac{p-i\theta}{2}}{\cosh\frac{\theta}{2}}. \tag{3.96}$$

Furthermore, one can notice from Eqs.(3.86) and (3.87)

$$\left[\operatorname*{Res}_{q^*=\tilde{q}^*} S^{AA}{}^{\mathbf{1}\mathbf{1}_Q}_{\mathbf{1}\mathbf{1}_Q}(q^*,p)\right] S^{AB}{}^{\mathbf{1}\mathbf{1}_Q}_{\mathbf{1}\mathbf{1}_Q}(q^*,p) = \operatorname*{Res}_{q^*=\tilde{q}^*} S_{\rm SYM}{}^{\mathbf{1}\mathbf{1}_Q}_{\mathbf{1}\mathbf{1}_Q}(q^*,p) \tag{3.97}$$

where $S_{\rm SYM}$ is the S-matrix of the $\mathcal{N} = 4$ SYM theory. Explicit evaluation of the residue term becomes in the leading order

$$-\frac{8ige^{-ip}\sin^2\frac{p}{2}}{\sin\frac{p-i\theta}{2}}\exp\left[-\frac{2e^{-\theta/2}\sin\frac{p}{2}}{\sin\frac{p-i\theta}{2}}\right]\left(\frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})}\right)^2\left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^{2Q}. \tag{3.98}$$

Combining all these together, we get

$$\begin{aligned}
\delta E_\mu &= -\frac{8ge^{-ip}\sin^3\frac{p}{2}}{\cosh\frac{\theta}{2}}\exp\left[-\frac{2e^{-\theta/2}\sin\frac{p}{2}}{\sin\frac{p-i\theta}{2}} - \frac{L}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right]\left(\frac{\eta(X^{(Q)})}{\tilde{\eta}(X^{(Q)})}\right)^2\left(\frac{\eta(y)}{\tilde{\eta}(y)}\right)^{2Q} \qquad (3.99)\\
&= -\frac{32g\sin^3\frac{p}{2}\,e^{i\alpha}}{\cosh\frac{\theta}{2}}\exp\left[-\frac{2\sin^2\frac{p}{2}\cosh^2\frac{\theta}{2}}{\sin^2\frac{p}{2}+\sinh^2\frac{\theta}{2}}\left(\frac{L-Q}{2g\sin\frac{p}{2}\cosh\frac{\theta}{2}}+1\right)\right]\\
&= -\frac{32g^2\sin^4\frac{p}{2}\,e^{i\alpha}}{\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}\exp\left[-\frac{2\sin^2\frac{p}{2}\left(L+\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}\right)\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}{Q^2+16g^2\sin^4\frac{p}{2}}\right].
\end{aligned}$$

The phase factor $e^{i\alpha}$ includes various phases arising in the computation as well as the frame dependence of η. As argued in [74], we will drop this phase assuming that this cancels out with appropriate prescription for the Lüscher formula.

The t-channel pole at $y^+ = X^{(Q)+}$ gives exactly the same contribution up to a phase factor. Therefore, combining together, we finally obtain the finite-size effect of the two symmetric DGM configuration as follows:

$$\delta E_\mu = -\frac{64g^2\sin^4\frac{p}{2}}{\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}\exp\left[-\frac{2\sin^2\frac{p}{2}\left(L+\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}\right)\sqrt{Q^2+16g^2\sin^2\frac{p}{2}}}{Q^2+16g^2\sin^4\frac{p}{2}}\right].$$

This is exactly what we have derived in Eq.(3.80) if we identify $J_1 = L$, $J_2 = Q$ and $g = \sqrt{\lambda/2}$.

3.1.4 Finite-size dyonic giant magnons in TsT-transformed $AdS_5 \times S^5$

Investigations on AdS/CFT duality for the cases with reduced or without supersymmetry is of obvious interest and importance. An interesting example of such correspondence between gauge and string theory models with reduced supersymmetry is provided by an exactly marginal deformation of $\mathcal{N}=4$ SYM theory [75] and string theory on a β-deformed $AdS_5 \times S^5$ background suggested in [76]. When $\beta \equiv \gamma$ is real, the deformed background can be obtained from $AdS_5 \times S^5$ by the so-called TsT transformation. It includes T-duality on one angle variable, a shift of another isometry variable, then a second T-duality on the first angle [76, 77]. Taking into account that the five-sphere has three isometric coordinates, one can consider generalization of the above procedure, consisting of chain of three TsT transformations. The result is a regular three-parameter deformation of $AdS_5 \times S^5$ string background, dual to a non-supersymmetric deformation of $\mathcal{N}=4$ SYM [77], which is conformal in the planar limit to any order of perturbation theory [78]. The action for this γ_i-deformed ($i = 1, 2, 3$) gauge theory can be obtained from the initial one after replacement of the usual product with associative $*$-product [76, 77, 79].

An essential property of the TsT transformation is that it preserves the classical integrability of string theory on $AdS_5 \times S^5$ [77]. The γ-dependence enters only through the *twisted* boundary conditions and the *level-matching* condition. The last one is modified since a closed string in the deformed background corresponds to an open string on $AdS_5 \times S^5$ in general.

The finite-size correction to the GM energy-charge relation, in the γ-deformed background, has been found in [80], by using conformal gauge and the string sigma model reduced to $R_t \times S^3$. For the deformed case, this is the smallest consistent reduction due to the *twisted* boundary conditions. It turns out that even for the three-parameter deformation, the reduced model depends only on one of them - γ_3. As far as there are two isometry angles ϕ_1, ϕ_2 on S^3, the solution can carry two non-vanishing angular momenta J_1, J_2. Then, the GM is an open string solution with only one charge $J_1 \neq 0$. The momentum p of the magnon excitation in the corresponding spin chain is identified with the angular difference $\Delta\phi_1$ between the end-points of the string. The other angle satisfies the following *twisted* boundary conditions [80]

$$\Delta\phi_2 = 2\pi(n_2 - \gamma_3 J_1),$$

where n_2 is an integer winding number of the string in the second isometry direction of the deformed sphere S^3_γ.

An interesting extension of this study is the dyonic giant magnon. This state corresponds to bound states of the fundamental magnons and stable even in the deformed theory. Understanding its string theory analog in the strong coupling limit can be helpful to extend the AdS/CFT duality to the deformed theories.

In [18] we investigated dyonic giant magnons propagating on γ-deformed $AdS_5 \times S^5$ by Neumann-Rosochatius reduction method with twisted boundary conditions. We compute finite-size effect of the dispersion relations of dyonic giant magnons, which generalizes the previously known case of the giant magnons with one angular momentum found by Bykov and Frolov.

The bosonic part of the Green-Schwarz action for strings on the γ-deformed $AdS_5 \times S^5_\gamma$ reduced to $R_t \times S^5_\gamma$ can be written as (the common radius R of AdS_5 and S^5_γ is set to 1) [81]

$$\begin{aligned} S &= -\frac{T}{2}\int d\tau d\sigma \left\{\sqrt{-\gamma}\gamma^{ab}\left[-\partial_a t\partial_b t + \partial_a r_i\partial_b r_i + G r_i^2\partial_a\varphi_i\partial_b\varphi_i\right.\right. \\ &\quad \left. + G r_1^2 r_2^2 r_3^2\left(\hat{\gamma}_i\partial_a\varphi_i\right)\left(\hat{\gamma}_j\partial_b\varphi_j\right)\right] \\ &\quad \left. -2G\,\epsilon^{ab}\left(\hat{\gamma}_3 r_1^2 r_2^2\partial_a\varphi_1\partial_b\varphi_2 + \hat{\gamma}_1 r_2^2 r_3^2\partial_a\varphi_2\partial_b\varphi_3 + \hat{\gamma}_2 r_3^2 r_1^2\partial_a\varphi_3\partial_b\varphi_1\right)\right\}, \end{aligned} \tag{3.100}$$

where φ_i are the three isometry angles of the deformed S^5_γ, and

$$\sum_{i=1}^{3} r_i^2 = 1, \qquad G^{-1} = 1 + \hat{\gamma}_3 r_1^2 r_2^2 + \hat{\gamma}_1 r_2^2 r_3^2 + \hat{\gamma}_2 r_1^2 r_3^2. \tag{3.101}$$

The deformation parameters $\hat{\gamma}_i$ are related to γ_i which appear in the dual gauge theory as follows

$$\hat{\gamma}_i = 2\pi T\gamma_i = \sqrt{\lambda}\gamma_i.$$

When $\hat{\gamma}_i = \hat{\gamma}$ this becomes the supersymmetric background of [76], and the deformation parameter γ enters the $\mathcal{N} = 1$ SYM superpotential in the following way

$$W \propto tr\left(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2\right).$$

By using the TsT transformations which map the string theory on $AdS_5 \times S^5$ to the γ_i-deformed theory, one can relate the angle variables ϕ_i on S^5 to the angles φ_i of the γ_i-deformed geometry [77]:

$$p_i = \pi_i, \qquad r_i^2\phi_i' = r_i^2\left(\varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k\right), \qquad i = 1, 2, 3, \tag{3.102}$$

where p_i, π_i are the momenta conjugated to ϕ_i, φ_i respectively, and the summation is over j, k. The equality $p_i = \pi_i$ implies that the charges

$$J_i = \int d\sigma p_i$$

are invariant under the TsT transformation.

If none of the variables r_i is vanishing on a given string solution, from (3.102) one gets

$$\phi_i' = \varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k.$$

Integrating the above equations and taking into account that for a closed string in the γ-deformed background

$$\Delta\varphi_i = \varphi_i(r) - \varphi_i(-r) = 2\pi n_i, \quad n_i \in \mathbb{Z},$$

one finds the *twisted* boundary conditions for the angles ϕ_i on the original S^5 space

$$\Delta\phi_i = \phi_i(r) - \phi_i(-r) = 2\pi\left(n_i - \nu_i\right), \quad \nu_i = \varepsilon_{ijk}\gamma_j J_k.$$

It is obvious that if the *twists* ν_i are not integer, then a closed string on the deformed background is mapped to an open string on $AdS_5 \times S^5$.

As we already explained, instead of considering strings on the γ-deformed background $AdS_5 \times S^5_\gamma$, we can consider strings on the original $AdS_5 \times S^5$ space, but with *twisted* boundary conditions. Actually, here we are interested in string configurations living in the $R_t \times S^3$ subspace, which can be described by the NR integrable system. Afer the NR reduction one obtains the following expressions for the conserved charges

$$\begin{aligned}
\mathcal{E} &= \frac{\kappa}{\alpha}\int_{-r}^{r} d\xi = \frac{(1-v^2)w}{\sqrt{1-u^2}}\int_{\chi_{min}}^{\chi_{max}} \frac{d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}},\\
\mathcal{J}_1 &= \frac{1}{\sqrt{1-u^2}}\int_{\chi_{min}}^{\chi_{max}} \frac{\left[1-v^2\left(w^2-u^2 j\right)-\chi\right]d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}},\\
\mathcal{J}_2 &= \frac{u}{\sqrt{1-u^2}}\int_{\chi_{min}}^{\chi_{max}} \frac{\left(\chi - v^2 j\right)d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}},
\end{aligned} \tag{3.103}$$

and for the angular differences

$$p = \Delta\phi_1 = \phi_1(r) - \phi_1(-r), \quad \delta = \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = 2\pi\left(n_2 - \gamma_3 J_1\right).$$

$$\begin{aligned}
p &= \int_{-r}^{r} d\xi f_1' = \frac{\beta\omega_1}{\alpha^2(1-v^2)}\int_{-r}^{r}\left(1-\frac{w^2-u^2 j}{r_1^2}\right)d\xi \\
&= \frac{v}{\sqrt{1-u^2}}\int_{\chi_{min}}^{\chi_{max}}\left(\frac{w^2-u^2 j}{1-\chi}-1\right)\frac{d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}},
\end{aligned} \tag{3.104}$$

$$
\begin{aligned}
\delta &= \int_{-r}^{r} d\xi f_2' = \frac{\beta\omega_2}{\alpha^2(1-v^2)} \int_{-r}^{r} \left(1 - \frac{j}{r_2^2}\right) d\xi \\
&= \frac{uv}{\sqrt{1-u^2}} \int_{\chi_{min}}^{\chi_{max}} \left(\frac{j}{\chi} - 1\right) \frac{d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}}.
\end{aligned} \tag{3.105}
$$

By using that

$$
\begin{aligned}
&\int_{\chi_{min}}^{\chi_{max}} \frac{d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}} = \frac{2}{\sqrt{\chi_{max}-\chi_n}} \mathbf{K}(1-\epsilon), \\
&\int_{\chi_{min}}^{\chi_{max}} \frac{\chi d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}} \\
&= \frac{2\chi_n}{\sqrt{\chi_{max}-\chi_n}} \mathbf{K}(1-\epsilon) + 2\sqrt{\chi_{max}-\chi_n} \mathbf{E}(1-\epsilon), \\
&\int_{\chi_{min}}^{\chi_{max}} \frac{d\chi}{\chi\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}} = \frac{2}{\chi_{max}\sqrt{\chi_{max}-\chi_n}} \mathbf{\Pi}\left(1 - \frac{\chi_{min}}{\chi_{max}} | 1-\epsilon\right), \\
&\int_{\chi_{min}}^{\chi_{max}} \frac{d\chi}{(1-\chi)\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}} \\
&= \frac{2}{(1-\chi_{max})\sqrt{\chi_{max}-\chi_n}} \mathbf{\Pi}\left(-\frac{\chi_{max}-\chi_{min}}{1-\chi_{max}} | 1-\epsilon\right),
\end{aligned}
$$

where

$$
\epsilon = \frac{\chi_{min}-\chi_n}{\chi_{max}-\chi_n},
$$

from (3.103), (3.104) and (3.105) one finds

$$
\begin{aligned}
\mathcal{E} &= \frac{4\tilde{\kappa}}{\sqrt{(1-\chi_n)(1-\tilde{v}^2)}} \mathbf{K}(1-\epsilon), \\
\mathcal{J}_1 &= \frac{4\tilde{\kappa}}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}} \left[\left(\omega(1-\chi_n) - \frac{v^2}{\omega}(1+\nu A_2)\right) \mathbf{K}(1-\epsilon) \right. \\
&\left. - \omega(1-\chi_n)(1-\tilde{v}^2)\mathbf{E}(1-\epsilon)\right], \\
\mathcal{J}_2 &= \frac{4\tilde{\kappa}}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}} \left[\left(v^2 A_2 + \nu\chi_n\right) \mathbf{K}(1-\epsilon) \right. \\
&\left. + \nu(1-\chi_n)(1-\tilde{v}^2)\mathbf{E}(1-\epsilon)\right], \\
p &= \frac{4\tilde{\kappa}v}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}} \left[\frac{1+\nu A_2}{\omega(1-\chi_n)\tilde{v}^2} \mathbf{\Pi}\left(\frac{\tilde{v}^2-1}{\tilde{v}^2}(1-\epsilon) | 1-\epsilon\right) - \omega\mathbf{K}(1-\epsilon)\right], \\
\delta &= -\frac{2\tilde{\kappa}v}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}} \left[\frac{A_2}{(1-\tilde{v}^2)\left(1+\chi_n\frac{\tilde{v}^2}{1-\tilde{v}^2}\right)} \mathbf{\Pi}\left(\frac{1-\chi_n}{1+\chi_n\frac{\tilde{v}^2}{1-\tilde{v}^2}}(1-\epsilon) | 1-\epsilon\right) \right. \\
&\left. + \nu\mathbf{K}(1-\epsilon)\right], \\
\tilde{\kappa} &= \frac{1-v^2}{2\sqrt{\omega^2-\nu^2}}.
\end{aligned} \tag{3.106}
$$

In the above equalities we introduced the new parameters

$$\tilde{v}^2 = \frac{1-\chi_{max}}{1-\chi_n}, \qquad \epsilon = \frac{\chi_{min}-\chi_n}{\chi_{max}-\chi_n}$$

instead of χ_{max} and χ_{min}.

In order to obtain the finite-size correction to the energy-charge relation, we have to consider the limit $\epsilon \to 0$ in (3.106). For the parameters in (3.106), we make the following ansatz

$$\begin{aligned} &v = v_0 + v_1\epsilon + v_2\epsilon\log(\epsilon), \quad \tilde{v} = \tilde{v}_0 + \tilde{v}_1\epsilon + \tilde{v}_2\epsilon\log(\epsilon), \quad \omega = 1+\omega_1\epsilon, \\ &\nu = \nu_0 + \nu_1\epsilon + \nu_2\epsilon\log(\epsilon), \quad A_2 = A_{21}\epsilon, \quad \chi_n = \chi_{n1}\epsilon. \end{aligned} \tag{3.107}$$

We insert all these expansions into (3.106) and impose the conditions:

1. p - finite
2. $\mathcal{J}_2$ - finite
3. $\mathcal{E} - \mathcal{J}_1 = \frac{2\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2} - \frac{(1-v_0^2-\nu_0^2)^{3/2}}{2(1-\nu_0^2)}\cos(\Phi)\epsilon$

From the first two conditions, we obtain the relations

$$p = \arcsin\left(\frac{2v_0\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2}\right), \quad \tilde{v}_0 = \frac{v_0}{\sqrt{1-\nu_0^2}}, \quad \mathcal{J}_2 = \frac{2\nu_0\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2}, \tag{3.108}$$

as well as six more equations. The third condition gives another two equations for the coefficients in (3.107). Thus, we have a system of eight equations, from which we can find all remaining coefficients in (3.107), except A_{21}. A_{21} can be found from the equation for δ to be

$$A_{21} = -\Lambda\frac{(1-v_0^2-\nu_0^2)^{3/2}}{v_0(1-\nu_0^2)}\sin(\Phi),$$

where Λ is constant with respect to Φ (actually, Λ can be fixed to 1). The equations (3.108) are solved by

$$v_0 = \frac{\sin(p)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}, \quad \tilde{v}_0 = \cos(p/2), \quad \nu_0 = \frac{\mathcal{J}_2}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}. \tag{3.109}$$

Replacing (3.109) into the solutions for the other coefficients, one can obtain the expressions for the remaining parameters in terms of physical quantities.

To the leading order, the equation for $\mathcal{J}_1$ gives

$$\epsilon = 16\exp\left[-\frac{2\left(\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\sin^2(p/2)}{\mathcal{J}_2^2+4\sin^4(p/2)}\right].$$

Accordingly, to the leading order again, the equation for δ reads

$$2\pi\left(n_2-\gamma_3\frac{\sqrt{\lambda}}{2\pi}\mathcal{J}_1\right)+\mathcal{J}_2\frac{\mathcal{J}_1+\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}{\mathcal{J}_2^2+4\sin^4(p/2)}\sin(p)=\Lambda\Phi. \tag{3.110}$$

Finally, the dispersion relation, including the leading finite-size correction, takes the form

$$\begin{aligned}\mathcal{E}-\mathcal{J}_1=&\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}-\frac{16\sin^4(p/2)}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}\cos(\Phi) \\ &\exp\left[-\frac{2\left(\mathcal{J}_1+\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\sin^2(p/2)}{\mathcal{J}_2^2+4\sin^4(p/2)}\right].\end{aligned} \tag{3.111}$$

For $\mathcal{J}_2=0$, (3.111) reduces to the result found in [80][8].

3.1.5 Finite-size giant magnons on $AdS_4\times CP^3_\gamma$

In [21] we investigated finite-size giant magnons propagating on γ-deformed $AdS_4\times CP^3_\gamma$ type IIA string theory background, dual to one parameter deformation of the $\mathcal{N}=6$ super Chern-Simoms-matter theory (ABJM theory) [66]. The resulting theory has $\mathcal{N}=2$ supersymmetry and the modified superpotential is [82]

$$W_\gamma\propto Tr\left(e^{-i\pi\gamma/2}A_1B_1A_2B_2-e^{i\pi\gamma/2}A_1B_2A_2B_1\right). \tag{3.112}$$

Here the chiral superfields A_i, B_i, $(i=1,2)$ represent the matter part of the theory. As in the $\mathcal{N}=4$ SYM case, the marginality of the deformation translates into the fact that AdS_4 part of the background is untouched. Taking into account that CP^3 has three isometric coordinates, one can consider a chain of three TsT transformations. The result is a regular three-parameter deformation of $AdS_4\times CP^3$ string background, dual to a non-supersymmetric deformation of ABJM theory, which reduces to the supersymmetric one by putting $\gamma_1=\gamma_2=0$ and $\gamma_3=\gamma$ [82].

The dispersion relation for the GM in the γ-deformed $AdS_4\times CP^3_\gamma$ background, carrying two nonzero angular momenta, has been found in [83]. Here we are interested in obtaining the finite-size correction to it. Analyzing the finite-size effect on the dispersion relation, we found that it is modified compared to the undeformed case, acquiring γ dependence.

[8] We want to point out that our result is different from [80] which has extra $\cos^3(p/4)$ in the denominator of the phase Φ.

Let us first write down the deformed background. It is given by [82][9]

$$ds^2_{IIA} = R^2\left(\frac{1}{4}ds^2_{AdS_4} + ds^2_{CP^3_\gamma}\right),$$

$$ds^2_{CP^3_\gamma} = d\psi^2 + G\sin^2\psi\cos^2\psi\left(\frac{1}{2}\cos\theta_1 d\phi_1 - \frac{1}{2}\cos\theta_2 d\phi_2 + d\phi_3\right)^2$$
$$+\frac{1}{4}\cos^2\psi\left(d\theta_1^2 + G\sin^2\theta_1 d\phi_1^2\right) + \frac{1}{4}\sin^2\psi\left(d\theta_2^2 + G\sin^2\theta_2 d\phi_2^2\right)$$
$$+\tilde{\gamma}G\sin^4\psi\cos^4\psi\sin^2\theta_1\sin^2\theta_2 d\phi_3^2,$$

$$B_2 = -R^2\tilde{\gamma}G\sin^2\psi\cos^2\psi$$
$$\times\left[\frac{1}{2}\cos^2\psi\sin^2\theta_1\cos\theta_2 d\phi_3\wedge d\phi_1 + \frac{1}{2}\sin^2\psi\sin^2\theta_2\cos\theta_1 d\phi_3\wedge d\phi_2\right.$$
$$\left.+\ \frac{1}{4}\left(\sin^2\theta_1\sin^2\theta_2 + \cos^2\psi\sin^2\theta_1\cos^2\theta_2 + \sin^2\psi\sin^2\theta_2\cos^2\theta_1\right)d\phi_1\wedge d\phi_2\right],$$

where

$$G^{-1} = 1 + \tilde{\gamma}^2\sin^2\psi\cos^2\psi\left(\sin^2\theta_1\sin^2\theta_2 + \cos^2\psi\sin^2\theta_1\cos^2\theta_2 + \sin^2\psi\sin^2\theta_2\cos^2\theta_1\right).$$

The deformation parameter $\tilde{\gamma}$ above is given by $\tilde{\gamma} = \frac{R^2}{4}\gamma$, where γ appears in the dual field theory superpotential (3.112).

Further on, we restrict our attention to the $R_t \times RP^3_\gamma$ subspace of $AdS_4 \times CP^3_\gamma$, where $\theta_1 = \theta_2 = \pi/2$, $\phi_3 = 0$, and

$$ds^2 = R^2\left(-\frac{1}{4}dt^2 + d\psi^2 + \frac{G}{4}\cos^2\psi d\phi_1^2 + \frac{G}{4}\sin^2\psi d\phi_2^2\right),$$
$$B_2 = b_{\phi_1\phi_2}d\phi_1\wedge d\phi_2 = -\frac{R^2}{4}\tilde{\gamma}G\sin^2\psi\cos^2\psi d\phi_1\wedge d\phi_2,$$
$$G^{-1} = 1 + \tilde{\gamma}^2\sin^2\psi\cos^2\psi.$$

To find the string solutions we are interested in, we use the ansatz ($j = 1, 2$)

$$t(\tau,\sigma) = \kappa\tau, \quad \psi(\tau,\sigma) = \psi(\xi), \quad \phi_j(\tau,\sigma) = \omega_j\tau + f_j(\xi),$$
$$\xi = \alpha\sigma + \beta\tau, \quad \kappa, \omega_j, \alpha, \beta = \text{constants}. \tag{3.113}$$

It leads to reduction of the string dynamics to the one of the γ-deformed NR system. The string Lagrangian becomes (prime is used for $d/d\xi$)

$$\mathcal{L}_s = -\frac{TR^2}{2}(\alpha^2 - \beta^2)\left[\psi'^2 + \frac{G}{4}\cos^2\psi\left(f_1' - \frac{\beta\omega_1}{\alpha^2-\beta^2}\right)^2 + \frac{G}{4}\sin^2\psi\left(f_2' - \frac{\beta\omega_2}{\alpha^2-\beta^2}\right)^2\right.$$
$$\left.-\ \frac{G\alpha^2}{4(\alpha^2-\beta^2)^2}\left(\omega_1^2\cos^2\psi + \omega_2^2\sin^2\psi\right) + \frac{\alpha\tilde{\gamma}G}{2}\sin^2\psi\cos^2\psi\frac{\omega_1 f_2' - \omega_2 f_1'}{\alpha^2-\beta^2}\right], \tag{3.114}$$

[9]There are also nontrivial dilaton and fluxes F_2, F_4, but since the fundamental string does not interact with them at the classical level, we do not need to know the corresponding expressions.

while the Virasoro constraints acquire the form

$$\begin{aligned}
&\psi'^2 + \frac{G}{4}\cos^2\psi\left(f_1'^2 + \frac{2\beta\omega_1}{\alpha^2+\beta^2}f_1' + \frac{\omega_1^2}{\alpha^2+\beta^2}\right)\\
&+\frac{G}{4}\sin^2\psi\left(f_2'^2 + \frac{2\beta\omega_2}{\alpha^2+\beta^2}f_2' + \frac{\omega_2^2}{\alpha^2+\beta^2}\right) = \frac{\kappa^2/4}{\alpha^2+\beta^2}, \qquad (3.115)\\
&\psi'^2 + \frac{G}{4}\cos^2\psi\left(f_1'^2 + \frac{\omega_1}{\beta}f_1'\right) + \frac{G}{4}\sin^2\psi\left(f_2'^2 + \frac{\omega_2}{\beta}f_2'\right) = 0.
\end{aligned}$$

The equations of motion for $f_j(\xi)$ following from (3.114) can be integrated once to give

$$\begin{aligned}
f_1' &= \frac{1}{\alpha^2-\beta^2}\left[\frac{C_1}{\cos^2\psi} + \beta\omega_1 + \tilde{\gamma}\left(\alpha\omega_2 + \tilde{\gamma}C_1\right)\sin^2\psi\right], \qquad (3.116)\\
f_2' &= \frac{1}{\alpha^2-\beta^2}\left[\frac{C_2}{\sin^2\psi} + \beta\omega_2 - \tilde{\gamma}\left(\alpha\omega_1 - \tilde{\gamma}C_2\right)\cos^2\psi\right],
\end{aligned}$$

where C_j are constants. Replacing (3.116) into (3.115), one can rewrite the Virasoro constraints as

$$\begin{aligned}
\psi'^2 &= \frac{1}{4(\alpha^2-\beta^2)^2}\Bigg[(\alpha^2+\beta^2)\kappa^2 - \frac{C_1^2}{\cos^2\psi} - \frac{C_2^2}{\sin^2\psi} \qquad (3.117)\\
&- \left(\alpha\omega_1 - \tilde{\gamma}C_2\right)^2\cos^2\psi - \left(\alpha\omega_2 + \tilde{\gamma}C_1\right)^2\sin^2\psi\Bigg],\\
&\omega_1 C_1 + \omega_2 C_2 + \beta\kappa^2 = 0. \qquad (3.118)
\end{aligned}$$

Let us point out that (3.117) is the first integral of the equation of motion for ψ. Integrating (3.116) and (3.117), one can find string solutions with very different properties. Particular examples are (dyonic) giant magnons and single-spike strings.

In the case at hand, the background metric does not depend on t and ϕ_j. The corresponding conserved quantities are the string energy E_s and two angular momenta J_j, given by

$$\begin{aligned}
E_s &= \frac{TR^2}{4}\frac{\kappa}{\alpha}\int d\xi, \qquad (3.119)\\
J_1 &= \frac{TR^2}{4}\frac{1}{\alpha^2-\beta^2}\int d\xi\left[\frac{\beta}{\alpha}C_1 + \left(\alpha\omega_1 - \tilde{\gamma}C_2\right)\cos^2\psi\right],\\
J_2 &= \frac{TR^2}{4}\frac{1}{\alpha^2-\beta^2}\int d\xi\left[\frac{\beta}{\alpha}C_2 + \left(\alpha\omega_2 + \tilde{\gamma}C_1\right)\sin^2\psi\right].
\end{aligned}$$

Let us remind that the relation between the string tension T and the t'Hooft coupling constant λ for the $\mathcal{N}=6$ super Chern-Simoms-matter theory is given by

$$TR^2 = 2\sqrt{2\lambda}.$$

If we introduce the variable

$$\chi = \cos^2\psi,$$

and use (3.118), the first integral (3.117) can be rewritten as

$$\chi'^2 = \frac{\Omega_2^2\,(1-u^2)}{\alpha^2(1-v^2)^2}(\chi_p-\chi)(\chi-\chi_m)(\chi-\chi_n),$$

where

$$\begin{aligned}
\chi_p+\chi_m+\chi_n &= \frac{2-(1+v^2)W-u^2}{1-u^2},\\
\chi_p\chi_m+\chi_p\chi_n+\chi_m\chi_n &= \frac{1-(1+v^2)W+(vW-uK)^2-K^2}{1-u^2},\\
\chi_p\chi_m\chi_n &= -\frac{K^2}{1-u^2},
\end{aligned} \tag{3.120}$$

and

$$v=-\frac{\beta}{\alpha},\quad u=\frac{\Omega_1}{\Omega_2},\quad W=\left(\frac{\kappa}{\Omega_2}\right)^2,\quad K=\frac{C_1}{\alpha\Omega_2},$$
$$\Omega_1=\omega_1\left(1-\tilde{\gamma}\frac{C_2}{\alpha\omega_1}\right),\quad \Omega_2=\omega_2\left(1+\tilde{\gamma}\frac{C_1}{\alpha\omega_2}\right).$$

We are interested in the case

$$0<\chi_m<\chi<\chi_p<1,\quad \chi_n<0,$$

which corresponds to the finite-size giant magnons.

In terms of the newly introduced variables, the conserved quantities (3.119) and the angular differences

$$p_1\equiv\Delta\phi_1=\phi_1(r)-\phi_1(-r),\quad p_2\equiv\Delta\phi_2=\phi_2(r)-\phi_2(-r), \tag{3.121}$$

transform to

$$\mathcal{E}\equiv\frac{E_s}{TR^2}=\frac{(1-v^2)\sqrt{W}}{\sqrt{1-u^2}}\frac{\mathbf{K}(1-\epsilon)}{\sqrt{\chi_p-\chi_n}}, \tag{3.122}$$

$$\mathcal{J}_1\equiv\frac{J_1}{TR^2}=\frac{1}{\sqrt{1-u^2}}\left[\frac{u\chi_n-vK}{\sqrt{\chi_p-\chi_n}}\mathbf{K}(1-\epsilon)+u\sqrt{\chi_p-\chi_n}\mathbf{E}(1-\epsilon)\right], \tag{3.123}$$

$$\mathcal{J}_2\equiv\frac{J_2}{TR^2}=\frac{1}{\sqrt{1-u^2}}\Big[\frac{1-\chi_n-v\,(vW-uK)}{\sqrt{\chi_p-\chi_n}}\mathbf{K}(1-\epsilon) \tag{3.124}$$
$$-\sqrt{\chi_p-\chi_n}\mathbf{E}(1-\epsilon)\Big],$$

$$p_1 = \frac{4}{\sqrt{1-u^2}} \tag{3.125}$$
$$\times\left\{\frac{K}{\chi_p\sqrt{\chi_p-\chi_n}}\mathbf{\Pi}\left(1-\frac{\chi_m}{\chi_p}|1-\epsilon\right) - \left[uv+\tilde{\gamma}v\left(vW-uK\right)-\tilde{\gamma}\left(1-\chi_n\right)\right]\frac{\mathbf{K}(1-\epsilon)}{\sqrt{\chi_p-\chi_n}}\right.$$
$$\left.-\tilde{\gamma}\sqrt{\chi_p-\chi_n}\mathbf{E}(1-\epsilon)\right\},$$

$$p_2 = \frac{4}{\sqrt{1-u^2}} \tag{3.126}$$
$$\times\left\{\frac{vW-uK}{(1-\chi_p)\sqrt{\chi_p-\chi_n}}\mathbf{\Pi}\left(-\frac{\chi_p-\chi_m}{1-\chi_p}|1-\epsilon\right) - \left[v\left(1-\tilde{\gamma}K\right)+\tilde{\gamma}u\chi_n\right]\frac{\mathbf{K}(1-\epsilon)}{\sqrt{\chi_p-\chi_n}}\right.$$
$$\left.-\tilde{\gamma}u\sqrt{\chi_p-\chi_n}\mathbf{E}(1-\epsilon)\right\},$$

where ϵ is given by

$$\epsilon = \frac{\chi_m-\chi_n}{\chi_p-\chi_n}. \tag{3.127}$$

From (3.122)-(3.124) one can see that the conserved charges are not affected by the γ-deformation as it should be. Only the angular differences are shifted.

Further on, we will consider the case when $\mathcal{E}$, $\mathcal{J}_2$ and p_1 are large, while $\mathcal{E}-\mathcal{J}_2$, $\mathcal{J}_1$ and p_2 are finite. To this end, we will introduce appropriate expansions.

Expansions

In order to find the leading finite-size correction to the energy-charge relation, we have to consider the limit $\epsilon\to 0$ in (3.120), (3.122)-(3.127)). We will use the following ansatz for the parameters $(\chi_p,\chi_m,\chi_n,v,u,W,K)$

$$\begin{aligned}
\chi_p &= \chi_{p0}+\left(\chi_{p1}+\chi_{p2}\log(\epsilon)\right)\epsilon,\\
\chi_m &= \chi_{m0}+\left(\chi_{m1}+\chi_{m2}\log(\epsilon)\right)\epsilon,\\
\chi_n &= \chi_{n0}+\left(\chi_{n1}+\chi_{n2}\log(\epsilon)\right)\epsilon,\\
v &= v_0+\left(v_1+v_2\log(\epsilon)\right)\epsilon,\\
u &= u_0+\left(u_1+u_2\log(\epsilon)\right)\epsilon,\\
W &= W_0+\left(W_1+W_2\log(\epsilon)\right)\epsilon,\\
K &= K_0+\left(K_1+K_2\log(\epsilon)\right)\epsilon.
\end{aligned} \tag{3.128}$$

A few comments are in order. To be able to reproduce the dispersion relation for the infinite-size giant magnons, we set

$$\chi_{m0}=\chi_{n0}=K_0=0, \quad W_0=1. \tag{3.129}$$

Also to reproduce the undeformed case [67] in the $\tilde{\gamma} \to 0$ limit, we need to fix

$$\chi_{m2} = \chi_{n2} = W_2 = K_2 = 0. \tag{3.130}$$

Replacing (3.128) into (3.120) and (3.127), one finds six equations for the coefficients in the expansions of χ_p, χ_m, χ_n and W. They are solved by

$$\chi_{p0} = 1 - \frac{v_0^2}{1-u_0^2}, \tag{3.131}$$

$$\begin{aligned}\chi_{p1} = & \frac{v_0}{(1-v_0^2)\,(1-u_0^2)\,(1-v_0^2-u_0^2)}\Big\{ -2v_0u_0(1-v_0^2)(1-v_0^2-u_0^2)u_1 \\ & +2\left(1-u_0^2\right)(1-v_0^2-u_0^2)\left[K_1u_0(1+v_0^2)-(1-v_0^2)v_1\right] \\ & +v_0(1-v_0^2-2u_0^2)\sqrt{(1-u_0^2-v_0^2)^4-4K_1^2(1-u_0^2)^2(1-u_0^2-v_0^2)}\Big\},\end{aligned}$$

$$\begin{aligned}\chi_{p2} &= -2v_0\frac{v_2+(v_0u_2-u_0v_2)u_0}{(1-u_0^2)^2} \\ \chi_{m1} &= \frac{u_0^4-2u_0^2(1-v_0^2)+(1-v_0^2)^2+\sqrt{(1-u_0^2-v_0^2)^4-4K_1^2(1-u_0^2)^2(1-u_0^2-v_0^2)}}{2(1-u_0^2)(1-v_0^2-u_0^2)}, \\ \chi_{n1} &= -\frac{u_0^4-2u_0^2(1-v_0^2)+(1-v_0^2)^2-\sqrt{(1-u_0^2-v_0^2)^4-4K_1^2(1-u_0^2)^2(1-u_0^2-v_0^2)}}{2(1-u_0^2)(1-v_0^2-u_0^2)}, \\ W_1 &= -\frac{2K_1u_0v_0(1-u_0^2)+\sqrt{(1-u_0^2-v_0^2)^4-4K_1^2(1-u_0^2)^2(1-u_0^2-v_0^2)}}{(1-u_0^2)(1-v_0^2)}.\end{aligned}$$

As a next step, we impose the conditions for $\mathcal{J}_1$, p_2 to be independent of ϵ. By expanding r.h.s. of (3.123), (3.255) in ϵ, one gets

$$\mathcal{J}_1 = \frac{u_0\sqrt{1-v_0^2-u_0^2}}{1-u_0^2}, \tag{3.132}$$

$$p_2 = 2\arcsin\left(\frac{2v_0\sqrt{1-v_0^2-u_0^2}}{1-u_0^2}\right) - 4\tilde{\gamma}u_0\frac{\sqrt{1-v_0^2-u_0^2}}{1-u_0^2}, \tag{3.133}$$

along with four more equations from the coefficients of ϵ and $\epsilon\log\epsilon$. The equalities (3.132), (3.133) lead to

$$v_0 = \frac{\sin\Psi}{2\sqrt{\mathcal{J}_1^2+\sin^2(\Psi/2)}}, \qquad u_0 = \frac{\mathcal{J}_1}{\sqrt{\mathcal{J}_1^2+\sin^2(\Psi/2)}}, \qquad p_2 = 2\left(\Psi - 2\tilde{\gamma}\mathcal{J}_1\right), \tag{3.134}$$

where the angle Ψ is defined as

$$\Psi = \arcsin\left(\frac{2v_0\sqrt{1-v_0^2-u_0^2}}{1-u_0^2}\right).$$

After the replacement of (3.131) into the remaining four equations, they can be solved with respect to v_1, v_2, u_1, u_2, leading to the following form of the dispersion relation in the considered approximation

$$\mathcal{E} - \mathcal{J}_2 = \frac{\sqrt{1 - v_0^2 - u_0^2}}{1 - u_0^2} - \frac{1}{4}\frac{\sqrt{(1 - v_0^2 - u_0^2)^3 - 4K_1^2(1 - u_0^2)^2}}{1 - u_0^2}\epsilon. \tag{3.135}$$

To the leading order, the expansion for $\mathcal{J}_2$ gives

$$\epsilon = 16 \exp\left[-\frac{2}{1 - v_0^2}\left(1 - \frac{v_0^2}{1 - u_0^2} + \mathcal{J}_2\sqrt{1 - v_0^2 - u_0^2}\right)\right]. \tag{3.136}$$

By using (3.134) and (3.136), (3.135) can be rewritten as

$$\mathcal{E} - \mathcal{J}_2 = \sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)} - 4\sqrt{\frac{\sin^8(\Psi/2)}{\mathcal{J}_1^2 + \sin^2(\Psi/2)} - 4K_1^2} \tag{3.137}$$
$$\exp\left[-\frac{2\left(\mathcal{J}_2 + \sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)}\right)\sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)}\sin^2(\Psi/2)}{\mathcal{J}_1^2 + \sin^4(\Psi/2)}\right].$$

The parameter K_1 in (3.137) can be related to the angular difference p_1. To see that, let us consider the leading order in the ϵ-expansion for it:

$$\begin{aligned} p_1 = &\frac{4K_1 \arctan\sqrt{\frac{\chi_{p0}}{\chi_{m1}} - 1}}{\sqrt{(1 - u_0^2)\chi_{p0}\chi_{m1}(\chi_{p0} - \chi_{m1})}} \\ &-\frac{2}{\sqrt{(1 - u_0^2)\chi_{p0}}}\left[u_0 v_0 \log(16) + \tilde{\gamma}\left(2\chi_{p0} - (1 - v_0^2)\log(16)\right)\right] \\ &+\frac{2}{\sqrt{(1 - u_0^2)\chi_{p0}}}\left[u_0 v_0 - \tilde{\gamma}(1 - v_0^2)\right]\log(\epsilon). \end{aligned} \tag{3.138}$$

So, it is natural to introduce the angle Φ as

$$\frac{\Phi}{2} = \arctan\sqrt{\frac{\chi_{p0}}{\chi_{m1}} - 1}. \tag{3.139}$$

On the solution for the other parameters this gives

$$K_1 = \frac{(1 - v_0^2 - u_0^2)^{3/2}}{2(1 - u_0^2)}\sin(\Phi) = \frac{\sin^4(\Psi/2)}{2\sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)}}\sin(\Phi). \tag{3.140}$$

As a result, the relation (3.138) between the angles p_1 and Φ becomes

$$\Phi = \frac{p_1}{2} - \left(2\tilde{\gamma} - \mathcal{J}_1\frac{\sin\Psi}{\mathcal{J}_1^2 + \sin^4(\Psi/2)}\right)\mathcal{J}_2 + \mathcal{J}_1\frac{\sin\Psi\sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)}}{\mathcal{J}_1^2 + \sin^4(\Psi/2)}, \tag{3.141}$$

where due to the periodicity condition we should set

$$p_1 = 2\pi n_1, \quad n_1 \in \mathbb{Z}.$$

Finally, in view of (3.140), the dispersion relation (3.137) for the dyonic giant magnons acquires the form

$$\mathcal{E} - \mathcal{J}_2 = \sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)} - \frac{4\sin^4(\Psi/2)}{\sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)}} \cos\Phi \tag{3.142}$$
$$\exp\left[-\frac{2\left(\mathcal{J}_2 + \sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)}\right)\sqrt{\mathcal{J}_1^2 + \sin^2(\Psi/2)}\sin^2(\Psi/2)}{\mathcal{J}_1^2 + \sin^4(\Psi/2)}\right].$$

Based on the Lüscher μ-term formula for the undeformed case [17], we propose to identify the angle $\Psi \left(= \frac{p_2}{2} + 2\tilde{\gamma}\mathcal{J}_1\right)$ with the momentum p of the magnon exitations in the dual spin chain.

Let us point out that (3.142) has the same form as the dispersion relation for dyonic giant magnons on $R_t \times S^3_\gamma$ subspace of the γ-deformed $AdS_5 \times S^5$ [17]. Actually, the two energy-charge relations coincide after appropriate normalization of the charges and after exchange of the indices 1 and 2. The only remaining difference is in the first terms in the expressions for the angle Φ:

$$R_t \times RP^3_\gamma : \Phi = \frac{p_1}{2} + \ldots$$
$$R_t \times S^3_\gamma : \Phi = p_2 + \ldots.$$

All of the above results simplify a lot when one consider giant magnons with one angular momentum, i.e. $\mathcal{J}_1 = 0$. In particular, the energy-charge relation (3.142) reduces to

$$\mathcal{E} - \mathcal{J}_2 = \sin\frac{p}{2}\left[1 - 4\sin^2\frac{p}{2}\cos\left(\pi n_1 - 2\tilde{\gamma}\mathcal{J}_2\right) e^{-2-2\mathcal{J}_2\csc\frac{p}{2}}\right]. \tag{3.143}$$

3.1.6 String solutions in $AdS_3 \times S^3 \times T^4$ with NS-NS B-field

In [27] we developed an approach for solving the string equations of motion and Virasoro constraints in any background which has some (unfixed) number of commuting Killing vector fields (see the beginning of Section **3.1**). It is based on a specific ansatz for the string embedding, which is the one given in (3.4).

Here, we apply the above mentioned approach for strings moving in $AdS_3 \times S^3 \times T^4$ with 2-form NS-NS B-field. We succeeded to find solutions for a large class of string configurations on this background. In particular, we derive dyonic giant magnon solutions in the $R_t \times S^3$ subspace, and obtain the leading finite-size correction to the dispersion relation.

Strings in $AdS_3 \times S^3 \times T^4$ with NS-NS B-field

The background geometry of this target space can be written in the following form [10]:

$$\begin{aligned}
&ds^2_{AdS_3} = -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 + r^2 d\phi^2, \quad b_{t\phi} = qr^2,\\
&ds^2_{S^3} = d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2, \quad b_{\phi_1\phi_2} = -q\cos^2\theta,\\
&ds^2_{T^4} = (d\varphi^i)^2, \quad i = 1,2,3,4.
\end{aligned}$$

According to our notations

$$\begin{aligned}
&X^\mu = \left(t, \phi, \phi_1, \phi_2, \varphi^i\right), \quad X^a = (r, \theta),\\
&g_{\mu\nu} = (g_{tt}, g_{\phi\phi}, g_{\phi_1\phi_1}, g_{\phi_2\phi_2}, g_{ij}), \quad g_{ab} = (g_{rr}, g_{\theta\theta}), \quad g_{a\mu} = 0, \quad h_{ab} = g_{ab},\\
&b_{\mu\nu} = (b_{t\phi}, b_{\phi_1\phi_2}), \quad b_{a\nu} = 0,\\
&A_a = 0,
\end{aligned} \tag{3.144}$$

where

$$\begin{aligned}
&g_{tt} = (g^{tt})^{-1} = -(1+r^2), \quad g_{\phi\phi} = (g^{\phi\phi})^{-1} = r^2, \quad g_{\phi_1\phi_1} = (g^{\phi_1\phi_1})^{-1} = \sin^2\theta,\\
&g_{\phi_2\phi_2} = (g^{\phi_2\phi_2})^{-1} = \cos^2\theta, \quad g_{ij} = \left(g^{ij}\right)^{-1} = \delta_{ij},\\
&g_{rr} = (g^{rr})^{-1} = (1+r^2)^{-1}, \quad g_{\theta\theta} = 1,\\
&b_{t\phi} = qr^2, \quad b_{\phi_1\phi_2} = -q\cos^2\theta.
\end{aligned} \tag{3.145}$$

Since $g_{a\mu} = 0$, the solutions (3.11) for the coordinates X^μ are simplified to

$$X^\mu(\tau,\sigma) = \Lambda^\mu\tau + \tilde{X}^\mu(\xi) = \Lambda^\mu\tau + \frac{1}{\alpha^2-\beta^2}\int d\xi \left[g^{\mu\nu}\left(C_\nu - \alpha\Lambda^\rho b_{\nu\rho}\right) + \beta\Lambda^\mu\right], \tag{3.146}$$

where $g^{\mu\nu}$ and $b_{\nu\rho}$ must be replaced from above.

Now, we want to find the solutions for the non-isometric string coordinates X^a. To this end we have to solve the equations (3.12), which in the case at hand reduce to

$$(\alpha^2-\beta^2)\left[g_{ab}\tilde{X}^{b''} + \Gamma_{a,bc}\tilde{X}^{b'}\tilde{X}^{c'}\right] + \partial_a\sum_{b=r,\theta} U_b = 0, \tag{3.147}$$

where the scalar potential U in (3.15) is represented as a sum of two parts: $U_r = U_r(r)$ for the AdS_3 subspace and $U_\theta = U_\theta(\theta)$ for the S^3 subspace of the background.

Taking into account that the metric g_{ab} is diagonal, one can find the following two first integrals of (3.147)

$$\tilde{X}^{a'} = \sqrt{\frac{C_a - 2U_a}{(\alpha^2-\beta^2)g_{aa}}}. \tag{3.148}$$

[10]The common radius R of the three subspaces is set to 1, and q is the parameter used in [84].

It follows from here that

$$d\xi = \frac{d\tilde{X}^a}{\sqrt{\frac{C_a-2U_a}{(\alpha^2-\beta^2)g_{aa}}}}. \tag{3.149}$$

So, we have two different expressions for $d\xi$, which obviously must coincide. This is a condition for self-consistency. It leads to

$$\int \frac{dr}{\sqrt{\frac{C_r-2U_r}{g_{rr}}}} = \int \frac{d\theta}{\sqrt{\frac{C_\theta-2U_\theta}{g_{\theta\theta}}}}, \tag{3.150}$$

which actually gives implicitly the "orbit" $r(\theta)$, i.e. how the radial coordinate r on AdS_3 depends on the angle θ in S^3.

Now, we have to check if the first integrals for $\tilde{X}^a(\xi)$ are compatible with the Virasoro constraints (3.16). Replacing $\tilde{X}^{a'}$ in the first of them, one finds

$$C_r + C_\theta = 0.$$

Thus, we found all first integrals of the string equations of motion, compatible with the Virasoro constraints, which reduce to algebraic relations between the embedding parameters and the integration constants.

Now, let us give the expressions for the conserved charges (3.17), corresponding to the isometric coordinates.

$$-Q_t \equiv E_s = \frac{T}{\alpha^2-\beta^2}\left[\left(\alpha\Lambda^t - \frac{\beta}{\alpha}C_t - q\ C_\phi\right)\int d\xi + \alpha(1-q^2)\Lambda^t\int d\xi r^2\right], \tag{3.151}$$

$$Q_\phi \equiv S = \frac{T}{\alpha^2-\beta^2}\left[\left(\frac{\beta}{\alpha}C_\phi + qC_t + q^2\alpha\Lambda^\phi\right)\int d\xi + (1-q^2)\alpha\Lambda^\phi\int d\xi r^2 - \left(qC_t + q^2\alpha\Lambda^\phi\right)\int \frac{d\xi}{1+r^2}\right], \tag{3.152}$$

$$Q_{\phi_1} \equiv J_1 = \frac{T}{\alpha^2-\beta^2}\left[\left(\frac{\beta}{\alpha}C_{\phi_1} + \alpha\Lambda^{\phi_1} - qC_{\phi_2}\right)\int d\xi - (1-q^2)\alpha\Lambda^{\phi_1}\int \cos^2\theta d\xi\right], \tag{3.153}$$

$$Q_{\phi_2} \equiv J_2 = \frac{T}{\alpha^2-\beta^2}\left[\left(\frac{\beta}{\alpha}C_{\phi_2} - q\left(C_{\phi_1} + q\alpha\Lambda^{\phi_2}\right)\right)\int d\xi + (1-q^2)\alpha\Lambda^{\phi_2}\int \cos^2\theta d\xi + q\left(C_{\phi_1} + q\alpha\Lambda^{\phi_2}\right)\int \frac{d\xi}{1-\cos^2\theta}\right],$$

$$Q_i \equiv J_i^T = \frac{T}{\alpha^2-\beta^2}\left(\frac{\beta}{\alpha}C_i + \alpha\Lambda^j\delta_{ij}\right)\int d\xi. \tag{3.154}$$

Here we used the following notations: E_s is the string energy, S is the spin in AdS_3, J_1 and J_2 are the two angular momenta in S^3, while J_i^T are the four angular momenta on T^4.

The explicit expressions for the string coordinates, the "orbit" $r(\theta)$, and the conserved charges in this background are given in Appendix C.

Giant magnon solutions

The giant magnon string solution was found in [60]. It is a specific string configuration, living in the $R_t \times S^2$ subspace of $AdS_5 \times S^5$ with an angular momentum J_1 which goes to ∞. A similar configuration, dyonic giant magnon, has been obtained in [58] which moves in $R_t \times S^3$ subspace with two angular momenta J_1, J_2 with $J_1 \to \infty$. These classical configurations have played an important role in understanding exact, quantum aspects of the AdS/CFT correspondence. In particular, corrections due to a large but finite J_1 obtained in [64] and [65] can provide a nontrivial check for the exact worldsheet S-matrix.

Here we provide similar string solutions in $AdS_3 \times S^3 \times T^4$ with NS-NS B-field for a large but finite J_1. Dyonic giant magnon solution with infinite angular momentum J_1 has been constructed in [84] with the following dispersion relation [11]

$$E_s - J_1 = \sqrt{(J_2 - qT\Delta\phi_1)^2 + 4T^2(1-q^2)\sin^2\frac{\Delta\phi_1}{2}}. \tag{3.155}$$

This relation is already quite different from those for the ordinary (dyonic) giant magnons. We will show that there exist even bigger differences for the finite-size corrections.

Exact results

In order to consider dyonic giant magnon solutions, we restrict our general ansatz (3.4) in the following way:

$$\begin{aligned}
&X^t \equiv t = \kappa\tau, \quad \text{i.e.} \quad \Lambda^t = \kappa, \quad \tilde{X}^t(\xi) = 0,\\
&X^\phi \equiv \phi = 0, \quad \text{i.e.} \quad \Lambda^\phi = 0, \quad \tilde{X}^\phi(\xi) = 0\\
&X^r \equiv r = \tilde{X}^r(\xi) = 0,\\
&X^{\phi_1} \equiv \phi_1 = \omega_1\tau + \tilde{X}^{\phi_1}(\xi), \quad \text{i.e.} \quad \Lambda^{\phi_1} = \omega_1,\\
&X^{\phi_2} \equiv \phi_2 = \omega_2\tau + \tilde{X}^{\phi_2}(\xi), \quad \text{i.e.} \quad \Lambda^{\phi_2} = \omega_2,\\
&X^\theta \equiv \theta = \tilde{X}^\theta(\xi), \quad X^{\varphi^i} \equiv \varphi^i = 0.
\end{aligned}$$

As a result, we can claim that

$$C_t = \beta\kappa,$$

[11] The terms proportional to q are due to the nonzero B-field on S^3.

which comes from $\frac{d\tilde{X}^t}{d\xi} = 0$.

Now, we can rewrite the first integrals for $\tilde{X}^\mu$ on S^3 as

$$\frac{d\tilde{X}^{\phi_1}}{d\xi} = \frac{1}{\alpha^2 - \beta^2}\left[(C_{\phi_1} + q\alpha\omega_2)\frac{1}{1-\chi} + \beta\omega_1 - q\alpha\omega_2\right], \quad (3.156)$$
$$\frac{d\tilde{X}^{\phi_2}}{d\xi} = \frac{1}{\alpha^2 - \beta^2}\left(\frac{C_{\phi_2}}{\chi} + \beta\omega_2 - q\alpha\omega_1\right),$$

where $\chi = \cos^2\theta$.

The first Virasoro constraint, which in the case under consideration is the first integral of the equation of motion for θ, reduces to

$$\left(\frac{d\chi}{d\xi}\right)^2 = \frac{4}{(\alpha^2-\beta^2)^2}\chi(1-\chi)\left[(\alpha^2+\beta^2)\kappa^2 - \frac{(C_{\phi_1}+q\alpha\omega_2\chi)^2}{1-\chi} \right. \quad (3.157)$$
$$\left. - \frac{(C_{\phi_2}-q\alpha\omega_1\chi)^2}{\chi} - \alpha^2(\omega_2^2-\omega_1^2)\chi - \alpha^2\omega_1^2\right].$$

Also, the second Virasoro constraint becomes

$$\omega_1 C_{\phi_1} + \omega_2 C_{\phi_2} + \beta\kappa^2 = 0. \quad (3.158)$$

Taking (3.158) into account, we can rewrite (3.157) as

$$\left(\frac{d\chi}{d\xi}\right)^2 = 4(1-q^2)\frac{\omega_1^2}{\alpha^2}\frac{1-u^2}{(1-v^2)^2}(\chi_p-\chi)(\chi-\chi_m)(\chi-\chi_n), \quad (3.159)$$

where

$$\chi_p+\chi_m+\chi_n = \frac{-\left(v^2W + (W+u^2-2+q^2)\right) + 2q\left(uvW + K(1-u^2)\right)}{(1-q^2)(1-u^2)}, \quad (3.160)$$
$$\chi_p\chi_m + \chi_p\chi_n + \chi_m\chi_n = -\frac{(1+v^2)W + K^2 - (vW-uK)^2 - 1 + 2qK}{(1-q^2)(1-u^2)},$$
$$\chi_p\chi_m\chi_n = -\frac{K^2}{(1-q^2)(1-u^2)},$$

and we introduced the notations

$$v = -\frac{\beta}{\alpha}, \quad u = \frac{\omega_2}{\omega_1}, \quad W = \left(\frac{\kappa}{\omega_1}\right)^2, \quad K = \frac{C_{\phi_2}}{\alpha\omega_1}.$$

This leads to

$$d\xi = \frac{\alpha}{2\omega_1}\frac{1-v^2}{\sqrt{(1-q^2)(1-u^2)}}\frac{d\chi}{\sqrt{(\chi_p-\chi)(\chi-\chi_m)(\chi-\chi_n)}}. \quad (3.161)$$

Integrating (3.161) and inverting $\xi(\chi)$ to $\chi(\xi) \equiv \cos^2[\theta(\xi)]$, one finds the following explicit solution

$$\chi = (\chi_p - \chi_n)\ \mathbf{dn}^2 \left[\frac{\sqrt{(1-q^2)(1-u^2)(\chi_p - \chi_n)}}{1-v^2}\ \omega_1(\sigma - v\tau), \frac{\chi_p - \chi_m}{\chi_p - \chi_n} \right] + \chi_n. \quad (3.162)$$

Next, we integrate (3.156), and according to our ansatz, obtain that the solutions for the isometric angles on S^3 are given by

$$\phi_1 = \omega_1 \tau + \frac{2}{\sqrt{(1-q^2)(1-u^2)(\chi_p - \chi_n)}} \quad (3.163)$$
$$\left[\frac{vW - Ku + qu}{1-\chi_p}\ \Pi\left(\arcsin\sqrt{\frac{\chi_p - \chi}{\chi_p - \chi_m}}, -\frac{\chi_p - \chi_m}{1-\chi_p}, \frac{\chi_p - \chi_m}{\chi_p - \chi_n} \right) \right.$$
$$\left. -(v+qu)F\left(\arcsin\sqrt{\frac{\chi_p - \chi}{\chi_p - \chi_m}}, \frac{\chi_p - \chi_m}{\chi_p - \chi_n} \right) \right]$$

$$\phi_2 = \omega_2 \tau + \frac{2}{\sqrt{(1-q^2)(1-u^2)(\chi_p - \chi_n)}} \quad (3.164)$$
$$\left[\frac{K}{\chi_p}\ \Pi\left(\arcsin\sqrt{\frac{\chi_p - \chi}{\chi_p - \chi_m}}, 1 - \frac{\chi_m}{\chi_p}, \frac{\chi_p - \chi_m}{\chi_p - \chi_n} \right) \right.$$
$$\left. -(uv+q)F\left(\arcsin\sqrt{\frac{\chi_p - \chi}{\chi_p - \chi_m}}, \frac{\chi_p - \chi_m}{\chi_p - \chi_n} \right) \right].$$

By using (3.161), one can find also the conserved quantities, namely, the string energy E_s and the two angular momenta J_1, J_2 :

$$E_s = 2T \frac{(1-v^2)\sqrt{W}}{\sqrt{(1-q^2)(1-u^2)(\chi_p - \chi_n)}}\ \mathbf{K}(1-\epsilon), \quad (3.165)$$

$$J_1 = \frac{2T}{\sqrt{(1-q^2)(1-u^2)(\chi_p - \chi_n)}} \left\{ \left[1 - v^2 W + K(uv - q)\right] \mathbf{K}(1-\epsilon) \right.$$
$$\left. -(1-q^2)\left[\chi_n\ \mathbf{K}(1-\epsilon) + (\chi_p - \chi_n)\ \mathbf{E}(1-\epsilon)\right] \right\}, \quad (3.166)$$

$$J_2 = \frac{2T}{\sqrt{(1-q^2)(1-u^2)(\chi_p - \chi_n)}} \left\{ (1-q^2)u\left[\chi_n\ \mathbf{K}(1-\epsilon) + (\chi_p - \chi_n)\ \mathbf{E}(1-\epsilon)\right] \right.$$
$$- \left[Kv + q\,(vW - Ku) + q^2 u\right]\ \mathbf{K}(1-\epsilon) \quad (3.167)$$
$$\left. +q \frac{vW - Ku + qu}{1-\chi_p} \mathbf{\Pi}\left(-\frac{\chi_p - \chi_n}{1-\chi_p}(1-\epsilon), 1-\epsilon \right) \right\}.$$

where ϵ is defined as

$$\epsilon = \frac{\chi_m - \chi_n}{\chi_p - \chi_n}. \tag{3.168}$$

We will need also the expression for the angular difference $\Delta\phi_1$. It can be found to be

$$\Delta\phi_1 = \frac{2}{\sqrt{(1-q^2)(1-u^2)(\chi_p - \chi_n)}} \tag{3.169}$$
$$\left[\frac{vW - Ku + qu}{1-\chi_p}\,\mathbf{\Pi}\left(-\frac{\chi_p - \chi_n}{1-\chi_p}(1-\epsilon), 1-\epsilon\right) - (v+qu)\,\mathbf{K}\,(1-\epsilon)\right].$$

The expressions (3.165), (3.166), (3.167), (3.169) are for the finite-size dyonic strings living in the $R_t \times S^3$ subspace of $AdS_3 \times S^3 \times T^4$.

Leading finite-size effect on the dispersion relation

In order to find the leading finite-size effect on the dispersion relation, we have to consider the limit $\epsilon \to 0$, since $\epsilon = 0$ corresponds to the infinite-size case. In this subsection we restrict ourselves to the particular case when $\chi_n = K = 0$[12]. Then the third equation in (3.160) is satisfied identically, while the other two simplify to

$$\chi_p + \chi_m = \frac{2 - (1+v^2)W - u^2 - 2q(uvW + \frac{q}{2})}{(1-q^2)(1-u^2)}, \tag{3.170}$$
$$\chi_p \chi_m = \frac{(1-W)(1-v^2W)}{(1-q^2)(1-u^2)},$$

and ϵ becomes

$$\epsilon = \frac{\chi_m}{\chi_p}. \tag{3.171}$$

The relevant expansions of the parameters are

$$\chi_p = \chi_{p0} + (\chi_{p1} + \chi_{p2}\log(\epsilon))\,\epsilon, \quad W = 1 + W_1\epsilon,$$
$$v = v_0 + (v_1 + v_2\log(\epsilon))\,\epsilon, \quad u = u_0 + (u_1 + u_2\log(\epsilon))\,\epsilon. \tag{3.172}$$

[12]As we will see later on, this choice allow us to reproduce the dispersion relation in the infinite volume limit [84].

Replacing (3.171), (3.172) into (3.170), one finds the following solutions in the small ϵ limit

$$\chi_{p0} = \frac{1 - v_0^2 - u_0^2 - 2q(u_0 v_0 + \frac{q}{2})}{(1-q^2)(1-u_0^2)}, \tag{3.173}$$
$$\chi_{p1} = -\frac{v_0 + q u_0}{(1-q^2)^2(1-v_0^2)(1-u_0^2)^2} \times$$
$$\Big[\Big(1 - v_0^2 - u_0^2 - 2q(u_0 v_0 + \frac{q}{2})\Big)\Big(v_0^3 + q u_0(1+3v_0^2) - v_0(1 - 2u_0^2 - 2q^2)\Big)$$
$$+2(1-q^2)(1-v_0^2)\big((1-u_0^2)v_1 + (u_0 v_0 + q)u_1\big)\Big]$$
$$\chi_{p2} = -\frac{2(v_0 + q u_0)\left((1-u_0^2)v_2 + (u_0 v_0 + q)u_2\right)}{(1-q^2)(1-u_0^2)^2}$$
$$W_1 = -\frac{\left(1 - v_0^2 - u_0^2 - 2q(u_0 v_0 + \frac{q}{2})\right)^2}{(1-q^2)(1-u_0^2)(1-v_0^2)}.$$

The coefficients in the expansions of v and u, will be obtained by imposing the conditions that J_2 and $\Delta\phi_1$ do not depend on ϵ, as in the cases without B-field ($AdS_5 \times S^5$ and $AdS_4 \times CP^3$) and their TsT-deformations, where the B-field is nonzero, but its contribution is different.

Expanding (3.167) and (3.169) to the leading order in ϵ (now $\chi_n = K = 0$), one finds that on the solutions (3.173)

$$J_2 = 2T\left(\frac{u_0\sqrt{1 - u_0^2 - v_0^2 - 2q(u_0 v_0 + \frac{q}{2})}}{1-u_0^2}\right.$$
$$\left. + q \arcsin\sqrt{\frac{1 - u_0^2 - v_0^2 - 2q(u_0 v_0 + \frac{q}{2})}{(1-q^2)(1-u_0^2)}}\right), \tag{3.174}$$

$$\Delta\phi_1 = 2\arcsin\sqrt{\frac{1 - u_0^2 - v_0^2 - 2q(u_0 v_0 + \frac{q}{2})}{(1-q^2)(1-u_0^2)}}, \tag{3.175}$$

$$u_1 = \frac{1 - u_0^2 - v_0^2 - 2q(u_0 v_0 + \frac{q}{2})}{4(1-q^2)(1-u_0^2)}$$
$$\times\left[u_0\left(1 - \log 16 - v_0^2(1 + \log 16)\right) - 2q v_0 \log 16\right], \tag{3.176}$$

$$v_1 = \frac{1 - u_0^2 - v_0^2 - 2q(u_0 v_0 + \frac{q}{2})}{4(1-q^2)(1-u_0^2)(1-v_0^2)}$$
$$\times\left[v_0\left((1-4q^2)(1-\log 16) - u_0^2(5 - \log 4096)\right)\right.$$
$$\left. - v_0^3\left(1 - \log 16 - u_0^2(1 + \log 16)\right) - 4q u_0\left(1 - \log 4 + v_0^2(1 - \log 64)\right)\right], \tag{3.177}$$

$$u_2 = \frac{(u_0(1+v_0^2) + 2qv_0)\left(1 - u_0^2 - v_0^2 - 2q(u_0v_0 + \frac{q}{2})\right)}{4(1-q^2)(1-v_0^2)}, \tag{3.178}$$

$$\begin{aligned} v_2 = & \frac{1 - u_0^2 - v_0^2 - 2q(u_0v_0 + \frac{q}{2})}{4(1-q^2)(1-u_0^2)(1-v_0^2)} \\ & \times \left[v_0\left(1 - v_0^2 - u_0^2(3+v_0^2)\right) - 2q\left(u_0(1+3v_0^2) + 2qv_0\right)\right]. \end{aligned} \tag{3.179}$$

Now, let us turn to the energy-charge relation. Expanding (3.165) and (3.166) in ϵ and taking into account the solutions (3.173), (3.176) -(3.179), we obtain

$$E_s - J_1 = 2T\frac{\sqrt{1 - u_0^2 - v_0^2 - 2q(u_0v_0 + \frac{q}{2})}}{1-u_0^2}\left(1 - \frac{1 - u_0^2 - v_0^2 - 2q(u_0v_0 + \frac{q}{2})}{4(1-q^2)}\ \epsilon\right). \tag{3.180}$$

The expression for ϵ can be found from the expansion of J_1. To the leading order, it is given by

$$\epsilon = 16\exp\left[-\frac{J_1}{T}\frac{\sqrt{1 - u_0^2 - v_0^2 - 2q(u_0v_0 + \frac{q}{2})}}{1-v_0^2} - 2\frac{1 - u_0^2 - v_0^2 - 2q(u_0v_0 + \frac{q}{2})}{(1-v_0^2)(1-u_0^2)}\right]. \tag{3.181}$$

Next, we would like to express the right hand side of (3.180) in terms of J_2 and $\Delta\phi_1$. To this end, we solve (3.174), (3.175) with respect to u_0, v_0. The result is

$$u_0 = \frac{J_2 - qT\Delta\phi_1}{\sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}}, \tag{3.182}$$

$$v_0 = \frac{T(1-q^2)\sin\Delta\phi_1 - q(J - qT\Delta\phi_1)}{\sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}}. \tag{3.183}$$

Replacing (3.182), (3.183) into (3.180), (3.181), one finds

$$\begin{aligned} E_s - J_1 = & \sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}} \\ & \left(1 - \frac{(1-q^2)T^2\sin^4\frac{\Delta\phi_1}{2}}{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}\ \epsilon\right), \end{aligned} \tag{3.184}$$

where

$$\epsilon = 16\ e^{-\frac{2\left(J_1 + \sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}\right)\sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}\sin^2\frac{\Delta\phi_1}{2}}{(J_2 - qT\Delta\phi_1)^2 + 4T^2\sin^4\frac{\Delta\phi_1}{2} + 2qT\sin\Delta\phi_1\left((J_2 - qT\Delta\phi_1) + \frac{q}{2}T\sin\Delta\phi_1\right)}}. \tag{3.185}$$

Our result [13] matches with that of [84] in (3.155) when we take $\epsilon \to 0$ limit by sending $J_1 \to \infty$. This dispersion relation is different from the ordinary giant magnon's one.

The dispersion relation for the ordinary giant magnon with one nonzero angular momentum cam be obtained by setting $J_2 = 1$ and taking the limit $T \to \infty$. To take into account the *leading* finite-size effect only, we restrict ourselves to the case when $\frac{J_1}{T} >> 1$. The result is the following:

$$E_s - J_1 = T\sqrt{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\left(1 - \frac{(1-q^2)\sin^4\frac{p}{2}}{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\,\epsilon\right), \tag{3.186}$$

where

$$\epsilon = 16\exp\left[\frac{-2}{q^2(p-\sin p)^2 + 4\sin^4\frac{p}{2}}\left(\frac{J_1}{T} + \sqrt{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\right)\right.$$
$$\left.\sqrt{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\;\sin^2\frac{p}{2}\right].$$

Our results on the leading finite-size correction to the dispersion relation can provide an important check for the exact integrability conjecture and S-matrix elements based on it.

3.1.7 Finite-size giant magnons on η-deformed $AdS_5 \times S^5$

A new integrable deformation of the type IIB $AdS_5 \times S^5$ superstring action, depending on one real parameter η, has been found recently in [86]. The bosonic part of the superstring sigma model Lagrangian on this η-deformed background was determined in [87]. Then the authors of [87] used it to compute the perturbative S-matrix of bosonic particles in the model.

Interesting new developments were made in [88]. There the spectrum of a string moving on η-deformed $AdS_5 \times S^5$ is considered. This is done by treating the corresponding worldsheet theory as integrable field theory. In particular, it was found that the dispersion relation for the infinite-size giant magnons [60] on this background, in the large string tension limit $g \to \infty$ is given by

$$E = \frac{2g\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right), \tag{3.187}$$

where $\tilde{\eta}$ is related to the deformation parameter η according to

$$\tilde{\eta} = \frac{2\eta}{1-\eta^2}. \tag{3.188}$$

[13] Eqs.(3.184) and (3.185) have been confirmed by an independent analysis based on algebraic curve method [85] after our result has been appeared in the arXiv.

Here, we are going to extend the result (3.187) to the case of finite-size giant magnons [28].

String Lagrangian and background fields

The bosonic part of the string Lagrangian $\mathcal{L}$ on the η-deformed $AdS_5 \times S^5$ found in [87] is given by a sum of the Lagrangians $\mathcal{L}_a$ and $\mathcal{L}_s$, for the AdS and sphere subspaces. Since there is nonzero B-field on both subspaces, which leads to the appearance of Wess-Zumino terms, these Lagrangians can be further decomposed as

$$\mathcal{L}_a = L_a^g + L_a^{WZ}, \quad \mathcal{L}_s = L_s^g + L_s^{WZ}, \tag{3.189}$$

where the superscript "g" is related to the dependence on the background metric. The explicit expressions for the Lagrangians in (3.189) are as follows [87]

$$\begin{aligned} L_a^g &= -\frac{T}{2}\gamma^{\alpha\beta}\left[-\frac{(1+\rho^2)\partial_\alpha t\partial_\beta t}{1-\tilde{\eta}^2\rho^2} + \frac{\partial_\alpha\rho\partial_\beta\rho}{(1+\rho^2)(1-\tilde{\eta}^2\rho^2)} + \frac{\rho^2\partial_\alpha\zeta\partial_\beta\zeta}{1+\tilde{\eta}^2\rho^4\sin^2\zeta}\right. \\ &\left. + \frac{\rho^2\cos^2\zeta\ \partial_\alpha\psi_1\partial_\beta\psi_1}{1+\tilde{\eta}^2\rho^4\sin^2\zeta} + \rho^2\sin^2\zeta\ \partial_\alpha\psi_2\partial_\beta\psi_2\right], \end{aligned} \tag{3.190}$$

$$L_a^{WZ} = \frac{T}{2}\tilde{\eta}\ \epsilon^{\alpha\beta}\frac{\rho^4\sin 2\zeta}{1+\tilde{\eta}^2\rho^4\sin^2\zeta}\ \partial_\alpha\psi_1\partial_\beta\zeta, \tag{3.191}$$

$$\begin{aligned} L_s^g &= -\frac{T}{2}\gamma^{\alpha\beta}\left[\frac{(1-r^2)\partial_\alpha\phi\partial_\beta\phi}{1+\tilde{\eta}^2r^2} + \frac{\partial_\alpha r\partial_\beta r}{(1-r^2)(1+\tilde{\eta}^2r^2)} + \frac{r^2\partial_\alpha\xi\partial_\beta\xi}{1+\tilde{\eta}^2r^4\sin^2\xi}\right. \\ &\left. + \frac{r^2\cos^2\xi\ \partial_\alpha\phi_1\partial_\beta\phi_1}{1+\tilde{\eta}^2r^4\sin^2\xi} + r^2\sin^2\xi\ \partial_\alpha\phi_2\partial_\beta\phi_2\right], \end{aligned} \tag{3.192}$$

$$L_s^{WZ} = -\frac{T}{2}\tilde{\eta}\ \epsilon^{\alpha\beta}\frac{r^4\sin 2\xi}{1+\tilde{\eta}^2r^4\sin^2\xi}\ \partial_\alpha\phi_1\partial_\beta\xi, \tag{3.193}$$

where we introduced the notation

$$T = g\sqrt{1+\tilde{\eta}^2}. \tag{3.194}$$

Comparing (3.190)-(3.193) with the Polyakov string Lagrangian, one can extract the components of the background fields. They are given by

$$\begin{aligned} g_{tt} &= -\frac{1+\rho^2}{1-\tilde{\eta}^2\rho^2}, \quad g_{\rho\rho} = \frac{1}{(1+\rho^2)(1-\tilde{\eta}^2\rho^2)}, \quad g_{\zeta\zeta} = \frac{\rho^2}{1+\tilde{\eta}^2\rho^4\sin^2\zeta} \\ g_{\psi_1\psi_1} &= \frac{\rho^2\cos^2\zeta}{1+\tilde{\eta}^2\rho^4\sin^2\zeta}, \quad g_{\psi_2\psi_2} = \rho^2\sin^2\zeta, \quad b_{\psi_1\zeta} = \tilde{\eta}\frac{\rho^4\sin 2\zeta}{1+\tilde{\eta}^2\rho^4\sin^2\zeta}. \end{aligned} \tag{3.195}$$

$$g_{\phi\phi} = \frac{1-r^2}{1+\tilde{\eta}^2 r^2}, \quad g_{rr} = \frac{1}{(1-r^2)(1+\tilde{\eta}^2 r^2)}, \quad g_{\xi\xi} = \frac{r^2}{1+\tilde{\eta}^2 r^4 \sin^2\xi} \tag{3.196}$$
$$g_{\phi_1\phi_1} = \frac{r^2\cos^2\xi}{1+\tilde{\eta}^2 r^4 \sin^2\xi}. \quad g_{\phi_2\phi_2} = r^2\sin^2\xi, \quad b_{\phi_1\xi} = -\tilde{\eta}\frac{r^4\sin 2\xi}{1+\tilde{\eta}^2 r^4 \sin^2\xi}.$$

GM solutions

Since we are going to consider giant magnon solutions, we restrict ourselves to the $R_t \times S^3_\eta$ subspace, which corresponds to the following choice in AdS_η

$$\rho = 0, \quad \zeta = 0, \quad \psi_1 = \psi_2 = 0 \;\Rightarrow\; b_{\psi_1\zeta} = 0.$$

On S^5_η we first introduce the angle $\tilde{\theta}$ in the following way

$$r = \sin\tilde{\theta},$$

which leads to

$$\begin{aligned} ds^2_{S^5_\eta} &= \frac{\cos^2\tilde{\theta}}{1+\tilde{\eta}^2\sin^2\tilde{\theta}}\, d\phi^2 + \frac{d\tilde{\theta}^2}{1+\tilde{\eta}^2\sin^2\tilde{\theta}} + \frac{\sin^2\tilde{\theta}}{1+\tilde{\eta}^2\sin^4\tilde{\theta}\sin^2\xi}\, d\xi^2 \\ &+ \frac{\sin^2\tilde{\theta}\cos^2\xi}{1+\tilde{\eta}^2\sin^4\tilde{\theta}\sin^2\xi}\, d\phi_1^2 + \sin^2\tilde{\theta}\;\sin^2\xi\; d\phi_2^2, \end{aligned}$$

$$b_{\phi_1\xi} = -\tilde{\eta}\frac{\sin^4\tilde{\theta}\sin 2\xi}{1+\tilde{\eta}^2\sin^4\tilde{\theta}\sin^2\xi}.$$

Now, to go to S^3_η, we can safely set $\phi = 0$, $\tilde{\theta} = \frac{\pi}{2}$ (we also exchange ϕ_1 and ϕ_2 and replace ξ with θ). Thus, the background seen by the string moving in the $R_t \times S^3_\eta$ subspace can be written as

$$\begin{aligned} &g_{tt} = -1, \quad g_{\phi_1\phi_1} = \sin^2\theta, \quad g_{\phi_2\phi_2} = \frac{\cos^2\theta}{1+\tilde{\eta}^2\sin^2\theta}, \\ &g_{\theta\theta} = \frac{1}{1+\tilde{\eta}^2\sin^2\theta}, \quad b_{\phi_2\theta} = -\tilde{\eta}\frac{\sin 2\theta}{1+\tilde{\eta}^2\sin^2\theta}. \end{aligned} \tag{3.197}$$

Working in conformal wordsheet gauge, we impose the following ansatz for the string embedding

$$t(\tau,\sigma) = \kappa\tau, \quad \phi_i(\tau,\sigma) = \omega_i\tau + F_i(\xi), \quad \theta(\tau,\sigma) = \theta(\xi), \quad \xi = \alpha\sigma + \beta\tau,\; i = 1,2, \tag{3.198}$$

where τ and σ are the string world-sheet coordinates, $F_i(\xi)$, $\theta(\xi)$ are arbitrary functions of ξ, and $\kappa, \omega_i, \alpha, \beta$ are parameters.

Then one can find the following solutions of the equations of motion for $\phi_i(\tau,\sigma)$ (we introduced the notation $\chi \equiv \cos^2\theta$)

$$\phi_1(\tau,\sigma) = \omega_1\tau + \frac{1}{\alpha^2-\beta^2}\int d\xi \left(\frac{C_1}{1-\chi} + \beta\omega_1\right), \tag{3.199}$$

$$\phi_2(\tau,\sigma) = \omega_2\tau + \frac{1}{\alpha^2-\beta^2}\int d\xi \left[\frac{(1+\tilde{\eta}^2)C_2}{\chi} + \beta\omega_2 - \tilde{\eta}^2 C_2\right], \tag{3.200}$$

where C_1, C_2 are integration constants.

By using (3.199), (3.200), one can show that the Virasoro constraints take the form

$$\left(\frac{d\chi}{d\xi}\right)^2 = \frac{4\chi(1-\chi)\left[1+\tilde{\eta}^2(1-\chi)\right]}{(\alpha^2-\beta^2)^2}\left[(\alpha^2+\beta^2)\kappa^2 - \frac{C_1^2}{1-\chi} - C_2^2\frac{1+\tilde{\eta}^2(1-\chi)}{\chi} - \alpha^2\omega_1^2(1-\chi) - \alpha^2\omega_2^2\frac{\chi}{1+\tilde{\eta}^2(1-\chi)}\right], \tag{3.201}$$

$$\omega_1 C_1 + \omega_2 C_2 + \beta\kappa^2 = 0. \tag{3.202}$$

Next, we solve (3.202) with respect to C_1 and replace the solution into (3.201). The result is

$$\left(\frac{d\chi}{d\xi}\right)^2 = \frac{4}{(\alpha^2-\beta^2)^2}\alpha^2\tilde{\eta}^2\omega_1^2(\chi_\eta-\chi)(\chi_p-\chi)(\chi-\chi_m)(\chi-\chi_n), \tag{3.203}$$

where

$$\chi_\eta + \chi_p + \chi_m + \chi_n = -\frac{\alpha^2\left[\omega_2^2 - \omega_1^2 + \tilde{\eta}^2(\kappa^2 - 3\omega_1^2)\right] + \tilde{\eta}^2\beta^2\kappa^2 + \tilde{\eta}^4 C_2^2}{\alpha^2\tilde{\eta}^2\omega_1^2}, \tag{3.204}$$

$$\begin{aligned}&\chi_p\chi_\eta + (\chi_p+\chi_\eta)\chi_n + \chi_m(\chi_p+\chi_\eta+\chi_n) = \\ &\frac{1}{\tilde{\eta}^2\alpha^2\omega_1^4}\left\{\beta^2\kappa^2\left[\tilde{\eta}^2(\kappa^2-2\omega_1^2)-\omega_1^2\right] + 2C_2\beta\tilde{\eta}^2\kappa^2\omega_2\right. \\ &+ \alpha^2\omega_1^2\left[\left(2+3\tilde{\eta}^2\right)\omega_1^2 - \omega_2^2 - \left(1+2\tilde{\eta}^2\right)\kappa^2\right] \\ &\left.+ C_2^2\tilde{\eta}^2\left(\omega_2^2 - \left(2+3\tilde{\eta}^2\right)\omega_1^2\right)\right\},\end{aligned} \tag{3.205}$$

$$\begin{aligned}&\chi_m\chi_n\chi_p + \chi_m\chi_n\chi_\eta + \chi_m\chi_p\chi_\eta + \chi_n\chi_p\chi_\eta = \\ &-\frac{1+\tilde{\eta}^2}{\tilde{\eta}^2\alpha^2\omega_1^4}\left[C_2^2(1+3\tilde{\eta}^2)\omega_1^2 - 2C_2\beta\kappa^2\omega_2 - C_2^2\omega_2^2 - (\kappa^2-\omega_1^2)(\beta^2\kappa^2-\alpha^2\omega_1^2)\right],\end{aligned} \tag{3.206}$$

$$\chi_m\chi_n\chi_p\chi_\eta = -\frac{C_2^2(1+\tilde{\eta}^2)^2}{\tilde{\eta}^2\alpha^2\omega_1^2}. \tag{3.207}$$

The solution $\xi(\chi)$ of (3.203) is

$$\xi(\chi) = \frac{\alpha^2 - \beta^2}{\tilde{\eta}\alpha\omega_1\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \times \tag{3.208}$$
$$F\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, \frac{(\chi_p - \chi_m)(\chi_\eta - \chi_n)}{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}\right),$$

where

$$\chi_\eta > \chi_p > \chi > \chi_m > \chi_n.$$

Inverting $\xi(\chi)$ to $\chi(\xi)$, one finds

$$\chi(\xi) = \frac{\chi_\eta(\chi_p - \chi_n)\ \mathbf{dn}^2(x|m) + (\chi_\eta - \chi_p)\chi_n}{(\chi_p - \chi_n)\ \mathbf{dn}^2(x|m) + \chi_\eta - \chi_p}, \tag{3.209}$$

where

$$x = \frac{\tilde{\eta}\alpha\omega_1\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}}{\alpha^2 - \beta^2}\ \xi,$$
$$m = \frac{(\chi_p - \chi_m)(\chi_\eta - \chi_n)}{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}.$$

By using (3.203) we can find the explicit solutions for the isometric angles ϕ_1, ϕ_2. They are given by

$$\phi_1(\tau,\sigma) = \omega_1\tau + \frac{1}{\tilde{\eta}\alpha\omega_1^2(\chi_\eta - 1)\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \times \tag{3.210}$$
$$\left\{\left[\beta\left(\kappa^2 + \omega_1^2(\chi_\eta - 1) + C_2\omega_2\right)\right]\ F\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, m\right)\right.$$
$$\left. - \frac{(\chi_\eta - \chi_p)(\beta\kappa^2 + C_2\omega_2)}{1 - \chi_p}\ \Pi\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, -\frac{(\chi_\eta - 1)(\chi_p - \chi_m)}{(1 - \chi_p)(\chi_\eta - \chi_m)}, m\right)\right\}.$$

$$\phi_2(\tau,\sigma) = \omega_2\tau + \frac{1}{\tilde{\eta}\alpha\omega_1\chi_\eta\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \times \tag{3.211}$$
$$\left\{\left[C_2\left(1 - \tilde{\eta}^2(\chi_\eta - 1)\right) + \beta\omega_2\chi_\eta\right]\ F\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, m\right)\right.$$
$$+ \frac{C_2(1 + \tilde{\eta}^2)(\chi_\eta - \chi_p)}{\chi_p} \times$$
$$\left.\Pi\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, \frac{\chi_\eta(\chi_p - \chi_m)}{(\chi_\eta - \chi_m)\chi_p}, m\right)\right\}.$$

Now, let us go to the computations of the conserved charges Q_μ, i.e. the string energy E_s and the two angular momenta J_1, J_2. Starting with

$$Q_\mu = \int d\sigma \frac{\partial \mathcal{L}}{\partial\left(\partial_\tau X^\mu\right)}, \qquad X^\mu = (t, \phi_1, \phi_2),$$

and applying the ansatz (3.4), one finds

$$E_s = \frac{T}{\tilde{\eta}}\left(1 - \frac{\beta^2}{\alpha^2}\right)\frac{\kappa}{\omega_1}\int_{\chi_m}^{\chi_p}\frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}, \tag{3.212}$$

$$\begin{aligned} J_1 &= \frac{T}{\tilde{\eta}}\left[\left(1 - \frac{\beta(\beta\kappa^2 + C_2\omega_2)}{\alpha^2\omega_1^2}\right)\int_{\chi_m}^{\chi_p}\frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}\right. \\ &\left. - \int_{\chi_m}^{\chi_p}\frac{\chi d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}\right], \end{aligned} \tag{3.213}$$

$$\begin{aligned} J_2 &= \frac{T}{\tilde{\eta}^3}\left[\left(1 + \frac{1}{\tilde{\eta}^2}\right)\frac{\omega_2}{\omega_1}\int_{\chi_m}^{\chi_p}\frac{d\chi}{\left(1 + \frac{1}{\tilde{\eta}^2} - \chi\right)\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}\right. \\ &\left. - \left(\frac{\omega_2}{\omega_1} - \tilde{\eta}^2\frac{\beta C_2}{\alpha^2\omega_1}\right)\int_{\chi_m}^{\chi_p}\frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}\right]. \end{aligned} \tag{3.214}$$

We will need also the expression for the angular difference $\Delta\phi_1$. The computations give the following result

$$\begin{aligned} \Delta\phi_1 &= \frac{1}{\tilde{\eta}}\left[\frac{\beta}{\alpha}\int_{\chi_m}^{\chi_p}\frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}\right. \\ &\left. - \left(\frac{\beta\kappa^2}{\alpha\omega_1^2} + \frac{\omega_2 C_2}{\alpha\omega_1^2}\right)\int_{\chi_m}^{\chi_p}\frac{d\chi}{(1-\chi)\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}\right]. \end{aligned} \tag{3.215}$$

Solving the integrals in (3.212)-(3.215) and introducing the notations

$$v = -\frac{\beta}{\alpha},\ u = \frac{\omega_2}{\omega_1},\ W = \frac{\kappa^2}{\omega_1^2},\ K_2 = \frac{C_2}{\alpha\omega_1},\ \epsilon = \frac{(\chi_\eta - \chi_p)(\chi_m - \chi_n)}{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}, \tag{3.216}$$

we finally obtain

$$E_s = \frac{2T}{\tilde{\eta}}\frac{(1 - v^2)\sqrt{W}}{\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}}\,\mathbf{K}(1 - \epsilon), \tag{3.217}$$

$$J_1 = \frac{2T}{\tilde{\eta}\sqrt{(\chi_\eta-\chi_m)(\chi_p-\chi_n)}}\Bigg[\left(1-v^2W+K_2uv-\chi_\eta\right)\ \mathbf{K}(1-\epsilon) \tag{3.218}$$
$$+(\chi_\eta-\chi_p)\ \mathbf{\Pi}\left(\frac{\chi_p-\chi_m}{\chi_\eta-\chi_m},1-\epsilon\right)\Bigg],$$

$$J_2 = \frac{2T}{\tilde{\eta}^3\sqrt{(\chi_\eta-\chi_m)(\chi_p-\chi_n)}}\Bigg\{\frac{\left(1+\frac{1}{\tilde{\eta}^2}\right)u}{\left(1+\frac{1}{\tilde{\eta}^2}-\chi_\eta\right)}\times \tag{3.219}$$
$$\left[\mathbf{K}(1-\epsilon)-\frac{\chi_\eta-\chi_p}{1+\frac{1}{\tilde{\eta}^2}-\chi_p}\ \mathbf{\Pi}\left(\frac{(\chi_p-\chi_m)\left(1+\frac{1}{\tilde{\eta}^2}-\chi_\eta\right)}{(\chi_\eta-\chi_m)\left(1+\frac{1}{\tilde{\eta}^2}-\chi_p\right)},1-\epsilon\right)\right]$$
$$-\left(u+\tilde{\eta}^2K_2v\right)\ \mathbf{K}(1-\epsilon)\Bigg\},$$

$$\Delta\phi_1 = \frac{2}{\tilde{\eta}\sqrt{(\chi_\eta-\chi_m)(\chi_p-\chi_n)}}\times \tag{3.220}$$
$$\Bigg\{\frac{vW-K_2u}{(\chi_\eta-1)(1-\chi_p)}\Bigg[(\chi_\eta-\chi_p)\ \mathbf{\Pi}\left(-\frac{(\chi_\eta-1)(\chi_p-\chi_m)}{(\chi_\eta-\chi_m)(1-\chi_p)},1-\epsilon\right)$$
$$-(1-\chi_p)\ \mathbf{K}(1-\epsilon)\Bigg]-v\ \mathbf{K}(1-\epsilon)\Bigg\}.$$

Small ϵ-expansions and dispersion relation

In this section we restrict ourselves to the simpler case of giant magnons with one nonzero angular momentum. To this end, we set the second isometric angle $\phi_2=0$. From the solution (3.211) it is clear that ϕ_2 is zero when

$$\omega_2 = C_2 = 0,$$

or equivalently (see (3.216))

$$u = K_2 = 0.$$

Then it follows from (3.207) that $\chi_n=0$ because $\chi_\eta>\chi_p>\chi_m>0$ for the finite-size case. In addition, we express χ_m through the other parameters

$$\chi_m = \frac{\chi_\eta\chi_p}{\chi_\eta-(1-\epsilon)\chi_p}\ \epsilon.$$

As a consequence (3.204)-(3.206) take the form

$$\frac{(1-\epsilon)\chi_p^2-2\epsilon\chi_p\chi_\eta-\chi_\eta^2}{\chi_\eta-(1-\epsilon)\chi_p}+3-(1+v^2)W+\frac{1}{\tilde{\eta}^2}=0, \tag{3.221}$$

$$\chi_p\chi_\eta + \frac{\epsilon\chi_p\chi_\eta(\chi_p+\chi_\eta)}{\chi_\eta-(1-\epsilon)\chi_p} - \frac{2-(1+v^2)W+\left(3-\left(2+v^2(2-W)\right)W\right)\tilde{\eta}^2}{\tilde{\eta}^2} = 0, \quad (3.222)$$

$$\frac{\epsilon\chi_p^2\chi_\eta^2}{\chi_\eta-(1-\epsilon)\chi_p} - \frac{(1+\tilde{\eta}^2)(1-W)(1-v^2W)}{\tilde{\eta}^2} = 0. \quad (3.223)$$

In order to obtain the leading finite-size effect on the dispersion relation, we consider the limit $\epsilon \to 0$ in (3.221)-(3.223) first. We will use the following small ϵ-expansions for the remaining parameters

$$\begin{aligned} \chi_\eta &= \chi_{\eta 0} + (\chi_{\eta 1} + \chi_{\eta 2}\log\epsilon)\epsilon \\ \chi_p &= \chi_{p0} + (\chi_{p1} + \chi_{p2}\log\epsilon)\epsilon, \\ v &= v_0 + (v_1 + v_2\log\epsilon)\epsilon, \\ W &= 1 + W_1\epsilon. \end{aligned} \quad (3.224)$$

Replacing (3.224) into (3.221)-(3.223) and expanding in ϵ one finds the following solution of the resulting equations

$$\begin{aligned} &\chi_{p0} = 1-v_0^2, \quad \chi_{p1} = 1-v_0^2-2v_0v_1-\frac{(1-v_0^2)^2}{1+\tilde{\eta}^2v_0^2}, \quad \chi_{p2} = -2v_0v_2, \\ &\chi_{\eta 0} = 1+\frac{1}{\tilde{\eta}^2}, \quad \chi_{\eta 1} = \chi_{\eta 2} = 0, \\ &W_1 = -\frac{(1+\tilde{\eta}^2)(1-v_0^2)}{1+\tilde{\eta}^2v_0^2}. \end{aligned} \quad (3.225)$$

Next, we expand $\Delta\phi_1$ in ϵ and impose the condition that the resulting expression does not depend on ϵ. After using (3.225) this gives

$$\Delta\phi_1 = 2\ \mathrm{arccot}\left(v_0\sqrt{\frac{1+\tilde{\eta}^2}{1-v_0^2}}\right) \quad (3.226)$$

and two equations with solution

$$v_1 = \frac{v_0(1-v_0^2)\left[1-\log 16+\tilde{\eta}^2\left(2-v_0^2(1+\log 16)\right)\right]}{4(1+\tilde{\eta}^2v_0^2)}, \quad v_2 = \frac{1}{4}v_0(1-v_0^2). \quad (3.227)$$

Solving (3.226) with respect to v_0 one finds

$$v_0 = \frac{\cot\frac{\Delta\phi_1}{2}}{\sqrt{\tilde{\eta}^2+\csc^2\frac{\Delta\phi_1}{2}}}. \quad (3.228)$$

Now let us go to the ϵ-expansion of the difference $E_s - J_1$. Taking into account the solutions for the parameters, it can be written as

$$E_s - J_1 = 2g\sqrt{1+\tilde{\eta}^2}\left[\frac{1}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right) - \frac{(1+\tilde{\eta}^2)\sin^3\frac{p}{2}}{4\sqrt{1+\tilde{\eta}^2\sin^2\frac{p}{2}}}\,\epsilon\right]. \tag{3.229}$$

where the expression for ϵ can be found from the expansion of J_1. To the leading order, the result is

$$\epsilon = 16\ \exp\left[-\left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde{\eta}^2\sin^2\frac{p}{2}}{(1+\tilde{\eta}^2)\sin^2\frac{p}{2}}}\right]. \tag{3.230}$$

In writing (3.229), (3.230), we used (3.194) and identified the angular difference $\Delta\phi_1$ with the magnon momentum p in the dual spin chain.

For $\epsilon = 0$, (3.229) reduces to the dispersion relation for the infinite-size giant magnon obtained in [88] for the large g case. In the limit $\tilde{\eta} \to 0$, (3.229) gives the correct result for the undeformed case found in [64].

3.2 Membrane results

3.2.1 M2-brane solutions in $AdS_7 \times S^4$

In [7] different M2-brane configurations in the M-theory $AdS_7 \times S^4$ background with field theory dual $A_{N-1}(2,0)$ $SCFT$ have been considered. New membrane solutions are found and compared with the known ones. Here we will give an example of such solution chosen among the ones obtained there.

We use the following coordinates for the $AdS_7 \times S^4$ metric

$$\begin{aligned} l_p^{-2} ds^2_{AdS_7\times S^4} &= 4R^2\left\{-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\psi_1^2 + \cos^2\psi_1 d\psi_2^2 + \sin^2\psi_1 d\Omega_3^2\right)\right. \\ &+ \left.\frac{1}{4}\left[d\alpha^2 + \cos^2\alpha d\theta^2 + \sin^2\alpha\left(d\beta^2 + \cos^2\beta d\gamma^2\right)\right]\right\}, \\ d\Omega_3^2 &= d\psi_3^2 + \cos^2\psi_3 d\psi_4^2 + \cos^2\psi_3\cos^2\psi_4 d\psi_5^2, \end{aligned} \tag{3.231}$$

and embed the membrane according to the ansatz

$$\begin{aligned} &X^0(\tau,\delta,\sigma) \equiv t(\tau,\delta,\sigma) = \Lambda_1^0\delta + \Lambda_2^0\sigma + Y^0(\tau), \\ &X^1(\tau,\delta,\sigma) = Y^1(\tau) = \rho(\tau), \\ &X^2(\tau,\delta,\sigma) \equiv \psi_2(\tau,\delta,\sigma) = \Lambda_1^2\delta + \Lambda_2^2\sigma + Y^2(\tau), \\ &X^3(\tau,\delta,\sigma) \equiv \psi_5(\tau,\delta,\sigma) = \Lambda_1^3\delta + \Lambda_2^3\sigma + Y^3(\tau), \\ &X^4(\tau,\delta,\sigma) = Y^4(\tau) = \alpha(\tau), \\ &X^5(\tau,\delta,\sigma) \equiv \theta(\tau,\delta,\sigma) = \Lambda_1^5\delta + \Lambda_2^5\sigma + Y^5(\tau), \\ &X^\mu = X^{0,2,3,5}, X^a = X^{1,4}. \end{aligned} \tag{3.232}$$

Then the background seen by the membrane is ($\psi_1 = \pi/4$)

$$ds^2 = (2l_pR)^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \frac{1}{2}\sinh^2\rho\left(d\psi_2^2 + d\psi_5^2\right) + \frac{1}{4}\left(d\alpha^2 + \cos^2\alpha d\theta^2\right)\right], \quad (3.233)$$

and in our notations $X^\mu = X^{0,2,3,5}, X^a = X^{1,4}$.

Performing the necessary computations, one finds the following two first integrals of the equations of motion for $\rho(\tau)$ and $\alpha(\tau)$

$$(g_{11}\dot{\rho})^2 = \frac{(2\lambda^0)^2}{(2\pi)^4}\left[\frac{E^2}{\cosh^2\rho} + \frac{2(p_2^2 + p_3^2)}{\sinh^2\rho}\right] + (2l_pR)^6\left(\lambda^0 T_2\right)^2 \quad (3.234)$$

$$\times\left[2\left(\Delta_{02}^2 + \Delta_{03}^2\right)\cosh^2\rho - \Delta_{23}^2\sinh^2\rho\right]\sinh^2\rho - 4d \equiv F_1(\rho) \geq 0,$$

$$(g_{44}\dot{\alpha})^2 = d + (l_pR)^6\left(4\lambda^0 T_2\Delta_{05}\right)^2\cos^2\alpha - \frac{(2\lambda^0 p_5)^2}{(2\pi)^4\cos^2\alpha} \equiv F_4(\alpha) \geq 0. \quad (3.235)$$

The general solutions of the above two equations are

$$\tau(\rho) = (2l_pR)^2\int\frac{d\rho}{\sqrt{F_1(\rho)}}, \quad \tau(\alpha) = (l_pR)^2\int\frac{d\alpha}{\sqrt{F_4(\alpha)}}.$$

One can also find the orbit $\rho = \rho(\alpha)$:

$$4\int\frac{d\rho}{\sqrt{F_1(\rho)}} = \int\frac{d\alpha}{\sqrt{F_4(\alpha)}}. \quad (3.236)$$

The solutions for the M2-brane coordinates X^μ are given by

$$X^0(\rho,\alpha;\delta,\sigma) \equiv t(\rho,\alpha;\delta,\sigma) = \Lambda_1^0\left[\lambda^1\tau(\rho) + \delta\right] + \Lambda_2^0\left[\lambda^2\tau(\rho) + \sigma\right]$$
$$+\frac{2\lambda^0 E}{(2\pi)^2}\int\frac{d\rho}{\cosh^2\rho\sqrt{F_1(\rho)}} + f^0[C(\rho,\alpha)],$$

$$X^2(\rho,\alpha;\delta,\sigma) \equiv \psi_2(\rho,\alpha;\delta,\sigma) = \Lambda_1^2\left[\lambda^1\tau(\rho) + \delta\right] + \Lambda_2^2\left[\lambda^2\tau(\rho) + \sigma\right]$$
$$+\frac{4\lambda^0 p_2}{(2\pi)^2}\int\frac{d\rho}{\sinh^2\rho\sqrt{F_1(\rho)}} + f^2[C(\rho,\alpha)],$$

$$X^3(\rho,\alpha;\delta,\sigma) \equiv \psi_5(\rho,\alpha;\delta,\sigma) = \Lambda_1^3\left[\lambda^1\tau(\rho) + \delta\right] + \Lambda_2^3\left[\lambda^2\tau(\rho) + \sigma\right]$$
$$+\frac{4\lambda^0 p_3}{(2\pi)^2}\int\frac{d\rho}{\sinh^2\rho\sqrt{F_1(\rho)}} + f^3[C(\rho,\alpha)],$$

$$X^5(\rho,\alpha;\delta,\sigma) \equiv \theta(\rho,\alpha;\delta,\sigma) = \frac{1}{p_5}\left\{\left(\Lambda_1^0 E - \Lambda_1^2 p_2 - \Lambda_1^3 p_3\right)\left[\lambda^1\tau(\alpha) + \delta\right]\right.$$
$$\left. + \left(\Lambda_2^0 E - \Lambda_2^2 p_2 - \Lambda_2^3 p_3\right)\left[\lambda^2\tau(\alpha) + \sigma\right]\right\}$$
$$+\frac{2\lambda^0 p_5}{(2\pi)^2}\int\frac{d\alpha}{\cos^2\alpha\sqrt{F_4(\alpha)}} + f^5[C(\rho,\alpha)],$$

where $f^\mu[C(\rho,\alpha)]$ are arbitrary functions of $C(\rho,\alpha)$. In turn, $C(\rho,\alpha)$ is the first integral of the equation (3.236).

3.2.2 M2-brane solutions in $AdS_4 \times S^7$

The metric and the three-form gauge field are given by

$$ds^2_{AdS_4\times S^7} = l_{11}^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\alpha^2 + \sin^2\alpha d\beta^2\right) + B^2 ds_7^2\right],$$
$$b_3 = -\frac{k}{3}\sinh^3\rho\sin\alpha dt\wedge d\alpha\wedge d\beta, \quad k = const,$$

where B is the relative radius of AdS_4 with respect to the seven-sphere. We choose to parameterize S^7 as

$$\begin{aligned} ds_7^2 = 4d\xi^2 &+ \cos^2\xi\left(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi\right) \\ &+ \sin^2\xi\left(d\theta_1^2 + d\phi_1^2 + d\psi_1^2 + 2\cos\theta_1 d\phi_1 d\psi_1\right). \end{aligned}$$

Now, consider the following membrane embedding [8]

$$\begin{aligned} &X^0(\tau,\delta,\sigma) \equiv t(\tau,\delta,\sigma) = \Lambda_0^0\tau, \\ &X^1(\tau,\delta,\sigma) = Z^1(\sigma) = \rho(\sigma), \\ &X^2(\tau,\delta,\sigma) = Z^2(\sigma) = \alpha(\sigma), \\ &X^3(\tau,\delta,\sigma) \equiv \beta(\tau,\delta,\sigma) = \Lambda_0^3\tau + \Lambda_1^3\delta + \Lambda_2^3\sigma, \\ &X^4(\tau,\delta,\sigma) \equiv \phi(\tau,\delta,\sigma) = \Lambda_0^4\tau + \Lambda_1^4\delta + \Lambda_2^4\sigma, \\ &X^5(\tau,\delta,\sigma) \equiv \psi(\tau,\delta,\sigma) = \Lambda_0^5\tau + \Lambda_1^5\delta + \Lambda_2^5\sigma, \end{aligned}$$

where in our notations $\mu = 0,3,4,5$, $a = 1,2$. The relevant background seen by the membrane is

$$\begin{aligned} ds^2 &= l_{11}^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\alpha^2 + \sin^2\alpha d\beta^2\right)\right. \\ &+ \left. B^2\left(d\phi^2 + d\psi^2 + 2d\phi d\psi\right)\right] \\ &b_{023} = -\frac{k}{3}\sinh^3\rho\sin\alpha. \end{aligned}$$

For simplicity, we choose to work in diagonal worldvolume gauge $\lambda^i = 0$ in which we must have $G_{0i} = 0$. There exist four types of solutions for these constraints, for the membrane embedding used:

$$\Lambda_i^3 = 0, \quad \Lambda_i^5 = -\Lambda_i^4, \tag{3.237}$$
$$\Lambda_i^3 = 0, \quad \Lambda_0^5 = -\Lambda_0^4, \tag{3.238}$$
$$\Lambda_0^3 = 0, \quad \Lambda_i^5 = -\Lambda_i^4,$$
$$\Lambda_0^3 = 0, \quad \Lambda_0^5 = -\Lambda_0^4.$$

Let us note that in the first two cases, which will be considered here, the induced B-field is zero, while for the last two, it is not.

Working in the framework of ansatz (3.237), one obtains that $\det G_{mn} = \mathbf{G} = 0$, i.e. this case corresponds to tensionless membrane. Then one finds that the membrane trajectory $\rho = \rho(\alpha)$ is given by

$$\sinh\rho(\alpha) = \left[\frac{(B/\Lambda_0^0)^2(\Lambda_0^4+\Lambda_0^5)^2 - 1}{1-(\Lambda_0^3/\Lambda_0^0)^2\sin^2\alpha}\right]^{1/2}.$$

In the case under consideration the conserved charges are connected with each other by the equality

$$\Lambda_0^0 E = \Lambda_0^3 S + \frac{2\lambda^0}{l_{11}^2 B^2}J^2,$$

where E, S and J are the membrane energy, spin and angular momentum respectively.

In the framework of ansatz (3.238), one obtains that the conserved quantities read

$$E = \frac{l_{11}^2}{2\lambda^0}\Lambda_0^0\int d^2\xi\cosh^2\rho,$$
$$S = \frac{l_{11}^2}{2\lambda^0}\Lambda_0^3\int d^2\xi\sinh^2\rho\sin^2\alpha,$$
$$J = 0.$$

Taking $\alpha = 0$, we find the solution

$$\rho(\sigma) = \ln\tan(\sigma/2A_\rho), \quad A_\rho = 2\lambda^0 T_2 l_{11} B\frac{\Lambda_1^4+\Lambda_1^5}{\Lambda_0^0}.$$

For $\alpha = \alpha_0 \neq 0, \pi$, there exist two different solutions:

$$\rho(\sigma) = \frac{1}{2}\ln\frac{1+\mathbf{sn}(\sigma/A_\rho)}{1-\mathbf{sn}(\sigma/A_\rho)}, \quad k^2 = 1-\left(\Lambda_0^3/\Lambda_0^0\right)^2\sin^2\alpha_0 \in (0,1);$$
$$\tanh\rho(\sigma) = \frac{1}{\sqrt{1+k^2}}\mathbf{sn}\left(\frac{\sqrt{1+k^2}}{A_\rho}\sigma\right), \quad k^2 = \left(\Lambda_0^3/\Lambda_0^0\right)^2\sin^2\alpha_0 - 1 \in (0,1),$$

Fixing $\rho = \rho_0 \neq 0$, one obtains

$$\alpha(\sigma) = \arcsin\left[\mathbf{sn}(\sigma/A_\alpha)\right], \quad \alpha \in (-\pi/2, \pi/2),$$
$$A_\alpha = A_\rho\tanh\rho_0, \quad k^2 = \left(\Lambda_0^3/\Lambda_0^0\right)^2\tanh^2\rho_0 \in (0,1).$$

Now, let us turn to the general case, when none of the coordinates ρ and α are kept fixed. In order to be able to give *explicit* solution, we set $\Lambda_0^3 = 0$ and find

$$\cosh\rho(\sigma) = \frac{A}{\mathbf{cn}(C\sigma)}.$$

$$A=\sqrt{\frac{1}{2}\left[1+\sqrt{1+\left(\frac{2d}{l_{11}\Lambda_0^0 K}\right)^2}\right]},\quad C=\frac{l_{11}\Lambda_0^0}{K}\left[1+\left(\frac{2d}{l_{11}\Lambda_0^0 K}\right)^2\right]^{1/4},$$

and d is an arbitrary constant. The solution for the membrane trajectory is the following

$$\alpha(\rho)=\frac{d}{(A^2-1)CK^2}\left[A^2\Pi\left(\varphi,-\frac{1}{A^2-1},k\right)-\left(A^2-1\right)F(\varphi,k)\right],$$

where

$$\varphi=\arccos\left(\frac{A}{\cosh\rho}\right),\quad k=\left[1+\left(\frac{2d}{l_{11}\Lambda_0^0 K}\right)^2\right]^{-1/4}\sqrt{\frac{1}{2}\left[\sqrt{1+\left(\frac{2d}{l_{11}\Lambda_0^0 K}\right)^2}-1\right]}.$$

The condition $\Lambda_0^3=0$ leads to $S=0$, and the membrane energy E remains the only nontrivial conserved quantity. On the obtained solution, it is given by

$$E=A^2E_0+\frac{\pi CK^2}{2\lambda^0\Lambda_0^0}\left[\frac{\mathbf{sn}(2\pi C)\mathbf{dn}(2\pi C)}{\mathbf{cn}(2\pi C)}-\mathbf{E}(k)\right],\quad E_0=E_{\rho=0}.$$

3.2.3 Exact rotating membrane solutions on a G_2 manifold and their semiclassical limits

We obtain exact rotating membrane solutions and explicit expressions for the conserved charges on a manifold with exactly known metric of G_2 holonomy in M-theory, with four dimensional $\mathcal{N}=1$ gauge theory dual. After that, we investigate their semiclassical limits and derive different relations between the energy and the other conserved quantities, which is a step towards M-theory lift of the semiclassical string/gauge theory correspondence for $\mathcal{N}=1$ field theories [9].

To our knowledge, the only paper devoted to rotating membranes on G_2 manifolds at that time is [52], where various membrane configurations on different G_2 holonomy backgrounds have been studied systematically, but not exactly. In the semiclassical limit (large conserved charges), the following relations between the energy and the corresponding conserved charge K have been obtained: $E\sim K^{1/2}$, $E\sim K^{2/3}$, $E-K\sim K^{1/3}$, $E-K\sim\ln K$.

Here, our approach will be different. Taking into account that only a small number of G_2 holonomy metrics are known *exactly*, we choose to search for rotating membrane solutions on one of these metrics. Namely, the one discovered in [53]. First, we describe the G_2 holonomy background of [53]. Second, we obtain a number of exact rotating membrane solutions and the explicit expressions for the corresponding conserved charges. Then, we take the semiclassical limit and derive different energy-charge relations. They reproduce and generalize part of the results obtained in [52], for the case of more than two conserved quantities.

The background we are interested in is a one-parameter family of G_2 holonomy metrics (parameterized by r_0), which play an important role as supergravity dual of the large N limit of four dimensional $\mathcal{N}=1$ SYM. These metrics describe the M theory lift of the supergravity solution corresponding to a collection of D6-branes wrapping the supersymmetric three-cycle of the deformed conifold geometry for any value of the string coupling constant. The explicit expression for the metric with $SU(2)\times SU(2)\times U(1)\times Z_2$ symmetry is given by [53]

$$ds_7^2=\sum_{a=1}^{7}e^a\otimes e^a, \tag{3.239}$$

with the following vielbeins

$$\begin{aligned}
e^1 &= A(r)(\sigma_1-\Sigma_1)\ ,\quad e^2=A(r)(\sigma_2-\Sigma_2)\ ,\\
e^3 &= D(r)(\sigma_3-\Sigma_3)\ ,\quad e^4=B(r)(\sigma_1+\Sigma_1)\ ,\\
e^5 &= B(r)(\sigma_2+\Sigma_2)\ ,\quad e^6=r_0C(r)(\sigma_3+\Sigma_3)\ ,\\
e^7 &= dr/C(r),
\end{aligned} \tag{3.240}$$

where

$$\begin{aligned}
A &= \frac{1}{\sqrt{12}}\sqrt{(r-3r_0/2)(r+9r_0/2)},\qquad B=\frac{1}{\sqrt{12}}\sqrt{(r+3r_0/2)(r-9r_0/2)},\\
C &= \sqrt{\frac{(r-9r_0/2)(r+9r_0/2)}{(r-3r_0/2)(r+3r_0/2)}},\qquad D=r/3,
\end{aligned} \tag{3.241}$$

and

$$\begin{aligned}
&\sigma_1=\sin\psi\sin\theta d\phi+\cos\psi d\theta, &&\Sigma_1=\sin\tilde\psi\sin\tilde\theta d\tilde\phi+\cos\tilde\psi d\tilde\theta,\\
&\sigma_2=\cos\psi\sin\theta d\phi-\sin\psi d\theta, &&\Sigma_2=\cos\tilde\psi\sin\tilde\theta d\tilde\phi-\sin\tilde\psi d\tilde\theta,\\
&\sigma_3=\cos\theta d\phi+d\psi, &&\Sigma_3=\cos\tilde\theta d\tilde\phi+d\tilde\psi.
\end{aligned} \tag{3.242}$$

This metric is Ricci flat and complete for $r\geq 9r_0/2$. It has a G_2-structure given by the following covariantly constant three-form

$$\begin{aligned}
\Phi &= \frac{9r_0^3}{16}\epsilon_{abc}\ (\sigma_a\wedge\sigma_b\wedge\sigma_c-\Sigma_a\wedge\Sigma_b\wedge\Sigma_c)\\
&+\ d\left[\frac{r}{18}\left(r^2-\frac{27r_0^2}{4}\right)(\sigma_1\wedge\Sigma_1+\sigma_2\wedge\Sigma_2)+\frac{r_0}{3}\left(r^2-\frac{81r_0^2}{8}\right)\sigma_3\wedge\Sigma_3\right],
\end{aligned}$$

which guarantees the existence of a unique covariantly constant spinor [53].

The M-theory background for our case can be written as

$$l_{11}^{-2}ds_{11}^2=-dt^2+\delta_{IJ}dx^Idx^J+ds_7^2, \tag{3.243}$$

where l_{11} is the eleven dimensional Planck length, (*I,J=1,2,3*) and ds_7^2 is given in (3.239)-(3.242). In other words, the background is direct product of flat, four dimensional space-time, and a seven dimensional G_2 manifold.

We will search for solutions, for which the background felt by the membrane depends on only one coordinate. This will be the radial coordinate r, i.e. the rotating membrane embedding along this coordinate has the form $r = r(\sigma)$. Then, the remaining membrane coordinates, which are not fixed, will depend linearly on the worldvolume coordinates τ, δ and σ. The membrane configurations considered below are all for which, we were able to obtain *exact* solutions under the described conditions.

First type of membrane embedding [9]

Let us consider the following membrane configuration:

$$X^0 \equiv t = \Lambda_0^0\tau + \frac{1}{\Lambda_0^0}\left[(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)\,\delta + (\mathbf{\Lambda}_0.\mathbf{\Lambda}_2)\,\sigma\right], \quad X^I = \Lambda_0^I\tau + \Lambda_1^I\delta + \Lambda_2^I\sigma,$$
$$X^4 \equiv r(\sigma), \quad X^6 \equiv \theta = \Lambda_0^6\tau, \quad X^9 \equiv \tilde{\theta} = \Lambda_0^9\tau; \quad (\mathbf{\Lambda}_0.\mathbf{\Lambda}_i) = \delta_{IJ}\Lambda_0^I\Lambda_i^J. \tag{3.244}$$

It corresponds to membrane extended in the radial direction r, and rotating in the planes given by the angles θ and $\tilde{\theta}$. In addition, it is nontrivially spanned along X^0 and X^I. The relations between the parameters in X^0 and X^I guarantee that the constraints are identically satisfied. At the same time, the membrane moves along t-coordinate with constant energy E, and along X^I with constant momenta P_I. In this case, the target space metric seen by the membrane becomes

$$g_{00} \equiv g_{tt} = -l_{11}^2, \quad g_{IJ} = l_{11}^2\delta_{IJ}, \quad g_{44} \equiv g_{rr} = \frac{l_{11}^2}{C^2(r)},$$
$$g_{66} \equiv g_{\theta\theta} = l_{11}^2\left[A^2(r) + B^2(r)\right], \quad g_{99} \equiv g_{\tilde{\theta}\tilde{\theta}} = l_{11}^2\left[A^2(r) + B^2(r)\right],$$
$$g_{69} \equiv g_{\theta\tilde{\theta}} = -l_{11}^2\left[A^2(r) - B^2(r)\right]. \tag{3.245}$$

Therefore, in our notations, we have $\mu = (0, I, 6, 9) \equiv (t, I, \theta, \tilde{\theta})$, $a = 4 \equiv r$. The metric induced on the membrane worldvolume is

$$G_{00} = -l_{11}^2\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^-)^2A^2 - (\Lambda_0^+)^2B^2\right],$$
$$G_{11} = l_{11}^2M_{11}, \quad G_{12} = l_{11}^2M_{12}, \quad G_{22} = l_{11}^2\left[M_{22} + \frac{r'^2}{C^2}\right],$$

where

$$M_{ij} = (\mathbf{\Lambda}_i.\mathbf{\Lambda}_j) - \frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_i)\,(\mathbf{\Lambda}_0.\mathbf{\Lambda}_j)}{\left(\Lambda_0^0\right)^2}, \quad \Lambda_0^{\pm} = \Lambda_0^6 \pm \Lambda_0^9. \tag{3.246}$$

The constants of the motion $\mathcal{P}_\mu^2$, introduced in [8], are given by

$$\mathcal{P}_0^2 = -\frac{2\lambda^0T_2^2l_{11}^4}{\Lambda_0^0}\left[(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)\,M_{12} - (\mathbf{\Lambda}_0.\mathbf{\Lambda}_2)\,M_{11}\right], \tag{3.247}$$
$$\mathcal{P}_I^2 = 2\lambda^0T_2^2l_{11}^4\left(\Lambda_1^IM_{12} - \Lambda_2^IM_{11}\right), \quad \mathcal{P}_6^2 = \mathcal{P}_9^2 = 0.$$

The membrane Lagrangian takes the form

$$\mathcal{L}^A(\sigma) = \frac{1}{4\lambda^0}\left(K_{rr}r'^2 - V\right), \qquad K_{rr} = -(2\lambda^0 T_2 l_{11}^2)^2\frac{M_{11}}{C^2},$$
$$V = (2\lambda^0 T_2 l_{11}^2)^2 \det M_{ij} + l_{11}^2\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^-)^2 A^2 - (\Lambda_0^+)^2 B^2\right].$$

Let us first consider the particular case when $\Lambda_0^- = 0$, i.e. $\theta = \tilde{\theta}$. From the first integral of the equation of motion for $r(\sigma)$

$$K_{rr}r'^2 + U = 0, \qquad U = V + 4\lambda^0\Lambda_2^\mu \mathcal{P}_\mu^2,$$

one obtains the turning points of the effective one-dimensional periodic motion by solving the equation $r' = 0$. In the case under consideration, the result is

$$r_{min} = 3l, \qquad r_{max} = r_1 = l\left(2\sqrt{1 + \frac{3u_0^2}{l^2(\Lambda_0^+)^2}} + 1\right) > 3l,$$
$$r_2 = -l\left(2\sqrt{1 + \frac{3u_0^2}{l^2(\Lambda_0^+)^2}} - 1\right) < 0, \qquad l = 3r_0/2,$$

where we have introduced the notation

$$\begin{aligned} u_0^2 &= (2\lambda^0 T_2 l_{11})^2 \det M_{ij} + (\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 + 4\lambda^0\Lambda_2^\mu \mathcal{P}_\mu^2/l_{11}^2 \\ &= (\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (2\lambda^0 T_2 l_{11})^2 \det M_{ij}. \end{aligned} \tag{3.248}$$

Now, we can write down the following expression for the membrane solution ($\Delta r = r - 3l$)

$$\sigma(r) = \int_{3l}^{r}\left[-\frac{K_{rr}(t)}{U(t)}\right]^{1/2} dt = \frac{16\lambda^0 T_2 l_{11}}{\Lambda_0^+}\left[\frac{M_{11} l\Delta r}{(r_1 - 3l)(3l - r_2)}\right]^{1/2} \times$$
$$F_D^{(5)}\left(1/2; -1/2, -1/2, 1/2, 1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{r_1 - 3l}\right), \tag{3.249}$$

where the following normalization condition must be satisfied ($\Delta r_1 = r_1 - 3l$) [9]

$$2\pi = 2\int_{3l}^{r_1}\left[-\frac{K_{rr}(t)}{U(t)}\right]^{1/2} dt = \frac{32\lambda^0 T_2 l_{11}(M_{11} l)^{1/2}}{\Lambda_0^+ (3l - r_2)^{1/2}} \times$$
$$F_D^{(5)}\left(1/2; -1/2, -1/2, 1/2, 1/2, 1/2; 3/2; -\frac{\Delta r_1}{2l}, -\frac{\Delta r_1}{4l}, -\frac{\Delta r_1}{6l}, -\frac{\Delta r_1}{3l - r_2}, 1\right) =$$
$$\frac{16\pi\lambda^0 T_2 l_{11}(M_{11} l)^{1/2}}{\Lambda_0^+ (3l - r_2)^{1/2}} F_D^{(4)}\left(1/2; -1/2, -1/2, 1/2, 1/2, ; 1; -\frac{\Delta r_1}{2l}, -\frac{\Delta r_1}{4l}, -\frac{\Delta r_1}{6l}, -\frac{\Delta r_1}{3l - r_2}\right)$$
$$= \frac{16\pi\lambda^0 T_2 l_{11}(M_{11} l)^{1/2}}{\Lambda_0^+ (3l - r_2)^{1/2}}\left(1 + \frac{\Delta r_1}{2l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{4l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{6l}\right)^{-1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2}$$
$$\times F_D^{(4)}\left(1/2; -1/2, -1/2, 1/2, 1/2, ; 1; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right). \tag{3.250}$$

Now, we can compute the conserved charges on the obtained solution. They are:

$$E = -P_0 = \frac{\pi^2 l_{11}^2}{\lambda^0}\Lambda_0^0, \quad \mathbf{P} = \frac{\pi^2 l_{11}^2}{\lambda^0}\mathbf{\Lambda}_0, \tag{3.251}$$

$$P_\theta = P_{\tilde{\theta}} = \frac{\pi l_{11}^2}{\lambda^0}\Lambda_0^+ \int_{3l}^{r_1}\left[-\frac{K_{rr}(t)}{U(t)}\right]^{1/2} B^2(t)dt = \frac{4\pi^2 T_2 l_{11}^3\left(M_{11}l^3\right)^{1/2}}{3\left(3l-r_2\right)^{1/2}} \times$$
$$\Delta r_1 F_D^{(4)}\left(3/2;-1/2,-3/2,1/2,1/2,;2;-\frac{\Delta r_1}{2l},-\frac{\Delta r_1}{4l},-\frac{\Delta r_1}{6l},-\frac{\Delta r_1}{3l-r_2}\right)$$
$$= \frac{4\pi^2 T_2 l_{11}^3\left(M_{11}l^3\right)^{1/2}}{3\left(3l-r_2\right)^{1/2}}\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}$$
$$\times F_D^{(4)}\left(1/2;-1/2,-3/2,1/2,1/2,;2;\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right). \tag{3.252}$$

Our next task is to find the relation between the energy E and the other conserved quantities $\mathbf{P}$, $P_\theta = P_{\tilde{\theta}}$ in the semiclassical limit (large conserved charges). This corresponds to $r_1 \to \infty$, which in the present case leads to $3u_0^2/[l^2(\Lambda_0^+)^2] \to \infty$. In this limit, the condition (3.250) reduces to

$$\Lambda_0^+ = 2\sqrt{3}\lambda^0 T_2 l_{11} M_{11}^{1/2},$$

while the expression (3.252) for the momentum P_θ, takes the form

$$P_\theta = P_{\tilde{\theta}} = \sqrt{3}\pi^2 T_2 l_{11}^3 M_{11}^{1/2}\frac{u_0^2}{(\Lambda_0^+)^2}.$$

Combining these results with (3.21), one obtains

$$\left\{E^2\left(E^2-\mathbf{P}^2\right) - (2\pi^2 T_2 l_{11}^3)^2\left\{(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)^2 E^2 - [(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\times\mathbf{P}]^2\right\}\right\}^2 \tag{3.253}$$
$$-(4\sqrt{3}\pi^2 T_2 l_{11}^3)^2 E^2\left[\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1.\mathbf{P})^2\right]P_\theta^2 = 0, \quad (\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)_I = \varepsilon_{IJK}\Lambda_1^J\Lambda_2^K.$$

This is *fourth* order algebraic equation for E^2. Its positive solutions give the explicit dependence of the energy on $\mathbf{P}$ and P_θ: $E^2 = E^2(\mathbf{P}, P_\theta)$.

Let us consider a few particular cases. In the simplest case, when $\Lambda_0^I = 0$, i.e. $\mathbf{P} = 0$, and $\Lambda_2^I = c\Lambda_1^I$, which corresponds to the membrane embedding (see (3.244))

$$X^0 \equiv t = \Lambda_0^0\tau, \quad X^I = \Lambda_1^I(\delta + c\sigma), \quad X^4 \equiv r(\sigma), \quad X^6 \equiv \theta = \Lambda_0^6\tau = X^9 \equiv \tilde{\theta} = \Lambda_0^9\tau,$$

(3.253) simplifies to

$$E^2 = 4\sqrt{3}\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_\theta. \tag{3.254}$$

This is the relation $E \sim K^{1/2}$ obtained for G_2-manifolds in [27]. If we impose only the conditions $\Lambda_0^I = 0$, and Λ_i^I remain independent, (3.253) gives

$$E^2 = (2\pi^2 T_2 l_{11}^3)^2(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)^2 + 4\sqrt{3}\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_\theta. \tag{3.255}$$

Now, let us take $\Lambda_0^I \neq 0$, $\Lambda_2^I = c\Lambda_1^I$. Then, (3.253) reduces to

$$E^2 \left[\left(E^2 - \mathbf{P}^2\right)^2 - (4\sqrt{3}\pi^2 T_2 l_{11}^3)^2 \mathbf{\Lambda}_1^2 P_\theta^2\right] + (4\sqrt{3}\pi^2 T_2 l_{11}^3)^2 \left(\mathbf{\Lambda}_1.\mathbf{P}\right)^2 P_\theta^2 = 0,$$

which is *third* order algebraic equation for E^2. If the three-dimensional vectors $\mathbf{\Lambda}_1$ and $\mathbf{P}$ are orthogonal to each other, i.e. $(\mathbf{\Lambda}_1.\mathbf{P}) = 0$, the above relation simplifies to

$$E^2 = \mathbf{P}^2 + 4\sqrt{3}\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_\theta. \tag{3.256}$$

The obvious conclusion is that in the framework of a given embedding, one can obtain different relations between the energy and the other conserved charges, depending on the choice of the embedding parameters.

Now, we will consider the general case, when $\Lambda_0^- \neq 0$, i.e. $\theta \neq \tilde{\theta}$. The turning points are given by

$$r_{min} = 3l, \quad r_{max} = r_1 = l\left[2\sqrt{\frac{k^2+3}{4} + \frac{3u_0^2}{l^2\left((\Lambda_0^+)^2 + (\Lambda_0^-)^2\right)}} + k\right],$$
$$r_2 = -l\left[2\sqrt{\frac{k^2+3}{4} + \frac{3u_0^2}{l^2\left((\Lambda_0^+)^2 + (\Lambda_0^-)^2\right)}} - k\right], \quad k = \frac{(\Lambda_0^+)^2 - (\Lambda_0^-)^2}{(\Lambda_0^+)^2 + (\Lambda_0^-)^2} \in [0,1].$$

Now the solution for $\sigma(r)$ is

$$\sigma(r) = \int_{3l}^{r}\left[-\frac{K_{rr}(t)}{U(t)}\right]^{1/2} dt = \frac{16\lambda^0 T_2 l_{11}}{\left[(\Lambda_0^+)^2 + (\Lambda_0^-)^2\right]^{1/2}}\left[\frac{M_{11} l \Delta r}{(r_1 - 3l)(3l - r_2)}\right]^{1/2} \times$$
$$F_D^{(5)}\left(1/2; -1/2, -1/2, 1/2, 1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{r_1 - 3l}\right). \tag{3.257}$$

The normalization condition reads

$$\frac{8\lambda^0 T_2 l_{11} \left(M_{11} l\right)^{1/2}}{\left[(\Lambda_0^+)^2 + (\Lambda_0^-)^2\right]^{1/2}(3l - r_2)^{1/2}} \times$$
$$F_D^{(4)}\left(1/2; -1/2, -1/2, 1/2, 1/2, ; 1; -\frac{\Delta r_1}{2l}, -\frac{\Delta r_1}{4l}, -\frac{\Delta r_1}{6l}, -\frac{\Delta r_1}{3l - r_2}\right) =$$
$$\frac{8\lambda^0 T_2 l_{11} \left(M_{11} l\right)^{1/2}}{\left[(\Lambda_0^+)^2 + (\Lambda_0^-)^2\right]^{1/2}(3l - r_2)^{1/2}} \times$$
$$\left(1 + \frac{\Delta r_1}{2l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{4l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{6l}\right)^{-1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times$$
$$F_D^{(4)}\left(1/2; -1/2, -1/2, 1/2, 1/2, ; 1; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right) = 1. \tag{3.258}$$

Computing the conserved momenta, one obtains the same expressions for E and $\mathbf{P}$ as in (3.21)[14],

[14]Actually, these expressions for E and $\mathbf{P}$ are always valid for the background we use.

and

$$\frac{1}{2}\left(P_{\theta}+P_{\tilde{\theta}}\right)=\frac{4\pi^2T_2l_{11}^3\Lambda_0^+\left(M_{11}l^3\right)^{1/2}}{3\left[(\Lambda_0^+)^2+(\Lambda_0^-)^2\right]^{1/2}(3l-r_2)^{1/2}}\times$$
$$\Delta r_1 F_D^{(4)}\left(3/2;-1/2,-3/2,1/2,1/2,;2;-\frac{\Delta r_1}{2l},-\frac{\Delta r_1}{4l},-\frac{\Delta r_1}{6l},-\frac{\Delta r_1}{3l-r_2}\right)$$
$$=\frac{4\pi^2T_2l_{11}^3\Lambda_0^+\left(M_{11}l^3\right)^{1/2}}{3\left[(\Lambda_0^+)^2+(\Lambda_0^-)^2\right]^{1/2}(3l-r_2)^{1/2}}\times$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(4)}\left(1/2;-1/2,-3/2,1/2,1/2,;2;\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),\quad(3.259)$$

$$\frac{1}{2}\left(P_{\theta}-P_{\tilde{\theta}}\right)=\frac{8\pi^2T_2l_{11}^3\Lambda_0^-\left(M_{11}l^5\right)^{1/2}}{\left[(\Lambda_0^+)^2+(\Lambda_0^-)^2\right]^{1/2}(3l-r_2)^{1/2}}\times$$
$$F_D^{(4)}\left(1/2;-3/2,-1/2,-1/2,1/2,;1;-\frac{\Delta r_1}{2l},-\frac{\Delta r_1}{4l},-\frac{\Delta r_1}{6l},-\frac{\Delta r_1}{3l-r_2}\right)$$
$$=\frac{8\pi^2T_2l_{11}^3\Lambda_0^-\left(M_{11}l^5\right)^{1/2}}{\left[(\Lambda_0^+)^2+(\Lambda_0^-)^2\right]^{1/2}(3l-r_2)^{1/2}}\times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(4)}\left(1/2;-3/2,-1/2,-1/2,1/2,;1;\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).\quad(3.260)$$

Now, we go to the semiclassical limit $r_1\to\infty$. The normalization condition gives

$$\left[(\Lambda_0^+)^2+(\Lambda_0^-)^2\right]^{1/2}=2\sqrt{3}\lambda^0T_2l_{11}M_{11}^{1/2},$$

whereas (3.259) and (3.260) take the form

$$\frac{1}{2}\left(P_{\theta}\pm P_{\tilde{\theta}}\right)=\frac{\sqrt{3}\pi^2T_2l_{11}^3\Lambda_0^{\pm}M_{11}^{1/2}u_0^2}{\left[(\Lambda_0^+)^2+(\Lambda_0^-)^2\right]^{3/2}}.$$

The above expressions, together with (3.21), lead to the following connection between the energy and the conserved momenta

$$\left\{E^2\left(E^2-\mathbf{P}^2\right)-(2\pi^2T_2l_{11}^3)^2\left\{\left(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2\right)^2E^2-\left[(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\times\mathbf{P}\right]^2\right\}\right\}^2\quad(3.261)$$
$$-6(2\pi^2T_2l_{11}^3)^2E^2\left[\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2\right]\left(P_{\theta}^2+P_{\tilde{\theta}}^2\right)=0.$$

Obviously, (3.261) is the generalization of (3.253) for the case $P_\theta \neq P_{\tilde\theta}$ and for $P_\theta = P_{\tilde\theta}$ coincides with it, as it should be. The particular cases (3.254), (3.255) and (3.256) now generalize to

$$\begin{aligned}
&E^2 = 2\sqrt{6}\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid \left(P_\theta^2 + P_{\tilde\theta}^2\right)^{1/2},\\
&E^2 = (2\pi^2 T_2 l_{11}^3)^2 \left(\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2\right)^2 + 2\sqrt{6}\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid \left(P_\theta^2 + P_{\tilde\theta}^2\right)^{1/2},\\
&E^2 = \mathbf{P}^2 + 2\sqrt{6}\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid \left(P_\theta^2 + P_{\tilde\theta}^2\right)^{1/2}. \qquad (3.262)
\end{aligned}$$

Finally, let us give the semiclassical limit of the membrane solution (3.257), which is

$$\begin{aligned}
\sigma_{scl}(r) &= \left\{\frac{32(4\pi^2 T_2 l_{11}^3)^2\left[\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1.\mathbf{P})^2\right]}{27E^2\left(P_\theta^2 + P_{\tilde\theta}^2\right)}\right\}^{1/4} (l\Delta r)^{1/2} \qquad (3.263)\\
&\times\ F_D^{(3)}\left(1/2; -1/2, -1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}\right)\\
&= \left\{\frac{32(4\pi^2 T_2 l_{11}^3)^2\left[\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1.\mathbf{P})^2\right]}{27E^2\left(P_\theta^2 + P_{\tilde\theta}^2\right)}\right\}^{1/4} (l\Delta r)^{1/2}\\
&\times \left(1 + \frac{\Delta r}{2l}\right)^{1/2}\left(1 + \frac{\Delta r}{4l}\right)^{1/2}\left(1 + \frac{\Delta r}{6l}\right)^{-1/2}\\
&\quad F_D^{(3)}\left(1; -1/2, -1/2, 1/2, ; 3/2; \frac{1}{1 + \frac{2l}{\Delta r}}, \frac{1}{1 + \frac{4l}{\Delta r}}, \frac{1}{1 + \frac{6l}{\Delta r}}\right).
\end{aligned}$$

Second type of membrane embedding [9]

Let us consider membrane, which is extended along the radial direction r and rotates in the planes defined by the angles θ and $\tilde\theta$, with angular momenta P_θ and $P_{\tilde\theta}$. Now we want to have nontrivial wrapping along X^6 and X^9. The embedding parameters in X^6 and X^9 have to be chosen in such a way that the constraints are satisfied identically. It turns out that the angular momenta P_θ and $P_{\tilde\theta}$ must be equal, and the constants of the motion $\mathcal{P}_\mu^2$ are identically zero for this case. In addition, we want the membrane to move along X^0 and X^I with constant energy E and constant momenta P_I respectively. All this leads to the following ansatz:

$$\begin{aligned}
&X^0 \equiv t = \Lambda_0^0\tau, \quad X^I = \Lambda_0^I\tau, \quad X^4 \equiv r(\sigma),\\
&X^6 \equiv \theta = \Lambda_0^6\tau + \Lambda_1^6\delta + \Lambda_2^6\sigma, \quad X^9 \equiv \tilde\theta = \Lambda_0^6\tau - (\Lambda_1^6\delta + \Lambda_2^6\sigma). \qquad (3.264)
\end{aligned}$$

The background felt by the membrane is the same as in (3.245), but the metric induced on the membrane worldvolume is different and is given by

$$\begin{aligned}
&G_{00} = -l_{11}^2\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^+)^2 B^2\right], \quad G_{11} = 4l_{11}^2(\Lambda_1^6)^2 A^2,\\
&G_{12} = 4l_{11}^2\Lambda_1^6\Lambda_2^6 A^2, \quad G_{22} = l_{11}^2\left[\frac{r'^2}{C^2} + 4(\Lambda_2^6)^2 A^2\right].
\end{aligned}$$

For the present case, the membrane Lagrangian reduces to

$$\mathcal{L}^A(\sigma) = \frac{1}{4\lambda^0}\left(K_{rr}r'^2 - V\right), \quad K_{rr} = -(4\lambda^0 T_2 l_{11}^2)^2 (\Lambda_1^6)^2 \frac{A^2}{C^2},$$
$$V = U = l_{11}^2\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^+)^2 B^2\right].$$

The turning points of the effective one-dimensional periodic motion

$$K_{rr}r'^2 + V = 0,$$

are given by

$$r_{min} = 3l, \quad r_{max} = r_1 = l\left(2\sqrt{1+\frac{3v_0^2}{l^2(\Lambda_0^+)^2}}+1\right) > 3l,$$
$$r_2 = -l\left(2\sqrt{1+\frac{3v_0^2}{l^2(\Lambda_0^+)^2}}-1\right) < 0, \quad v_0^2 = (\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2. \tag{3.265}$$

Now, the membrane solution reads:

$$\sigma(r) = \int_{3l}^{r}\left[-\frac{K_{rr}(t)}{V(t)}\right]^{1/2} dt = \frac{32\lambda^0 T_2 l_{11}\Lambda_1^6}{\Lambda_0^+}\left[\frac{l^3 \Delta r}{(r_1-3l)(3l-r_2)}\right]^{1/2} \times$$
$$F_D^{(4)}\left(1/2; -1, -1/2, 1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{3l-r_2}, \frac{\Delta r}{r_1-3l}\right). \tag{3.266}$$

The normalization condition leads to the following relation between the parameters

$$\frac{16\lambda^0 T_2 l_{11}\Lambda_1^6 l^{3/2}}{\Lambda_0^+ (3l-r_2)^{1/2}} F_D^{(3)}\left(1/2; -1, -1/2, 1/2; 1; -\frac{\Delta r_1}{2l}, -\frac{\Delta r_1}{4l}, -\frac{\Delta r_1}{3l-r_2}\right)$$
$$= \frac{16\lambda^0 T_2 l_{11}\Lambda_1^6 l^{3/2}}{\Lambda_0^+ (3l-r_2)^{1/2}}\left(1+\frac{\Delta r_1}{2l}\right)\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}$$
$$\times F_D^{(3)}\left(1/2; -1, -1/2, 1/2, ; 1; \frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right) = 1. \tag{3.267}$$

In the case under consideration, the conserved quantities are E, $\mathbf{P}$ and $P_\theta = P_{\tilde{\theta}}$. We derive the following result for $P_\theta = P_{\tilde{\theta}}$

$$P_\theta = P_{\tilde{\theta}} = \frac{8\pi^2 T_2 l_{11}^3 \Lambda_1^6 l^{5/2}}{3(3l-r_2)^{1/2}}\Delta r_1 F_D^{(3)}\left(3/2; -1, -3/2, 1/2; 2; -\frac{\Delta r_1}{2l}, -\frac{\Delta r_1}{4l}, -\frac{\Delta r_1}{3l-r_2}\right)$$
$$= \frac{8\pi^2 T_2 l_{11}^3 \Lambda_1^6 l^{5/2}}{3(3l-r_2)^{1/2}}\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}$$
$$\times F_D^{(3)}\left(1/2; -1, -3/2, 1/2, ; 2; \frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right). \tag{3.268}$$

In the semiclassical limit, (3.267) and (3.268) reduce to

$$(\Lambda_0^+)^2 = \frac{8\sqrt{3}}{\pi}\lambda^0 T_2 l_{11}\Lambda_1^6\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2\right]^{1/2}, \quad P_\theta = P_{\tilde{\theta}} = \frac{16\pi T_2 l_{11}^3 \Lambda_1^6}{\sqrt{3}(\Lambda_0^+)^3}\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2\right]^{3/2}.$$

From here and (3.251), one obtains the relation

$$E^2 = \mathbf{P}^2 + 3^{5/3}(2\pi T_2 l_{11}^3 \Lambda_1^6)^{2/3} P_\theta^{4/3}. \tag{3.269}$$

In the particular case when $\mathbf{P} = 0$, (3.269) coincides with the energy-charge relation $E \sim K^{2/3}$, first obtained for G_2-manifolds in [27]. For the given embedding (3.264), the semiclassical limit of the membrane solution (3.266) is as follows

$$\begin{aligned}
\sigma_{scl}(r) &= 8\pi^{1/3}\left(\frac{2\pi^2 T_2 l_{11}^3 \Lambda_1^6}{9P_\theta}\right)^{2/3}\left(l^3\Delta r\right)^{1/2} F_D^{(2)}\left(1/2; -1, -1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}\right) \\
&= 8\pi^{1/3}\left(\frac{2\pi^2 T_2 l_{11}^3 \Lambda_1^6}{9P_\theta}\right)^{2/3}\left(l^3\Delta r\right)^{1/2}\left(1+\frac{\Delta r}{2l}\right)\left(1+\frac{\Delta r}{4l}\right)^{1/2} \\
&\times F_D^{(2)}\left(1; -1, -1/2; 3/2; \frac{1}{1+\frac{2l}{\Delta r}}, \frac{1}{1+\frac{4l}{\Delta r}}\right).
\end{aligned} \tag{3.270}$$

Third type of membrane embedding [9]

Again, we want the membrane to move in the flat, four dimensional part of the eleven dimensional background metric (3.243), with constant energy E and constant momenta P_I. On the curved part of the metric, the membrane is extended along the radial coordinate r, rotates in the plane given by the angle $\psi_+ = \psi + \tilde{\psi}$, and is wrapped along the angular coordinate $\psi_- = \psi - \tilde{\psi}$. This membrane configuration is given by

$$\begin{aligned}
&X^0 \equiv t = \Lambda_0^0 \tau, \quad X^I = \Lambda_0^I \tau, \quad X^4 \equiv r(\sigma), \\
&\psi_+ = \Lambda_0^+ \tau, \quad \psi_- = \Lambda_1^- \delta + \Lambda_2^- \sigma, \quad \psi_\pm = \psi \pm \tilde{\psi}.
\end{aligned} \tag{3.271}$$

In this case, the target space metric seen by the membrane is

$$\begin{aligned}
&g_{00} \equiv g_{tt} = -l_{11}^2, \quad g_{IJ} = l_{11}^2 \delta_{IJ}, \quad g_{44} \equiv g_{rr} = \frac{l_{11}^2}{C^2(r)}, \\
&g_{++} = l_{11}^2\left(\frac{2l}{3}\right)^2 C^2(r), \quad g_{--} = l_{11}^2 D^2(r).
\end{aligned} \tag{3.272}$$

Hence, in our notations, we have $\mu = (0, I, +, -)$, $a = 4 \equiv r$. Now, the metric induced on the membrane worldvolume is

$$\begin{aligned}
&G_{00} = -l_{11}^2\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 C^2\right], \\
&G_{11} = l_{11}^2(\Lambda_1^-)^2 D^2, \quad G_{12} = l_{11}^2 \Lambda_1^- \Lambda_2^- D^2, \quad G_{22} = l_{11}^2\left[(\Lambda_2^-)^2 D^2 + \frac{r'^2}{C^2}\right].
\end{aligned}$$

The constraints are satisfied identically, and $\mathcal{P}_\mu^2 \equiv 0$. The Lagrangian takes the form

$$\begin{aligned}
&\mathcal{L}^A(\sigma) = \frac{1}{4\lambda^0}\left(K_{rr} r'^2 - V\right), \quad K_{rr} = -(2\lambda^0 T_2 l_{11}^2 \Lambda_1^-)^2 \frac{D^2}{C^2}, \\
&V = U = l_{11}^2\left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 C^2\right].
\end{aligned}$$

The turning points read

$$r_{min} = 3l, \quad r_{max} = r_1 = l\sqrt{1 + \frac{8}{1 - \frac{9v_0^2}{4l^2(\Lambda_0^+)^2}}} > 3l,$$
$$r_2 = -l\sqrt{1 + \frac{8}{1 - \frac{9v_0^2}{4l^2(\Lambda_0^+)^2}}} < 0, \quad v_0^2 = (\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2.$$

For the present embedding, we derive the following membrane solution

$$\sigma(r) = \int_{3l}^{r}\left[-\frac{K_{rr}(t)}{V(t)}\right]^{1/2} dt = \frac{2\lambda^0 T_2 l_{11}\Lambda_1^-}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - v_0^2\right]^{1/2}}\left[\frac{2^7 l^5 \Delta r}{3\,(r_1 - 3l)\,(3l - r_2)}\right]^{1/2} \times \tag{3.273}$$
$$F_D^{(6)}\left(1/2; -1, -1, -1, 1/2, 1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{3l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{r_1 - 3l}\right).$$

The normalization condition leads to

$$\frac{\lambda^0 T_2 l_{11}\Lambda_1^-}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - v_0^2\right]^{1/2}}\left[\frac{2^7 l^5}{3\,(3l - r_2)}\right]^{1/2} \times \tag{3.274}$$
$$F_D^{(5)}\left(1/2; -1, -1, -1, 1/2, 1/2; 1; -\frac{\Delta r_1}{2l}, -\frac{\Delta r_1}{3l}, -\frac{\Delta r_1}{4l}, -\frac{\Delta r_1}{6l}, -\frac{\Delta r_1}{3l - r_2}\right) =$$
$$\frac{\lambda^0 T_2 l_{11}\Lambda_1^-}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - v_0^2\right]^{1/2}}\left[\frac{2^7 l^5}{3\,(3l - r_2)}\right]^{1/2} \times$$
$$\left(1 + \frac{\Delta r_1}{2l}\right)\left(1 + \frac{\Delta r_1}{3l}\right)\left(1 + \frac{\Delta r_1}{4l}\right)\left(1 + \frac{\Delta r_1}{6l}\right)^{-1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times$$
$$F_D^{(5)}\left(1/2; -1, -1, -1, 1/2, 1/2, ; 1; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{3l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right) = 1.$$

The computation of the conserved momentum $P_+ \equiv P_{\psi_+}$ gives

$$P_+ = \frac{\pi^2 T_2 l_{11}^3 \Lambda_0^+ \Lambda_1^-}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - v_0^2\right]^{1/2}}\left[\frac{2^5 l^7}{3^3\,(3l - r_2)}\right]^{1/2} \times \tag{3.275}$$
$$\Delta r_1 F_D^{(3)}\left(3/2; -1, -1/2, 1/2; 2; -\frac{\Delta r_1}{3l}, -\frac{\Delta r_1}{6l}, -\frac{\Delta r_1}{3l - r_2}\right) =$$
$$\frac{\pi^2 T_2 l_{11}^3 \Lambda_0^+ \Lambda_1^-}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - v_0^2\right]^{1/2}}\left[\frac{2^5 l^7}{3^3\,(3l - r_2)}\right]^{1/2} \times$$
$$\Delta r_1\left(1 + \frac{\Delta r_1}{3l}\right)\left(1 + \frac{\Delta r_1}{6l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times$$
$$F_D^{(3)}\left(1/2; -1, -1/2, 1/2; 2; \frac{1}{1 + \frac{3l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right).$$

Let us note that for the embedding (3.271), the momentum P_{ψ_-} is zero.

Going to the semiclassical limit $r_1 \to \infty$, which in the case under consideration leads to $9v_0^2/[4l^2(\Lambda_0^+)^2] \to 1_-$, one obtains that (3.274) and (3.275) reduce to

$$\Lambda_0^+\left[1-\frac{9v_0^2}{4l^2(\Lambda_0^+)^2}\right]^{3/2} = 2\lambda^0 T_2 l_{11}\Lambda_1^- l, \quad P_+ = \frac{2^{5/2}\pi^2 T_2 l_{11}^3 \Lambda_1^- l^3}{9\left[1-\frac{9v_0^2}{4l^2(\Lambda_0^+)^2}\right]^{3/2}}.$$

These two equalities, together with (3.251), give the following relation between the energy and the conserved momenta

$$E^2 = \mathbf{P}^2 + \frac{9}{2l^2}P_+^2 - (6\pi^2 T_2 l_{11}^3 \Lambda_1^-)^{2/3} P_+^{4/3}. \tag{3.276}$$

In the particular case when $\mathbf{P} = 0$, (3.276) can be rewritten as

$$E = \frac{3}{\sqrt{2}l}P_+\sqrt{1-\left(\frac{4\sqrt{2}\pi^2 T_2 l_{11}^3 \Lambda_1^- l^3}{9P_+}\right)^{2/3}}.$$

Expanding the square root and neglecting the higher order terms, one derives energy-charge relation of the type $E - K \sim K^{1/3}$, first found for backgrounds of G_2-holonomy in [27].

Now, let us write down the semiclassical limit of our membrane solution (3.273):

$$\begin{aligned}
\sigma_{scl}(r) = {}& \frac{\pi^2 T_2 l_{11}^3 \Lambda_1^-}{P_+}\left(\frac{2^7 l^5}{3^3}\right)^{1/2} \times \\
& \Delta r^{1/2} F_D^{(4)}\left(1/2; -1,-1,-1,1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{3l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}\right) = \\
& \frac{\pi^2 T_2 l_{11}^3 \Lambda_1^-}{P_+}\left(\frac{2^7 l^5}{3^3}\right)^{1/2} \Delta r^{1/2}\left(1+\frac{\Delta r}{2l}\right)\left(1+\frac{\Delta r}{3l}\right)\left(1+\frac{\Delta r}{4l}\right)\left(1+\frac{\Delta r}{6l}\right)^{-1/2} \times \\
& F_D^{(4)}\left(1; -1,-1,-1,1/2; 3/2; \frac{1}{1+\frac{2l}{\Delta r}}, \frac{1}{1+\frac{3l}{\Delta r}}, \frac{1}{1+\frac{4l}{\Delta r}}, \frac{1}{1+\frac{6l}{\Delta r}}\right).
\end{aligned} \tag{3.277}$$

Forth type of membrane embedding [9]

Let us consider membrane configuration given by the following ansatz:

$$\begin{aligned}
& X^0 \equiv t = \Lambda_0^0\tau + \frac{1}{\Lambda_0^0}\left[(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)\,\delta + (\mathbf{\Lambda}_0.\mathbf{\Lambda}_2)\,\sigma\right], \quad X^I = \Lambda_0^I\tau + \Lambda_1^I\delta + \Lambda_2^I\sigma, \\
& X^4 \equiv r(\sigma), \quad \psi_+ = \Lambda_0^+\tau, \quad \psi_- = \Lambda_0^-\tau, \quad \psi_\pm = \psi \pm \tilde{\psi}.
\end{aligned} \tag{3.278}$$

It is analogous to (3.244), but now the rotations are in the planes defined by the angles $\psi_\pm = \psi \pm \tilde{\psi}$ instead of θ and $\tilde{\theta}$.

The background felt by the membrane is as given in (3.272). However, the metric induced on the membrane worldvolume is different and it is the following

$$G_{00} = -l_{11}^2 \left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^+)^2 \left(\frac{2l}{3}\right)^2 C^2 - (\Lambda_0^-)^2 D^2 \right],$$
$$G_{11} = l_{11}^2 M_{11}, \quad G_{12} = l_{11}^2 M_{12}, \quad G_{22} = l_{11}^2 \left[M_{22} + \frac{r'^2}{C^2} \right],$$

where M_{ij} are defined in (3.246). The constraints are identically satisfied, and the constants of the motion $\mathcal{P}_\mu^2$ are given by (3.247). The membrane Lagrangian now takes the form

$$\mathcal{L}^A(\sigma) = \frac{1}{4\lambda^0}\left(K_{rr}r'^2 - V\right), \quad K_{rr} = -(2\lambda^0 T_2 l_{11}^2)^2 \frac{M_{11}}{C^2},$$
$$V = (2\lambda^0 T_2 l_{11}^2)^2 \det M_{ij} + l_{11}^2 \left[(\Lambda_0^0)^2 - \mathbf{\Lambda}_0^2 - (\Lambda_0^+)^2 \left(\frac{2l}{3}\right)^2 C^2 - (\Lambda_0^-)^2 D^2 \right].$$

Let us first consider the particular case when $\Lambda_0^- = 0$, i.e. $\psi = \tilde{\psi}$. The turning points now are

$$r_{min} = 3l, \quad r_{max} = r_1 = l\sqrt{1 + \frac{8}{1 - \frac{9u_0^2}{4l^2(\Lambda_0^+)^2}}} > 3l, \quad r_2 = -l\sqrt{1 + \frac{8}{1 - \frac{9u_0^2}{4l^2(\Lambda_0^+)^2}}} < 0,$$

where u_0^2 is introduced in (3.248). Now one arrives at the following membrane solution

$$\sigma(r) = \frac{2\lambda^0 T_2 l_{11}}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - u_0^2\right]^{1/2}} \left[\frac{2^7 l^3 M_{11}\Delta r}{3(r_1 - 3l)(3l - r_2)}\right]^{1/2} \tag{3.279}$$
$$\times F_D^{(5)}\left(1/2; -1, -1, 1/2, 1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{r_1 - 3l}\right).$$

The normalization condition gives

$$\frac{\lambda^0 T_2 l_{11}}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - u_0^2\right]^{1/2}} \left[\frac{2^7 l^3 M_{11}}{3(3l - r_2)}\right]^{1/2} \times \tag{3.280}$$
$$F_D^{(4)}\left(1/2; -1, -1, 1/2, 1/2; 1; -\frac{\Delta r_1}{2l}, -\frac{\Delta r_1}{4l}, -\frac{\Delta r_1}{6l}, -\frac{\Delta r_1}{3l - r_2}\right) =$$
$$\frac{\lambda^0 T_2 l_{11}}{\left[(\Lambda_0^+)^2\left(\frac{2l}{3}\right)^2 - u_0^2\right]^{1/2}} \left[\frac{2^7 l^3 M_{11}}{3(3l - r_2)}\right]^{1/2} \times$$
$$\left(1 + \frac{\Delta r_1}{2l}\right)\left(1 + \frac{\Delta r_1}{4l}\right)\left(1 + \frac{\Delta r_1}{6l}\right)^{-1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times$$
$$F_D^{(4)}\left(1/2; -1, -1, 1/2, 1/2, ; 1; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right) = 1.$$

We derive for the conserved momentum $P_+ \equiv P_{\psi_+}$ the following expression ($P_- \equiv P_{\psi_-} = 0$ as a consequence of $\Lambda_0^- = 0$):

$$
\begin{aligned}
P_+ = & \frac{\pi^2 T_2 l_{11}^3 \Lambda_0^+}{\left[(\Lambda_0^+)^2 \left(\frac{2l}{3}\right)^2 - u_0^2\right]^{1/2}} \left[\frac{2^5 l^5 M_{11}}{3^3 (3l - r_2)}\right]^{1/2} \times \\
& \Delta r_1 F_D^{(2)} \left(3/2; -1/2, 1/2; 2; -\frac{\Delta r_1}{6l}, -\frac{\Delta r_1}{3l - r_2}\right) = \\
& \frac{\pi^2 T_2 l_{11}^3 \Lambda_0^+}{\left[(\Lambda_0^+)^2 \left(\frac{2l}{3}\right)^2 - u_0^2\right]^{1/2}} \left[\frac{2^5 l^5 M_{11}}{3^3 (3l - r_2)}\right]^{1/2} \Delta r_1 \left(1 + \frac{\Delta r_1}{6l}\right)^{1/2} \left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times \\
& F_D^{(2)} \left(1/2; -1/2, 1/2; 2; \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right). \qquad (3.281)
\end{aligned}
$$

In the semiclassical limit, (3.280) and (3.281) simplify to

$$
\pi \Lambda_0^+ \left[1 - \frac{9 u_0^2}{4 l^2 (\Lambda_0^+)^2}\right] = 2^{3/2} 3 \lambda^0 T_2 l_{11} M_{11}^{1/2}, \qquad P_+ = \frac{2^{7/2} \pi T_2 l_{11}^3 l^2 M_{11}^{1/2}}{3 \left[1 - \frac{9 u_0^2}{4 l^2 (\Lambda_0^+)^2}\right]}.
$$

Taking also into account (3.251), we obtain the following *fourth* order algebraic equation for E^2 as a function of $\mathbf{P}$ and P_+

$$
\begin{aligned}
& \left\{E^2 \left[E^2 - \mathbf{P}^2 - (3/l)^2 P_+^2\right] - (2\pi^2 T_2 l_{11}^3)^2 \left\{(\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2)^2 E^2 - \left[(\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2) \times \mathbf{P}\right]^2\right\}\right\}^2 \\
& - 2^7 (3\pi T_2 l_{11}^3)^2 E^2 \left[\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1 . \mathbf{P})^2\right] P_+^2 = 0. \qquad (3.282)
\end{aligned}
$$

Let us consider a few simple cases. When $\Lambda_0^I = 0$ and $\Lambda_2^I = c\Lambda_1^I$, (3.282) reduces to

$$
E^2 = (3/l)^2 P_+^2 + 2^{7/2} 3\pi T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_+, \qquad (3.283)
$$

or

$$
E = \frac{3}{l} P_+ \sqrt{1 + \frac{2^{7/2} \pi T_2 l_{11}^3 l^2 \mid \mathbf{\Lambda}_1 \mid}{3 P_+}}.
$$

Expanding the square root and neglecting the higher order terms, one derives energy-charge relation of the type $E - K \sim const.$ If we impose only the conditions $\Lambda_0^I = 0$, (3.282) gives

$$
E^2 = (2\pi^2 T_2 l_{11}^3)^2 (\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2)^2 + (3/l)^2 P_+^2 + 2^{7/2} 3\pi T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_+. \qquad (3.284)
$$

If we take $\Lambda_0^I \neq 0$, $\Lambda_2^I = c\Lambda_1^I$, (3.282) simplifies to

$$
E^2 \left\{\left[E^2 - \mathbf{P}^2 - (3/l)^2 P_+^2\right]^2 - 2^7 (3\pi T_2 l_{11}^3)^2 \mathbf{\Lambda}_1^2 P_+^2\right\} + 2^7 (3\pi T_2 l_{11}^3)^2 (\mathbf{\Lambda}_1 . \mathbf{P})^2 P_+^2 = 0,
$$

which is *third* order algebraic equation for E^2. Suppose that $\mathbf{\Lambda}_1$ and $\mathbf{P}$ are orthogonal to each other, i.e. $(\mathbf{\Lambda}_1 . \mathbf{P}) = 0$. Then, the above relation becomes

$$
E^2 = \mathbf{P}^2 + (3/l)^2 P_+^2 + 2^{7/2} 3\pi T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_+. \qquad (3.285)
$$

Finally, we give the semiclassical limit of the membrane solution (3.279)

$$\sigma_{scl}(r) = 2\pi^2 T_2 l_{11}^3 \left(\frac{4l}{3}\right)^{3/2} \left[\mathbf{\Lambda}_1^2 - \frac{1}{E^2}\left(\mathbf{\Lambda}_1.\mathbf{P}\right)^2\right]^{1/2} \frac{\Delta r^{1/2}}{P_+}$$
$$\times F_D^{(3)}\left(1/2; -1, -1, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}\right)$$
$$= 2\pi^2 T_2 l_{11}^3 \left(\frac{4l}{3}\right)^{3/2} \left[\mathbf{\Lambda}_1^2 - \frac{1}{E^2}\left(\mathbf{\Lambda}_1.\mathbf{P}\right)^2\right]^{1/2} \frac{\Delta r^{1/2}}{P_+} \left(1+\frac{\Delta r}{2l}\right)\left(1+\frac{\Delta r}{4l}\right)\left(1+\frac{\Delta r}{6l}\right)^{-1/2}$$
$$\times F_D^{(3)}\left(1; -1, -1, 1/2; 3/2; \frac{1}{1+\frac{2l}{\Delta r}}, \frac{1}{1+\frac{4l}{\Delta r}}, \frac{1}{1+\frac{6l}{\Delta r}}\right).$$

Now, we turn to the case $\Lambda_0^- \neq 0$, when the solutions of the equation $r' = 0$ are

$$r_{min} = 3l, \quad r_{max} = r_1 = \frac{l}{\sqrt{2}}\sqrt{1+u^2-\Lambda^2}\sqrt{1+\sqrt{1-\frac{4(u^2-9\Lambda^2)}{(1+u^2-\Lambda^2)^2}}},$$
$$r_2 = \frac{l}{\sqrt{2}}\sqrt{1+u^2-\Lambda^2}\sqrt{1-\sqrt{1-\frac{4(u^2-9\Lambda^2)}{(1+u^2-\Lambda^2)^2}}},$$
$$r_3 = -\frac{l}{\sqrt{2}}\sqrt{1+u^2-\Lambda^2}\sqrt{1+\sqrt{1-\frac{4(u^2-9\Lambda^2)}{(1+u^2-\Lambda^2)^2}}},$$
$$r_4 = -\frac{l}{\sqrt{2}}\sqrt{1+u^2-\Lambda^2}\sqrt{1-\sqrt{1-\frac{4(u^2-9\Lambda^2)}{(1+u^2-\Lambda^2)^2}}},$$
$$u^2 = \left(\frac{3u_0}{l\Lambda_0^-}\right)^2, \quad \Lambda^2 = \left(2\frac{\Lambda_0^+}{\Lambda_0^-}\right)^2.$$

Correspondingly, we obtain the following solution for $\sigma(r)$:

$$\sigma(r) = \frac{\lambda^0 T_2 l_{11}}{\Lambda_0^-}\left[\frac{2^9 3 l^3 M_{11}\Delta r}{(r_1-3l)(3l-r_2)(3l-r_3)(3l-r_4)}\right]^{1/2} \times$$
$$F_D^{(7)}\left(1/2; -1, -1, 1/2, 1/2, 1/2, 1/2, 1/2; 3/2;\right. \qquad (3.286)$$
$$\left.-\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l-r_2}, -\frac{\Delta r}{3l-r_3}, -\frac{\Delta r}{3l-r_4}, \frac{\Delta r}{r_1-3l}\right).$$

For the normalization condition, we derive the result

$$\frac{\lambda^0 T_2 l_{11}}{\Lambda_0^-}\left[\frac{2^7 3 l^3 M_{11}}{(3l-r_2)(3l-r_3)(3l-r_4)}\right]^{1/2}\times \tag{3.287}$$
$$F_D^{(6)}(1/2;-1,-1,1/2,1/2,1/2,1/2;1;$$
$$-\frac{\Delta r_1}{2l},-\frac{\Delta r_1}{4l},-\frac{\Delta r_1}{6l},-\frac{\Delta r_1}{3l-r_2},-\frac{\Delta r_1}{3l-r_3},-\frac{\Delta r_1}{3l-r_4}\Bigg)=$$
$$\frac{\lambda^0 T_2 l_{11}}{\Lambda_0^-}\left[\frac{2^7 3 l^3 M_{11}}{(3l-r_2)(3l-r_3)(3l-r_4)}\right]^{1/2}\left(1+\frac{\Delta r}{2l}\right)\left(1+\frac{\Delta r}{4l}\right)\left(1+\frac{\Delta r}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r}{3l-r_2}\right)^{-1/2}\left(1+\frac{\Delta r}{3l-r_3}\right)^{-1/2}\left(1+\frac{\Delta r}{3l-r_4}\right)^{-1/2}\times$$
$$F_D^{(6)}(1/2;-1,-1,1/2,1/2,1/2,1/2;1;$$
$$\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}},\frac{1}{1+\frac{3l-r_3}{\Delta r_1}},\frac{1}{1+\frac{3l-r_4}{\Delta r_1}}\Bigg)=1.$$

The computation of the conserved quantities P_+ and P_- gives

$$P_+=\pi^2 T_2 l_{11}^3\frac{\Lambda_0^+}{\Lambda_0^-}\left[\frac{2^5 l^5 M_{11}}{3(3l-r_2)(3l-r_3)(3l-r_4)}\right]^{1/2}\Delta r_1\times \tag{3.288}$$
$$F_D^{(4)}\left(3/2;-1/2,1/2,1/2,1/2;2;-\frac{\Delta r_1}{6l},-\frac{\Delta r_1}{3l-r_2},-\frac{\Delta r_1}{3l-r_3},-\frac{\Delta r_1}{3l-r_4}\right)=$$
$$\pi^2 T_2 l_{11}^3\frac{\Lambda_0^+}{\Lambda_0^-}\left[\frac{2^5 l^5 M_{11}}{3(3l-r_2)(3l-r_3)(3l-r_4)}\right]^{1/2}\times$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_3}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_4}\right)^{-1/2}\times$$
$$F_D^{(4)}\left(1/2;-1/2,1/2,1/2,1/2;2;\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}},\frac{1}{1+\frac{3l-r_3}{\Delta r_1}},\frac{1}{1+\frac{3l-r_4}{\Delta r_1}}\right),$$

$$P_{-}=\pi^2 T_2 l_{11}^3\left[\frac{2^7 3 l^7 M_{11}}{(3l-r_2)(3l-r_3)(3l-r_4)}\right]^{1/2}\times \tag{3.289}$$

$$F_D^{(7)}\left(1/2;-1,-2,-1,1/2,1/2,1/2,1/2;1;\right.$$

$$\left.-\frac{\Delta r_1}{2l},-\frac{\Delta r_1}{3l},-\frac{\Delta r_1}{4l},-\frac{\Delta r_1}{6l},-\frac{\Delta r_1}{3l-r_2},-\frac{\Delta r_1}{3l-r_3},-\frac{\Delta r_1}{3l-r_4}\right)=$$

$$\pi^2 T_2 l_{11}^3\left[\frac{2^7 3 l^7 M_{11}}{(3l-r_2)(3l-r_3)(3l-r_4)}\right]^{1/2}\times$$

$$\left(1+\frac{\Delta r_1}{2l}\right)\left(1+\frac{\Delta r_1}{3l}\right)^2\left(1+\frac{\Delta r_1}{4l}\right)\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$

$$\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_3}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_4}\right)^{-1/2}\times$$

$$F_D^{(7)}\left(1/2;-1,-2,-1,1/2,1/2,1/2,1/2;1;\right.$$

$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{3l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}},\frac{1}{1+\frac{3l-r_3}{\Delta r_1}},\frac{1}{1+\frac{3l-r_4}{\Delta r_1}}\right).$$

Let us now take the semiclassical limit $r_1\to\infty$. In this limit, (3.287), (3.288) and (3.289) reduce correspondingly to

$$\Lambda_0^-=3\lambda^0 T_2 l_{11} M_{11}^{1/2},\qquad P_+=\frac{4}{3}\pi^2 T_2 l_{11}^3 l^2 M_{11}^{1/2}\frac{\Lambda_0^+}{\Lambda_0^-},$$
$$P_-=\frac{1}{6}\pi^2 T_2 l_{11}^3 l^2 M_{11}^{1/2}\left[\left(\frac{3u_0}{l\Lambda_0^-}\right)^2-\left(2\frac{\Lambda_0^+}{\Lambda_0^-}\right)^2\right].$$

These equalities, together with (3.251), lead to the following relation between the energy E and the conserved charges $\mathbf{P}$, P_+ and P_-:

$$\left\{E^2\left[E^2-\mathbf{P}^2-(3/2l)^2P_+^2\right]-(2\pi^2T_2l_{11}^3)^2\left\{(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)^2E^2-[(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\times\mathbf{P}]^2\right\}\right\}^2$$
$$-(6\pi^2T_2l_{11}^3)^2E^2\left[\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2\right]P_-^2=0. \tag{3.290}$$

Let us point out that the above relation is only valid for $P_-\neq 0$, whereas we can always set $\mathbf{P}$ or P_+ equal to zero. Below, we give a few simple solutions of (3.290).

Choosing $\Lambda_0^I=0$ and $\Lambda_2^I=c\Lambda_1^I$, one obtains

$$E^2=(3/2l)^2P_+^2+6\pi^2T_2l_{11}^3\mid\mathbf{\Lambda}_1\mid P_-, \tag{3.291}$$

which can be rewritten as

$$E=\frac{3}{2l}P_+\sqrt{1+\frac{8\pi^2T_2l_{11}^3l^2\mid\mathbf{\Lambda}_1\mid P_-}{3P_+^2}}.$$

Expanding the square root and neglecting the higher order terms, one arrives at

$$E = \frac{3}{2l}P_+ + 2\pi^2 T_2 l_{11}^3 l \mid \mathbf{\Lambda}_1 \mid \frac{P_-}{P_+}.$$

If only the conditions $\Lambda_0^I = 0$ are imposed, (3.290) gives

$$E^2 = (2\pi^2 T_2 l_{11}^3)^2 \left(\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2\right)^2 + (3/2l)^2 P_+^2 + 6\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_-. \tag{3.292}$$

If we choose $\Lambda_0^I \neq 0$, $\Lambda_2^I = c\Lambda_1^I$, then (3.290) simplifies to a *third* order algebraic equation for E^2

$$E^2 \left\{ \left[E^2 - \mathbf{P}^2 - (3/2l)^2 P_+^2\right]^2 - (6\pi^2 T_2 l_{11}^3)^2 \mathbf{\Lambda}_1^2 P_-^2 \right\} + (6\pi^2 T_2 l_{11}^3)^2 \left(\mathbf{\Lambda}_1.\mathbf{P}\right)^2 P_-^2 = 0.$$

If $(\mathbf{\Lambda}_1.\mathbf{P}) = 0$, the above relation reduces to

$$E^2 = \mathbf{P}^2 + (3/2l)^2 P_+^2 + 6\pi^2 T_2 l_{11}^3 \mid \mathbf{\Lambda}_1 \mid P_-. \tag{3.293}$$

Finally, let us write down the semiclassical limit of the membrane solution (3.286):

$$\begin{aligned}
&\sigma_{scl}(r) = \left(\frac{2^8\pi^2 T_2 l_{11}^3 l}{3^4 P_-}\right)^{1/2} \left[\mathbf{\Lambda}_1^2 - \frac{1}{E^2}\left(\mathbf{\Lambda}_1.\mathbf{P}\right)^2\right]^{1/4} \Delta r^{1/2} \\
&\times F_D^{(4)}\left(1/2; -1, 1, -1, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{3l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}\right) \\
&= \left(\frac{2^8\pi^2 T_2 l_{11}^3 l}{3^4 P_-}\right)^{1/2} \left[\mathbf{\Lambda}_1^2 - \frac{1}{E^2}\left(\mathbf{\Lambda}_1.\mathbf{P}\right)^2\right]^{1/4} \\
&\times \Delta r^{1/2} \left(1 + \frac{\Delta r}{2l}\right)\left(1 + \frac{\Delta r}{3l}\right)^{-1}\left(1 + \frac{\Delta r}{4l}\right)\left(1 + \frac{\Delta r}{6l}\right)^{-1/2} \\
&\times F_D^{(4)}\left(1; -1, 1, -1, 1/2; 3/2; \frac{1}{1 + \frac{2l}{\Delta r}}, \frac{1}{1 + \frac{3l}{\Delta r}}, \frac{1}{1 + \frac{4l}{\Delta r}}, \frac{1}{1 + \frac{6l}{\Delta r}}\right).
\end{aligned}$$

More rotating membrane solutions in this eleven dimensional supergravity background can be found in Appendix B of [9].

Some mathematical results

It is known that the Lauricella hypergeometric functions of n variables $F_D^{(n)}$ are defined as [51]

$$\begin{aligned}
&F_D^{(n)}(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \\
&\sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b_1)_{k_1}\ldots(b_n)_{k_n}}{(c)_{k_1+\ldots+k_n}} \frac{z_1^{k_1}\ldots z_n^{k_n}}{k_1!\ldots k_n!}, \quad |z_j| < 1, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},
\end{aligned} \tag{3.294}$$

and have the following integral representation [51]

$$\begin{aligned}
&F_D^{(n)}(a, b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \\
&\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-1}(1-z_1 x)^{-b_1}\ldots(1-z_n x)^{-b_n} dx, \\
&Re(a) > 0, \quad Re(c-a) > 0.
\end{aligned} \tag{3.295}$$

However, in order to perform our calculations, we need to know more about the properties of these functions. That is why, we proved in [9] that the following equalities hold

1. $F_D^{(n)}(a; b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n; c; z_1, \ldots, z_i, \ldots, z_j, \ldots, z_n) =$
$F_D^{(n)}(a; b_1, \ldots, b_j, \ldots, b_i, \ldots, b_n; c; z_1, \ldots, z_j, \ldots, z_i, \ldots, z_n).$

2. $F_D^{(n)}(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) =$
$\prod_{i=1}^{n} (1 - z_i)^{-b_i} F_D^{(n)} \left(c - a; b_1, \ldots, b_n; c; \frac{z_1}{z_1 - 1}, \ldots, \frac{z_n}{z_n - 1} \right).$

3. $F_D^{(n)}(a; b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_n; c; z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n) =$
$\frac{\Gamma(c)\Gamma(c-a-b_i)}{\Gamma(c-a)\Gamma(c-b_i)} F_D^{(n-1)}(a; b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n; c - b_i; z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).$

4. $F_D^{(n)}(a; b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_n; c; z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_n) =$
$F_D^{(n-1)}(a; b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n; c; z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).$

5. $F_D^{(n)}(a; b_1, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_n; c; z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) =$
$F_D^{(n-1)}(a; b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n; c; z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).$

6. $F_D^{(n)}(a; b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n; c; z_1, \ldots, z_i, \ldots, z_i, \ldots, z_n) =$
$F_D^{(n-1)}(a; b_1, \ldots, b_i + b_j, \ldots, b_n; c; z_1, \ldots, z_i, \ldots, z_n).$

7. $F_D^{(2n+1)}(a; a - c + 1, b_2, b_2, \ldots, b_{2n}, b_{2n}; c; -1, z_2, -z_2 \ldots, z_{2n}, -z_{2n}) =$
$\frac{\Gamma(a/2)\Gamma(c)}{2\Gamma(a)\Gamma(c-a/2)} F_D^{(n)}(a/2; b_2, \ldots, b_{2n}; c - a/2; z_2^2, \ldots, z_{2n}^2).$

8. $F_D^{(2n+1)} \left(c - a; a - c + 1, b_2, b_2, \ldots, b_{2n}, b_{2n}; c; \right.$
$\left. 1/2, -\frac{z_2}{1 - z_2}, \frac{z_2}{1 + z_2}, \ldots, -\frac{z_{2n}}{1 - z_{2n}}, \frac{z_{2n}}{1 + z_{2n}} \right) =$
$\frac{\Gamma(a/2)\Gamma(c)}{2^{c-a}\Gamma(a)\Gamma(c-a/2)} F_D^{(n)} \left(c - a; b_2, \ldots, b_{2n}; c - a/2; -\frac{z_2^2}{1 - z_2^2}, \ldots, -\frac{z_{2n}^2}{1 - z_{2n}^2} \right).$

3.2.4 Rotating D2-branes in type IIA reduction of M-theory on G_2 manifold and their semiclassical limits

Here, we will consider D2-branes rotating in the background (3.18).

We begin with the following D2-brane embedding in the target space:

$$X^0 = \Lambda_0^0 \xi^0 + \frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)}{\Lambda_0^0} \left(\xi^1 + c\xi^2 \right), \quad X^I = \Lambda_0^I \xi^0 + \Lambda_1^I \left(\xi^1 + c\xi^2 \right), \tag{3.296}$$
$$r = r(\xi^2), \quad \theta_1 = \Lambda_0^{\theta_1} \xi^0, \quad \theta_2 = \Lambda_0^{\theta_2} \xi^0; \quad (\mathbf{\Lambda}_0.\mathbf{\Lambda}_1) = \delta_{IJ} \Lambda_0^I \Lambda_1^J, \quad c = constant.$$

It corresponds to D2-brane extended in the radial direction r, and rotating in the planes given by the angles θ_1 and θ_2 with constant angular momenta P_{θ_1} and P_{θ_2}. It is nontrivially spanned along x^0 and x^I and moves with constant energy E, and constant momenta P_I.

The expression for the D2-brane solution found in [10] for this case is given by

$$\xi^2(r) = \frac{16}{3}\lambda^0 T_{D2}\left[\frac{Ml}{\left(\Lambda_+^2+\Lambda_-^2\right)(3l-r_2)\,\Delta r_1}\right]^{1/2}(2\Delta r)^{3/4}\times \tag{3.297}$$
$$F_D^{(5)}\left(3/4;-1/4,-1/4,1/4,1/2,1/2;7/4;-\frac{\Delta r}{2l},-\frac{\Delta r}{4l},-\frac{\Delta r}{6l},-\frac{\Delta r}{3l-r_2},\frac{\Delta r}{\Delta r_1}\right).$$

Now, we compute the conserved momenta on the obtained solution:

$$\frac{E}{\Lambda_0^0}=\frac{P_I}{\Lambda_0^I}=8\pi^2 T_{D2}\left[\frac{Ml}{\left(\Lambda_+^2+\Lambda_-^2\right)(3l-r_2)}\right]^{1/2}\times \tag{3.298}$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(4)}\left(1/2;-1/2,-1/2,1/2,1/2;1;\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$P_{\theta_1}=\left(\Lambda_0^{\theta_1}-\Lambda_0^{\theta_2}\cos\psi_1^0\right)I_{A1}^D+\left(\Lambda_0^{\theta_1}+\Lambda_0^{\theta_2}\cos\psi_1^0\right)I_{B1}^D, \tag{3.299}$$
$$P_{\theta_2}=\left(\Lambda_0^{\theta_2}-\Lambda_0^{\theta_1}\cos\psi_1^0\right)I_{A1}^D+\left(\Lambda_0^{\theta_2}+\Lambda_0^{\theta_1}\cos\psi_1^0\right)I_{B1}^D,$$

where

$$I_{A1}^D = 8\pi^2 T_{D2}\left[\frac{Ml^5}{\left(\Lambda_+^2+\Lambda_-^2\right)(3l-r_2)}\right]^{1/2}\times \tag{3.300}$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(4)}\left(1/2;-3/2,-1/2,-1/2,1/2;1;\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$I_{B1}^D = \frac{4}{3}\pi^2 T_{D2}\left[\frac{Ml^3}{\left(\Lambda_+^2+\Lambda_-^2\right)(3l-r_2)}\right]^{1/2}\times \tag{3.301}$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(4)}\left(1/2;-1/2,-3/2,1/2,1/2;2;\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).$$

In the semiclassical limit, (3.298) - (3.301) simplify to

$$\frac{E}{\Lambda_0^0}=\frac{P_I}{\Lambda_0^I}=\frac{2}{3}\pi^2 T_{D2}\left(\frac{M}{\Lambda_+^2+\Lambda_-^2}\right)^{1/2},$$
$$P_{\theta_1}=2\Lambda_0^{\theta_1}I_{A1}^D,\qquad P_{\theta_2}=2\Lambda_0^{\theta_2}I_{A1}^D,\qquad I_{A1}^D=I_{B1}^D=\frac{\sqrt{3}\pi^2 T_{D2}M^{1/2}}{\left(\Lambda_+^2+\Lambda_-^2\right)^{3/2}}v_0^2.$$

From here, one obtains the following relation between the energy and the conserved charges

$$E^2\left(E^2-\mathbf{P}^2\right)^2-\frac{2^3}{3^5}(\pi^2 T_{D2})^2\left[\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2\right]\left(P_{\theta_1}^2+P_{\theta_2}^2\right)=0, \tag{3.302}$$

which is third order algebraic equation for E^2.

For $(\mathbf{\Lambda}_1.\mathbf{P})=0$, (3.302) reduces to

$$E^2=\mathbf{P}^2+\frac{2^{3/2}}{3^{5/2}}\pi^2 T_{D2}\mid\mathbf{\Lambda}_1\mid\left(P_{\theta_1}^2+P_{\theta_2}^2\right)^{1/2}.$$

This is the same type energy-charge relation as the one obtained for the string in (3.25).

Let us now consider the other possible D2-brane embedding for the same background metric. It is given by

$$\begin{aligned}&X^0=\Lambda_0^0\xi^0,\quad X^I=\Lambda_0^I\xi^0,\quad r=r(\xi^2),\\&\theta_1=\Lambda_0^\theta\xi^0+\Lambda_1^\theta\xi^1+\Lambda_2^\theta\xi^2,\quad \theta_2=\Lambda_0^\theta\xi^0-\Lambda_1^\theta\xi^1-\Lambda_2^\theta\xi^2.\end{aligned} \tag{3.303}$$

This ansatz describes D2-brane, which is extended along the radial direction r and rotates in the planes defined by the angles θ_1 and θ_2, with equal angular momenta $P_{\theta_1}=P_{\theta_2}=P_\theta$. Now we have nontrivial wrapping along θ_1 and θ_2. In addition, the D2-brane moves along x^0 and x^I with constant energy E and constant momenta P_I respectively.

Now, one finds the following D2-brane solution [10]:

$$\begin{aligned}&\xi^2(r)=\frac{8}{3}\lambda^0 T_{D2}\left[\frac{l\left(\Lambda_{1+}^2+\Lambda_{1-}^2\right)(3l-v_+)(3l-v_-)}{3\left(\check{\Lambda}_+^2+\check{\Lambda}_-^2\right)(3l-r_2)\,\Delta r_1}\right]^{1/2}(2\Delta r)^{3/4}\times\\&F_D^{(7)}\left(3/4;-1/4,-1/4,1/4,-1/2,-1/2,1/2,1/2;7/4;\right.\\&\left.-\frac{\Delta r}{2l},-\frac{\Delta r}{4l},-\frac{\Delta r}{6l},-\frac{\Delta r}{3l-v_+},-\frac{\Delta r}{3l-v_-},-\frac{\Delta r}{3l-r_2},\frac{\Delta r}{\Delta r_1}\right),\end{aligned} \tag{3.304}$$

where $v_\pm$ are the zeros of the polynomial

$$t^2-2l\frac{\Lambda_{1+}^2-\Lambda_{1-}^2}{\Lambda_{1+}^2+\Lambda_{1-}^2}t-3l^2=(t-v_+)(t-v_-).$$

In the case under consideration, the conserved quantities are E, P_I and P_θ. We derive the following result for them [10]

$$\begin{aligned}&\frac{E}{\Lambda_0^0}=\frac{P_I}{\Lambda_0^I}=4\pi^2 T_{D2}\left[\frac{l\left(\Lambda_{1+}^2+\Lambda_{1-}^2\right)(3l-v_+)(3l-v_-)}{3\left(\check{\Lambda}_+^2+\check{\Lambda}_-^2\right)(3l-r_2)}\right]^{1/2}\times\\&\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times\\&\left(1+\frac{\Delta r_1}{3l-v_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-v_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times\\&F_D^{(6)}\left(1/2;-1/2,-1/2,1/2,-1/2,-1/2,1/2;1;\right.\\&\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-v+}{\Delta r_1}},\frac{1}{1+\frac{3l-v-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),\end{aligned} \tag{3.305}$$

$$P_\theta = \Lambda_0^\theta \left[\left(1-\cos\psi_1^0\right) I_{A2}^D + \left(1+\cos\psi_1^0\right) I_{B2}^D\right],$$

where

$$I_{A2}^D = 4\pi^2 T_{D2}\left[\frac{l^5\left(\Lambda_{1+}^2+\Lambda_{1-}^2\right)(3l-v_+)(3l-v_-)}{3\left(\check{\Lambda}_+^2+\check{\Lambda}_-^2\right)(3l-r_2)}\right]^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-v_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-v_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2; -3/2, -1/2, -1/2, -1/2, -1/2, 1/2; 1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-v+}{\Delta r_1}}, \frac{1}{1+\frac{3l-v-}{\Delta r_1}} \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$I_{B2}^D = 2\pi^2 T_{D2}\left[\frac{l^3\left(\Lambda_{1+}^2+\Lambda_{1-}^2\right)(3l-v_+)(3l-v_-)}{3^3\left(\check{\Lambda}_+^2+\check{\Lambda}_-^2\right)(3l-r_2)}\right]^{1/2}\times$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-v_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-v_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2; -1/2, -3/2, 1/2, -1/2, -1/2, 1/2; 2;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-v+}{\Delta r_1}}, \frac{1}{1+\frac{3l-v-}{\Delta r_1}} \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).$$

Taking the semiclassical limit in the above expressions[15], we obtain the following dependence of the energy on P_I and P_θ:

$$E^2 = \mathbf{P}^2 + 3^{5/3}(2\pi T_{D2}\Lambda_1^\theta)^{2/3} P_\theta^{4/3}. \tag{3.306}$$

Now, we turn to the case of D2-brane embedded in the following way

$$X^0 = \Lambda_0^0\xi^0 + \frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)}{\Lambda_0^0}\left(\xi^1 + c\xi^2\right), \quad X^I = \Lambda_0^I\xi^0 + \Lambda_1^I\left(\xi^1 + c\xi^2\right), \tag{3.307}$$
$$r = r(\xi^2), \quad \theta_1 = \Lambda_0^\theta\xi^0, \quad \phi_2 = \Lambda_0^\phi\xi^0.$$

(3.307) is analogous to (3.296), but now the rotations are in the planes defined by the angles θ_1 and ϕ_2 instead of θ_1 and θ_2.

The solution $\xi^2(r)$ can be obtained from (3.297) by the replacement

$$\Lambda_+^2 + \Lambda_-^2 \to \bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3. \tag{3.308}$$

[15]In this limit $v_\pm$ remain finite.

The explicit expressions for E and P_I can be obtained in the same way from (3.298). The computation of the conserved angular momenta P_θ and P_ϕ gives

$$
\begin{aligned}
P_\theta &= \left(\Lambda_0^\theta - \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right) J_{A1}^D + \left(\Lambda_0^\theta + \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right) J_{B1}^D,\\
P_\phi &= \left(\Lambda_0^\phi \sin\theta_2^0 - \Lambda_0^\theta \sin\psi_1^0\right) \sin\theta_2^0 J_{A1}^D + \left(\Lambda_0^\phi \sin\theta_2^0 + \Lambda_0^\theta \sin\psi_1^0\right) \sin\theta_2^0 J_{B1}^D\\
&+\Lambda_0^\phi \cos^2\theta_2^0 J_{D1}^D,
\end{aligned}
$$

where one obtains J_{A1}^D, J_{B1}^D from (3.300), (3.301) by the replacement (3.308), and

$$
\begin{aligned}
J_{D1}^D &= 8\pi^2 T_{D2}\left[\frac{Ml^5}{\left(\bar\Lambda_+^2 + \bar\Lambda_-^2 + 4\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2} \times\\
&\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l}\right)^{2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times\\
&F_D^{(5)}\left(1/2; -1/2, -2, -1/2, 1/2, 1/2; 1;\right.\\
&\left.\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{3l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).
\end{aligned}
$$

Taking $r_1 \to \infty$ in the above expressions, one obtains that in the semiclassical limit the following energy-charge relation holds

$$
\frac{E^2\left(E^2-\mathbf{P}^2\right)^2}{\boldsymbol{\Lambda}_1^2 E^2 - (\boldsymbol{\Lambda}_1.\mathbf{P})^2} = \frac{2^3}{3^5}(\pi^2 T_{D2})^2\left(P_\theta^2 + \frac{3P_\phi^2}{3-\cos^2\theta_2^0}\right).
$$

Obviously, this is a generalization of the relation (3.302) and for $\theta_2^0 = \pi/2$ has the same form.

Another possible ansatz for the D2-brane embedding is

$$
X^0 = \Lambda_0^0\xi^0, \quad X^I = \Lambda_0^I\xi^0, \quad r = r(\xi^2), \quad \theta_1 = \Lambda_1^\theta\xi^1 + \Lambda_2^\theta\xi^2, \quad \phi_2 = \Lambda_0^\phi\xi^0, \tag{3.309}
$$

i.e., we have D2-brane extended in the radial direction r, wrapped along the angular coordinate θ_1 and rotating in the plane given by the angle ϕ_2.

Then one obtains the solution:

$$
\begin{aligned}
&\xi^2(r) = \frac{8}{3}\lambda^0 T_{D2}\Lambda_1^\theta\left[\frac{l(3l-w_+)(3l-w_-)}{\left(3\Lambda^2+2\Lambda_D^2\right)(3l-r_2)\,\Delta r_1}\right]^{1/2}(2\Delta r)^{3/4}\times\\
&F_D^{(7)}\left(3/4; -1/4, -1/4, 1/4, -1/2, -1/2, 1/2, 1/2; 7/4;\right.\\
&\left.-\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l-w_+}, -\frac{\Delta r}{3l-w_-}, -\frac{\Delta r}{3l-r_2}, \frac{\Delta r}{\Delta r_1}\right), \quad w_\pm = \pm\sqrt{3}l.
\end{aligned}
\tag{3.310}
$$

The computation of the conserved quantities E, P_I and $P_{\phi_2} \equiv P_\phi$, gives

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = 4\pi^2 T_{D2} \Lambda_1^\theta \left[\frac{l(3l-w_+)(3l-w_-)}{\left(3\Lambda^2 + 2\Lambda_D^2\right)(3l-r_2)} \right]^{1/2} \times \tag{3.311}$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{1/2} \left(1+\frac{\Delta r_1}{4l}\right)^{1/2} \left(1+\frac{\Delta r_1}{6l}\right)^{-1/2} \times$$
$$\left(1+\frac{\Delta r_1}{3l-w_+}\right)^{1/2} \left(1+\frac{\Delta r_1}{3l-w_-}\right)^{1/2} \left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2} \times$$
$$F_D^{(6)}\left(1/2; -1/2, -1/2, 1/2, -1/2, -1/2, 1/2; 1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-w+}{\Delta r_1}}, \frac{1}{1+\frac{3l-w-}{\Delta r_1}} \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$P_\phi = \sin^2\theta_2^0 \left(J_{A2}^D + J_{B2}^D\right) + \cos^2\theta_2^0 J_{D2}^D,$$

where

$$J_{A2}^D = 4\pi^2 T_{D2} \Lambda_0^\phi \Lambda_1^\theta \left[\frac{l^5(3l-w_+)(3l-w_-)}{\left(3\Lambda^2 + 2\Lambda_D^2\right)(3l-r_2)} \right]^{1/2} \times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2} \left(1+\frac{\Delta r_1}{4l}\right)^{1/2} \left(1+\frac{\Delta r_1}{6l}\right)^{1/2} \times$$
$$\left(1+\frac{\Delta r_1}{3l-w_+}\right)^{1/2} \left(1+\frac{\Delta r_1}{3l-w_-}\right)^{1/2} \left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2} \times$$
$$F_D^{(6)}\left(1/2; -3/2, -1/2, -1/2, -1/2, -1/2, 1/2; 1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-w+}{\Delta r_1}}, \frac{1}{1+\frac{3l-w-}{\Delta r_1}} \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$J_{B2}^D = \frac{2}{3}\pi^2 T_{D2} \Lambda_0^\phi \Lambda_1^\theta \left[\frac{l^3(3l-w_+)(3l-w_-)}{\left(3\Lambda^2 + 2\Lambda_D^2\right)(3l-r_2)} \right]^{1/2} \times$$
$$\Delta r_1 \left(1+\frac{\Delta r_1}{2l}\right)^{1/2} \left(1+\frac{\Delta r_1}{4l}\right)^{3/2} \left(1+\frac{\Delta r_1}{6l}\right)^{-1/2} \times$$
$$\left(1+\frac{\Delta r_1}{3l-w_+}\right)^{1/2} \left(1+\frac{\Delta r_1}{3l-w_-}\right)^{1/2} \left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2} \times$$
$$F_D^{(6)}\left(1/2; -1/2, -3/2, 1/2, -1/2, -1/2, 1/2; 2;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-w+}{\Delta r_1}}, \frac{1}{1+\frac{3l-w-}{\Delta r_1}} \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$
\begin{aligned}
J^D_{D2} &= 4\pi^2 T_{D2}\Lambda_0^{\phi}\Lambda_1^{\theta}\left[\frac{l^5(3l-w_+)(3l-w_-)}{\left(3\Lambda^2+2\Lambda_D^2\right)(3l-r_2)}\right]^{1/2}\times\\
&\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l}\right)^{2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times\\
&\left(1+\frac{\Delta r_1}{3l-w_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-w_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times\\
&F_D^{(7)}\left(1/2;-1/2,-2,-1/2,1/2,-1/2,-1/2,1/2;1;\right.\\
&\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{3l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-w+}{\Delta r_1}},\frac{1}{1+\frac{3l-w-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).
\end{aligned}
$$

Going to the semiclassical limit $r_1 \to \infty$ in the above expressions for the conserved quantities, one obtains the following relation between them

$$
E^2 = \mathbf{P}^2 + \frac{3^{7/3}}{2^{1/3}}\left(\frac{\pi T_{D2}\Lambda_1^{\theta}}{3-\cos^2\theta_2^0}\right)^{2/3} P_{\phi}^{4/3}. \tag{3.312}
$$

This is a generalization of the energy-charge relation received in (3.306).

Another admissible embedding is

$$
\begin{aligned}
&X^0 = \Lambda_0^0\xi^0 + \frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)}{\Lambda_0^0}\left(\xi^1+c\xi^2\right), \quad X^I = \Lambda_0^I\xi^0 + \Lambda_1^I\left(\xi^1+c\xi^2\right),\\
&r = r(\xi^2), \quad \phi_1 = \Lambda_0^{\phi_1}\xi^0, \quad \phi_2 = \Lambda_0^{\phi_2}\xi^0.
\end{aligned} \tag{3.313}
$$

It is analogous to (3.296) and (3.307), but now the rotations are in the planes given by the angles ϕ_1 and ϕ_2.

The solution $\xi^2(r)$, and the expressions for E, P_I, may be obtained from the corresponding quantities for the embedding (3.307) by the replacements $\bar{\Lambda}^2_{\mp} \to \tilde{\Lambda}^2_{\pm}$, $\Lambda^2_D \to \tilde{\Lambda}^2_D$. For the conserved angular momenta P_{ϕ_1} and P_{ϕ_2} one finds

$$
\begin{aligned}
P_{\phi_1} &= \left(\Lambda_0^{\phi_1}\sin\theta_1^0 + \Lambda_0^{\phi_2}\cos\psi_1^0\sin\theta_2^0\right)\sin\theta_1^0 K^D_{A1}\\
&+ \left(\Lambda_0^{\phi_1}\sin\theta_1^0 - \Lambda_0^{\phi_2}\cos\psi_1^0\sin\theta_2^0\right)\sin\theta_1^0 K^D_{B1}\\
&+ \left(\Lambda_0^{\phi_1}\cos\theta_1^0 + \Lambda_0^{\phi_2}\cos\theta_2^0\right)\cos\theta_1^0 K^D_{D1},
\end{aligned} \tag{3.314}
$$

$$
\begin{aligned}
P_{\phi_2} &= \left(\Lambda_0^{\phi_2}\sin\theta_2^0 + \Lambda_0^{\phi_1}\cos\psi_1^0\sin\theta_1^0\right)\sin\theta_2^0 K^D_{A1}\\
&+ \left(\Lambda_0^{\phi_2}\sin\theta_2^0 - \Lambda_0^{\phi_1}\cos\psi_1^0\sin\theta_1^0\right)\sin\theta_2^0 K^D_{B1}\\
&+ \left(\Lambda_0^{\phi_1}\cos\theta_1^0 + \Lambda_0^{\phi_2}\cos\theta_2^0\right)\cos\theta_2^0 K^D_{D1},
\end{aligned} \tag{3.315}
$$

where K^D_{A1}, K^D_{B1} and K^D_{D1} can be obtained from J^D_{A1}, J^D_{B1} and J^D_{D1} through the above mentioned replacements.

The calculations show that in the semiclassical limit, the dependence of the energy on the conserved charges, for the present case, is given by the equality:

$$\frac{E^2\left(E^2-\mathbf{P}^2\right)^2}{\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2}= \tag{3.316}$$
$$\frac{2^3}{3^5}(\pi^2T_{D2})^2\frac{\left(3-\cos^2\theta_2^0\right)P_{\phi_1}^2+\left(3-\cos^2\theta_1^0\right)P_{\phi_2}^2-4P_{\phi_1}P_{\phi_2}\cos\theta_1^0\cos\theta_2^0}{3-\cos^2\theta_1^0-\cos^2\theta_2^0-\cos^2\theta_1^0\cos^2\theta_2^0}.$$

Finally, let us consider the following possible D2-brane embedding

$$X^0=\Lambda_0^0\xi^0,\quad X^I=\Lambda_0^I\xi^0,\quad r=r(\xi^2), \tag{3.317}$$
$$\phi_1=\Lambda_0^\phi\xi^0+\Lambda_1^\phi\xi^1+\Lambda_2^\phi\xi^2,\quad \phi_2=\Lambda_0^\phi\xi^0-\Lambda_1^\phi\xi^1-\Lambda_2^\phi\xi^2.$$

It describes D2-brane configuration, which is analogous to the one in (3.303), but now the rotations are in the planes defined by the angles ϕ_1 and ϕ_2 instead of θ_1 and θ_2.

For this embedding, one obtains

$$\xi^2(r)=\frac{8}{3}\lambda^0T_{D2}\left[\frac{l\left(\hat\Lambda_{1+}^2+\hat\Lambda_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat\Lambda_+^2+\hat\Lambda_-^2+4\hat\Lambda_D^2/3\right)(3l-r_2)\,\Delta r_1}\right]^{1/2}(2\Delta r)^{3/4}\times$$
$$F_D^{(7)}(3/4;-1/4,-1/4,1/4,-1/2,-1/2,1/2,1/2;7/4;$$
$$\left.-\frac{\Delta r}{2l},-\frac{\Delta r}{4l},-\frac{\Delta r}{6l},-\frac{\Delta r}{3l-u_+},-\frac{\Delta r}{3l-u_-},-\frac{\Delta r}{3l-r_2},\frac{\Delta r}{\Delta r_1}\right),$$

where

$$u_\pm=l\left[\frac{\hat\Lambda_{1+}^2-\hat\Lambda_{1-}^2}{\hat\Lambda_{1+}^2+\hat\Lambda_{1-}^2}\pm\sqrt{3+\left(\frac{\hat\Lambda_{1+}^2-\hat\Lambda_{1-}^2}{\hat\Lambda_{1+}^2+\hat\Lambda_{1-}^2}\right)^2}\right].$$

The computation of the conserved charges results in

$$\frac{E}{\Lambda_0^0}=\frac{P_I}{\Lambda_0^I}=4\pi^2T_{D2}\left[\frac{l\left(\hat\Lambda_{1+}^2+\hat\Lambda_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat\Lambda_+^2+\hat\Lambda_-^2+4\hat\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2}\times \tag{3.318}$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}(1/2;-1/2,-1/2,1/2,-1/2,-1/2,1/2;1;$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-u+}{\Delta r_1}},\frac{1}{1+\frac{3l-u-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$P_\phi \equiv P_{\phi_1} = P_{\phi_2} =$$
$$\Lambda_0^\phi \left\{\sin^2\theta^0 \left[\left(1+\cos\psi_1^0\right) K_{A2}^D + \left(1-\cos\psi_1^0\right) K_{B2}^D\right] + 2\cos^2\theta^0 K_{D2}^D\right\},$$

where

$$K_{A2}^D = 4\pi^2 T_{D2} \left[\frac{l^5\left(\hat\Lambda_{1+}^2+\hat\Lambda_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat\Lambda_+^2+\hat\Lambda_-^2+4\hat\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2} \times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2;-3/2,-1/2,-1/2,-1/2,-1/2,1/2;1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-u+}{\Delta r_1}},\frac{1}{1+\frac{3l-u-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$K_{B2}^D = 2\pi^2 T_{D2} \left[\frac{l^3\left(\hat\Lambda_{1+}^2+\hat\Lambda_{1-}^2\right)(3l-u_+)(3l-u_-)}{3^3\left(\hat\Lambda_+^2+\hat\Lambda_-^2+4\hat\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2} \times$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2;-1/2,-3/2,1/2,-1/2,-1/2,1/2;2;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-u+}{\Delta r_1}},\frac{1}{1+\frac{3l-u-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$K_{D2}^D = 4\pi^2 T_{D2} \left[\frac{l^5\left(\hat\Lambda_{1+}^2+\hat\Lambda_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat\Lambda_+^2+\hat\Lambda_-^2+4\hat\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2} \times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l}\right)^{2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(7)}\left(1/2;-1/2,-2,-1/2,1/2,-1/2,-1/2,1/2;1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{3l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-u+}{\Delta r_1}},\frac{1}{1+\frac{3l-u-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).$$

Taking the semiclassical limit in the above expressions for E, P_I and P_ϕ, which in the case under consideration corresponds to

$$r_{1,2} \to \pm 2\sqrt{\frac{3v_0^2}{\hat{\Lambda}_+^2 + \hat{\Lambda}_-^2 + 4\hat{\Lambda}_D^2/3}} \to \infty,$$

we receive that the energy depends on P_I and P_ϕ as follows

$$E^2 = \mathbf{P}^2 + 3^{7/3}\left(\frac{2\pi T_{D2}\Lambda_1^\phi \sin\theta^0}{4-\sin^2\theta^0}\right)^{2/3} P_\phi^{4/3}. \tag{3.319}$$

This is another generalization of the energy-charge relation given in (3.306).

3.2.5 Integrable systems from membranes on $AdS_4 \times S^7$

It is known that large class of classical string solutions in the type IIB $AdS_5 \times S^5$ background is related to the Neumann and Neumann-Rosochatius integrable systems, including spiky strings and giant magnons [54]. It is also interesting if these integrable systems can be associated with some membrane configurations in M-theory. We explain here how this can be achieved by considering membrane embedding in $AdS_4 \times S^7$ solution of M-theory, with the desired properties [11].

On the other hand, we will show the existence of membrane configurations in $AdS_4 \times S^7$ [12], which correspond to the continuous limit of the $SU(2)$ integrable spin chain, arising in $\mathcal{N}=4$ SYM in four dimensions, dual to strings in $AdS_5 \times S^5$ [55].

Here we will work with the action (2.5) written for membranes ($p=2$) in diagonal worldvolume gauge $\lambda^i = 0$, in which the action and the constraints simplify to

$$S_M = \int d^3\xi \mathcal{L}_M = \int d^3\xi \left\{\frac{1}{4\lambda^0}\left[G_{00} - \left(2\lambda^0 T_2\right)^2 \det G_{ij}\right] + T_2 C_{012}\right\}, \tag{3.320}$$

$$G_{00} + \left(2\lambda^0 T_2\right)^2 \det G_{ij} = 0, \tag{3.321}$$

$$G_{0i} = 0. \tag{3.322}$$

Searching for membrane configurations in $AdS_4 \times S^7$, which correspond to the Neumann or Neumann-Rosochatius integrable systems, we should first eliminate the membrane interaction with the background 3-form field on AdS_4, to ensure more close analogy with the strings on $AdS_5 \times S^5$. To make our choice, let us write down the background. It can be parameterized as follows

$$\begin{aligned}
&ds^2 = (2l_p\mathcal{R})^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\alpha^2 + \sin^2\alpha d\beta^2\right) + 4d\Omega_7^2\right],\\
&d\Omega_7^2 = d\psi_1^2 + \cos^2\psi_1 d\varphi_1^2\\
&+ \sin^2\psi_1\left[d\psi_2^2 + \cos^2\psi_2 d\varphi_2^2 + \sin^2\psi_2\left(d\psi_3^2 + \cos^2\psi_3 d\varphi_3^2 + \sin^2\psi_3 d\varphi_4^2\right)\right],\\
&c_{(3)} = (2l_p\mathcal{R})^3 \sinh^3\rho \sin\alpha dt \wedge d\alpha \wedge d\beta.
\end{aligned}$$

Since we want the membrane to have nonzero conserved energy and spin on AdS, one possible choice, for which the interaction with the $c_{(3)}$ field disappears, is to fix the angle α: $\alpha = \alpha_0 = const$. The metric of the corresponding subspace of AdS_4 is

$$ds_{sub}^2 = (2l_p\mathcal{R})^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d(\beta\sin\alpha_0)^2\right].$$

The appropriate membrane embedding into ds^2_{sub} and S^7 is

$$Z_\mu = 2l_p\mathcal{R}\mathrm{r}_\mu(\xi^m)e^{i\phi_\mu(\xi^m)}, \quad \mu = (0,1), \quad \phi_\mu = (\phi_0,\phi_1) = (t,\beta\sin\alpha_0),$$
$$W_a = 4l_p\mathcal{R}r_a(\xi^m)e^{i\varphi_a(\xi^m)}, \quad a = (1,2,3,4),$$

where r_μ and r_a are real functions of ξ^m, while ϕ_μ and φ_a are the isometric coordinates on which the background metric does not depend. The six complex coordinates Z_μ, W_a are restricted by the two real embedding constraints

$$\eta^{\mu\nu}Z_\mu\bar{Z}_\nu + (2l_p\mathcal{R})^2 = 0, \quad \eta^{\mu\nu} = (-1,1), \quad \delta_{ab}W_a\bar{W}_b - (4l_p\mathcal{R})^2 = 0,$$

or equivalently

$$\eta^{\mu\nu}\mathrm{r}_\mu\mathrm{r}_\nu + 1 = 0, \quad \delta_{ab}r_ar_b - 1 = 0.$$

The coordinates r_μ, r_a are connected to the initial coordinates, on which the background depends, through the equalities

$$\mathrm{r}_0 = \cosh\rho, \quad \mathrm{r}_1 = \sinh\rho,$$
$$r_1 = \cos\psi_1, \quad r_2 = \sin\psi_1\cos\psi_2,$$
$$r_3 = \sin\psi_1\sin\psi_2\cos\psi_3, \quad r_4 = \sin\psi_1\sin\psi_2\sin\psi_3.$$

For the embedding described above, the induced metric is given by

$$G_{mn} = \eta^{\mu\nu}\partial_{(m}Z_\mu\partial_{n)}\bar{Z}_\nu + \delta_{ab}\partial_{(m}W_a\partial_{n)}\bar{W}_b = \tag{3.323}$$
$$(2l_p\mathcal{R})^2\left[\sum_{\mu,\nu=0}^{1}\eta^{\mu\nu}\left(\partial_m\mathrm{r}_\mu\partial_n\mathrm{r}_\nu + \mathrm{r}_\mu^2\partial_m\phi_\mu\partial_n\phi_\nu\right) + 4\sum_{a=1}^{4}\left(\partial_mr_a\partial_nr_a + r_a^2\partial_m\varphi_a\partial_n\varphi_a\right)\right].$$

Correspondingly, the membrane Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_M + \Lambda_A(\eta^{\mu\nu}\mathrm{r}_\mu\mathrm{r}_\nu + 1) + \Lambda_S(\delta_{ab}r_ar_b - 1),$$

where Λ_A and Λ_S are Lagrange multipliers.

Neumann and Neumann-Rosochatius integrable systems from membranes

Let us consider the following particular case of the above membrane embedding

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, Z_1 = 0, \quad W_a = 4l_p\mathcal{R}r_a(\xi,\eta)e^{i[\omega_a\tau+\mu_a(\xi,\eta)]}, \tag{3.324}$$
$$\xi = \alpha\sigma_1 + \beta\tau, \eta = \gamma\sigma_2 + \delta\tau,$$

which implies

$$\mathrm{r}_0 = 1, \quad \mathrm{r}_1 = 0, \quad \phi_0 = t = \kappa\tau, \quad \varphi_a(\xi^m) = \varphi_a(\tau,\sigma_1,\sigma_2) = \omega_a\tau + \mu_a(\xi,\eta). \tag{3.325}$$

Here κ, ω_a, α, β, γ, δ are parameters, whereas $r_a(\xi,\eta)$, $\mu_a(\xi,\eta)$ are arbitrary functions. As a consequence, the embedding constraint $\eta^{\mu\nu}\mathrm{r}_\mu\mathrm{r}_\nu + 1 = 0$ is satisfied identically. For this ansatz, the membrane Lagrangian takes the form ($\partial_\xi = \partial/\partial\xi$, $\partial_\eta = \partial/\partial\eta$)

$$\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\Big\{\left(8\lambda^0 T_2 l_p\mathcal{R}\alpha\gamma\right)^2 \sum_{a<b=1}^{4}\left[(\partial_\xi r_a\partial_\eta r_b - \partial_\eta r_a\partial_\xi r_b)^2\right.$$
$$+ (\partial_\xi r_a\partial_\eta \mu_b - \partial_\eta r_a\partial_\xi \mu_b)^2 r_b^2 + (\partial_\xi \mu_a\partial_\eta r_b - \partial_\eta \mu_a\partial_\xi r_b)^2 r_a^2$$
$$\left. + (\partial_\xi \mu_a\partial_\eta \mu_b - \partial_\eta \mu_a\partial_\xi \mu_b)^2 r_a^2 r_b^2\right]$$
$$+ \sum_{a=1}^{4}\left[\left(8\lambda^0 T_2 l_p\mathcal{R}\alpha\gamma\right)^2(\partial_\xi r_a\partial_\eta\mu_a - \partial_\eta r_a\partial_\xi\mu_a)^2 - (\beta\partial_\xi\mu_a + \delta\partial_\eta\mu_a + \omega_a)^2\right] r_a^2$$
$$- \sum_{a=1}^{4}(\beta\partial_\xi r_a + \delta\partial_\eta r_a)^2 + (\kappa/2)^2\Big\} + \Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right).$$

Now, we make the choice

$$r_1 = r_1(\xi), \quad r_2 = r_2(\xi), \quad \omega_3 = \pm\omega_4 = \omega,$$
$$r_3 = r_3(\eta) = \epsilon\sin(b\eta + c), \quad r_4 = r_4(\eta) = \epsilon\cos(b\eta + c),$$
$$\mu_1 = \mu_1(\xi), \quad \mu_2 = \mu_2(\xi), \quad \mu_3, \mu_4 = constants,$$

and receive (prime is used for $d/d\xi$)

$$\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\Big\{\sum_{a=1}^{2}\left[(A^2-\beta^2)r_a'^2 + (A^2-\beta^2)r_a^2\left(\mu_a' - \frac{\beta\omega_a}{A^2-\beta^2}\right)^2 - \frac{A^2}{A^2-\beta^2}\omega_a^2 r_a^2\right]$$
$$+ (\kappa/2)^2 - \epsilon^2(\omega^2 + b^2\delta^2)\Big\} + \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-\epsilon^2)\right],$$

where $A^2 \equiv \left(8\lambda^0 T_2 l_p\mathcal{R}\epsilon b\alpha\gamma\right)^2$. A single time integration of the equations of motion for μ_a following from the above Lagrangian gives

$$\mu_a' = \frac{1}{A^2-\beta^2}\left(\frac{C_a}{r_a^2} + \beta\omega_a\right),$$

where C_a are arbitrary constants. Taking this into account, one obtains the following effective Lagrangian for the coordinates $r_a(\xi)$

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[(A^2-\beta^2)r_a'^2 - \frac{1}{A^2-\beta^2}\frac{C_a^2}{r_a^2} - \frac{A^2}{A^2-\beta^2}\omega_a^2 r_a^2\right]$$
$$+\Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-\epsilon^2)\right].$$

This Lagrangian in full analogy with the string considerations corresponds to particular case of the n-dimensional *Neumann-Rosochatius integrable system*. For $C_a = 0$ one obtains *Neumann integrable system*, which describes two-dimensional harmonic oscillator, constrained to remain on a circle of radius $\sqrt{1-\epsilon^2}$.

Let us write down the three constraints (3.321), (3.322) for the present case. To achieve more close correspondence with the string on $AdS_5 \times S^5$, we want the third one, $G_{02} = 0$, to be satisfied identically. To this end, since $G_{02} \sim (ab)^2\gamma\delta$, we set $\delta = 0$, i.e. $\eta = \gamma\sigma_2$. Then, the first two constraints give

$$\sum_{a=1}^{2}\left[(A^2-\beta^2)r_a'^2 + \frac{1}{A^2-\beta^2}\frac{C_a^2}{r_a^2} + \frac{A^2}{A^2-\beta^2}\omega_a^2 r_a^2\right] = \frac{A^2+\beta^2}{A^2-\beta^2}\left[(\kappa/2)^2-(\epsilon\omega)^2\right],$$
$$\sum_{a=1}^{2}\omega_a C_a + \beta\left[(\kappa/2)^2-(\epsilon\omega)^2\right] = 0. \qquad (3.326)$$

Now, let us compute the energy and angular momenta for the membrane configuration we are considering. Due to the background isometries, there exist global conserved charges. In our case, the background does not depend on $\phi_0 = t$ and φ_a. Therefore, the corresponding conserved quantities are the membrane energy E and four angular momenta J_a, given as spatial integrals of the conjugated to these coordinates momentum densities

$$E = -\int d^2\sigma \frac{\partial\mathcal{L}}{\partial(\partial_0 t)}, \qquad J_a = \int d^2\sigma \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_a)}, \qquad a = 1,2,3,4.$$

E and J_a can be computed by using the expression (3.323) for the induced metric and the ansats (3.324), (3.325).

In order to reproduce the string case, we can set $\omega = 0$, and thus $J_3 = J_4 = 0$. The energy and the other two angular momenta are given by

$$E = \frac{4\pi(l_p\mathcal{R})^2\kappa}{\lambda^0\alpha}\int d\xi, \qquad J_a = \frac{\pi(4l_p\mathcal{R})^2}{\lambda^0\alpha(A^2-\beta^2)}\int d\xi\left(\beta C_a + A^2\omega_a r_a^2\right), \qquad a = 1,2.$$

From here, by using the constraints (3.326), one obtains the energy-charge relation

$$\frac{4}{A^2-\beta^2}\left[A^2(1-\epsilon^2) + \beta\sum_{a=1}^{2}\frac{C_a}{\omega_a}\right]\frac{E}{\kappa} = \sum_{a=1}^{2}\frac{J_a}{\omega_a},$$

in full analogy with the string case. Namely, for strings on $AdS_5 \times S^5$, the result in conformal gauge is [54]

$$\frac{1}{\alpha^2-\beta^2}\left(\alpha^2 + \beta\sum_a\frac{C_a}{\omega_a}\right)\frac{E}{\kappa} = \sum_a\frac{J_a}{\omega_a}.$$

$SU(2)$ spin chain from membrane

One of the predictions of AdS/CFT duality is that the string theory on $AdS_5 \times S^5$ should be dual to $\mathcal{N} = 4$ SYM theory in four dimensions. The spectrum of the string states and of the operators in SYM should be the same. The first checks of this conjecture *beyond* the supergravity approximation revealed that there exist string configurations, whose energies in the semiclassical limit are related to the anomalous dimensions of certain gauge invariant operators in the planar SYM. On the field theory side, it was found that the corresponding dilatation operator is connected to the Hamiltonian of integrable Heisenberg spin chain. On the other hand, it was established that there is agreement at the level of actions between the continuous limit of the $SU(2)$ spin chain arising in $\mathcal{N}=4$ SYM theory and a certain limit of the string action in $AdS_5\times S^5$ background. Shortly after, it was shown that such equivalence also holds for the $SU(3)$ and $SL(2)$ cases.

Here, we are interested in answering the question: is it possible to reproduce this type of string/spin chain correspondence from membranes on eleven dimensional curved backgrounds? It turns out that the answer is positive at least for the case of M2-branes on $AdS_4\times S^7$, as we will show below.

We will use our initial membrane embedding and fix

$$Z_0=2l_p\mathcal{R}e^{i\kappa\tau},\qquad Z_1=0,$$

which implies $\mathrm{r}_0=1$, $\mathrm{r}_1=0$, $\phi_0=t=\kappa\tau$. Let us now introduce new coordinates by setting

$$(\varphi_1,\varphi_2,\varphi_3,\varphi_4)=\left(\frac{\kappa}{2}\tau+\alpha+\varphi,\frac{\kappa}{2}\tau+\alpha-\varphi,\frac{\kappa}{2}\tau+\alpha+\phi,\frac{\kappa}{2}\tau+\alpha-\tilde{\phi}\right)$$

and take the limit $\kappa\to\infty$, $\partial_0\to 0$, $\kappa\partial_0$ - finite. In this limit, we obtain the following expression for the membrane Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{(2l_p\mathcal{R})^2}{\lambda^0}\kappa\left(\partial_0\alpha+\sum_{k=1}^{3}\nu_k\partial_0\rho_k\right)-\lambda^0T_2^2(4l_p\mathcal{R})^4\left\{\sum_{a<b=1}^{4}(\partial_1r_a\partial_2r_b-\partial_2r_a\partial_1r_b)^2\right.\\ &+ \sum_{a=1}^{4}\sum_{k=1}^{3}\mu_k(\partial_1r_a\partial_2\rho_k-\partial_2r_a\partial_1\rho_k)^2-\sum_{a=1}^{4}\left(\partial_1r_a\sum_{k=1}^{3}\nu_k\partial_2\rho_k-\partial_2r_a\sum_{k=1}^{3}\nu_k\partial_1\rho_k\right)^2\\ &+ \sum_{k<n=1}^{3}\mu_k\mu_n(\partial_1\rho_k\partial_2\rho_n-\partial_2\rho_k\partial_1\rho_n)^2\\ &- \left.\sum_{k=1}^{3}\mu_k\left(\partial_1\rho_k\sum_{n=1}^{3}\nu_n\partial_2\rho_n-\partial_2\rho_k\sum_{n=1}^{3}\nu_n\partial_1\rho_n\right)^2\right\}+\Lambda_S\left(\sum_{a=1}^{4}r_a^2-1\right),\end{aligned}$$

where

$$(\mu_1,\mu_2,\mu_3)=(r_1^2+r_2^2,r_3^2,r_4^2),(\nu_1,\nu_2,\nu_3)=(r_1^2-r_2^2,r_3^2,-r_4^2),(\rho_1,\rho_2,\rho_3)=(\varphi,\phi,\tilde{\phi}).$$

Now, we are ready to face our main problem: how to reduce this Lagrangian to the one corresponding to the thermodynamic limit of spin chain, *without shrinking the membrane to string*? We propose the following solution of this task:

$$\begin{aligned}&\alpha=\alpha(\tau,\sigma_1),\quad r_1=r_1(\tau,\sigma_1),\quad r_2=r_2(\tau,\sigma_1),\\ &r_3=r_3(\tau,\sigma_2)=\epsilon\sin[b\sigma_2+c(\tau)],\quad r_4=r_4(\tau,\sigma_2)=\epsilon\cos[b\sigma_2+c(\tau)],\\ &\varphi=\varphi(\tau,\sigma_1),\quad \epsilon,b,\phi,\tilde{\phi}=constants,\quad \epsilon^2<1.\end{aligned}$$

These restrictions lead to

$$\begin{aligned}\mathcal{L} &= \frac{(2l_p\mathcal{R})^2}{\lambda^0}\kappa\left[\partial_0\alpha+(r_1^2-r_2^2)\partial_0\varphi\right]-\lambda^0(\epsilon bT_2)^2(4l_p\mathcal{R})^4\left\{\sum_{a=1}^{2}(\partial_1r_a)^2\right.\\ &+ \left.\left[(r_1^2+r_2^2)-(r_1^2-r_2^2)^2\right](\partial_1\varphi)^2\right\}+\Lambda_S\left[\sum_{a=1}^{2}r_a^2-(1-\epsilon^2)\right].\end{aligned}$$

If we introduce the parametrization

$$r_1 = (1-\epsilon^2)^{1/2}\cos\psi, \qquad r_2 = (1-\epsilon^2)^{1/2}\sin\psi,$$

the new variable $\tilde{\alpha}$=$\alpha/(1-\epsilon^2)$, and take the limit $\epsilon^2 \to 0$ neglecting the terms of order higher than ϵ^2, we will receive

$$\frac{\mathcal{L}}{1-\epsilon^2} = \frac{(2l_p\mathcal{R})^2}{\lambda^0}\kappa\left[\partial_0\tilde{\alpha} + \cos(2\psi)\partial_0\varphi\right] - \lambda^0(\epsilon bT_2)^2(4l_p\mathcal{R})^4\left[(\partial_1\psi)^2 + \sin^2(2\psi)(\partial_1\varphi)^2\right].$$

As for the membrane action corresponding to the above Lagrangian, it can be represented in the form

$$S_M = \frac{\mathcal{J}}{2\pi}\int dtd\sigma\left[\partial_t\tilde{\alpha} + \cos(2\psi)\partial_t\varphi\right] - \frac{\tilde{\lambda}}{4\pi\mathcal{J}}\int dtd\sigma\left[(\partial_\sigma\psi)^2 + \sin^2(2\psi)(\partial_\sigma\varphi)^2\right],$$

where $\mathcal{J}$ is the angular momentum conjugated to $\tilde{\alpha}$, $t = \kappa\tau$ and

$$\tilde{\lambda} = 2^{15}(\pi^2\epsilon bT_2)^2(l_p\mathcal{R})^6.$$

This action corresponds to the thermodynamic limit of $SU(2)$ *integrable spin chain* [55].

3.2.6 M2-brane perspective on $\mathcal{N} = 6$ super Chern-Simons-matter theory at level k

In 2008, O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena (ABJM) proposed three-dimensional super Chern-Simons-matter theory, which at level k is supposed to describe the low energy limit of N M2-branes [66]. For large N and k, but fixed 't Hooft coupling $\lambda = N/k$, it is dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. For large N but *finite* k, it is dual to M theory on $AdS_4 \times S^7/Z_k$. Here, relying on the second duality, we find exact giant magnon and single spike solutions of membrane configurations on $AdS_4 \times S^7/Z_k$ by reducing the system to the Neumann-Rosochatius integrable model. We derive the dispersion relations and their finite-size corrections with explicit dependence on the level k [16].

Let us introduce the following complex coordinates on the S^7/Z_k subspace

$$z_1 = \cos\psi\cos\frac{\theta_1}{2}e^{i\left[\frac{\varphi}{k}+\frac{1}{2}(\phi_1+\phi_3)\right]}, \qquad z_2 = \cos\psi\sin\frac{\theta_1}{2}e^{i\left[\frac{\varphi}{k}-\frac{1}{2}(\phi_1-\phi_3)\right]},$$
$$z_3 = \sin\psi\cos\frac{\theta_2}{2}e^{i\left[\frac{\varphi}{k}+\frac{1}{2}(\phi_2-\phi_3)\right]}, \qquad z_4 = \sin\psi\sin\frac{\theta_2}{2}e^{i\left[\frac{\varphi}{k}-\frac{1}{2}(\phi_2+\phi_3)\right]}.$$

Obviously, they satisfy the relation

$$\sum_{a=1}^{4} z_a\bar{z}_a \equiv 1.$$

Next, we compute the metric

$$ds^2_{S^7/Z_k} = \sum_{a=1}^{4} dz_a d\bar{z}_a = \frac{1}{k^2}\left(d\varphi + kA_1\right)^2 + ds^2_{\mathbb{CP}^3},$$

where

$$A_1 = \frac{1}{2}\left[\cos^2\psi\cos\theta_1 d\phi_1 + \sin^2\psi\cos\theta_2 d\phi_2 + \left(\cos^2\psi - \sin^2\psi\right) d\phi_3\right],$$
$$ds^2_{\mathbb{CP}^3} = d\psi^2 + \sin^2\psi\cos^2\psi\left(\frac{1}{2}\cos\theta_1 d\phi_1 - \frac{1}{2}\cos\theta_2 d\phi_2 + d\phi_3\right)^2$$
$$+\frac{1}{4}\cos^2\psi\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\right) + \frac{1}{4}\sin^2\psi\left(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right). \quad (3.327)$$

The membrane embedding into $R_t \times S^7/Z_k$, appropriate for our purposes, is

$$X_0 = \frac{R}{2}t(\xi^m), \quad W_a = Rr_a(\xi^m)e^{i\varphi_a(\xi^m)}, \quad a = (1,2,3,4),$$

where t is the AdS time, r_a are real functions of ξ^m, while φ_a are the isometric coordinates on which the background metric does not depend. The four complex coordinates W_a are restricted by the real embedding condition

$$\sum_{a=1}^{4} W_a\bar{W}_a = R^2, \quad \text{or} \quad \sum_{a=1}^{4} r_a^2 = 1.$$

The coordinates r_a are connected to the initial coordinates, on which the background depends, in an obvious way.

For the embedding described above, the metric induced on the M2-brane worldvolume is given by

$$G_{mn} = \frac{R^2}{4}\left[-\partial_m t\partial_n t + 4\sum_{a=1}^{4}\left(\partial_m r_a\partial_n r_a + r_a^2\partial_m\varphi_a\partial_n\varphi_a\right)\right]. \quad (3.328)$$

Correspondingly, the membrane Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_M + \Lambda\left(\sum_{a=1}^{4} r_a^2 - 1\right),$$

where Λ is a Lagrange multiplier.

NR integrable system for M2-branes on $R_t \times S^7/Z_k$

Let us consider the following particular case of the above membrane embedding [11]

$$X_0 = \frac{R}{2}\kappa\tau, \quad W_a = Rr_a(\xi,\eta)e^{i[\omega_a\tau + \mu_a(\xi,\eta)]}, \quad (3.329)$$
$$\xi = \alpha\sigma_1 + \beta\tau, \quad \eta = \gamma\sigma_2 + \delta\tau,$$

which implies

$$t = \kappa\tau, \quad \varphi_a(\xi^m) = \varphi_a(\tau,\sigma_1,\sigma_2) = \omega_a\tau + \mu_a(\xi,\eta). \quad (3.330)$$

Here κ, ω_a, α, β, γ, δ are parameters. For this ansatz, the membrane Lagrangian takes the form ($\partial_\xi = \partial/\partial\xi$, $\partial_\eta = \partial/\partial\eta$)

$$\mathcal{L} = -\frac{R^2}{4\lambda^0}\Bigg\{\left(2\lambda^0 T_2 R\alpha\gamma\right)^2 \sum_{a<b=1}^{4}\Big[(\partial_\xi r_a\partial_\eta r_b - \partial_\eta r_a\partial_\xi r_b)^2$$
$$+ (\partial_\xi r_a\partial_\eta \mu_b - \partial_\eta r_a\partial_\xi \mu_b)^2 r_b^2 + (\partial_\xi \mu_a\partial_\eta r_b - \partial_\eta \mu_a\partial_\xi r_b)^2 r_a^2$$
$$+ (\partial_\xi \mu_a\partial_\eta \mu_b - \partial_\eta \mu_a\partial_\xi \mu_b)^2 r_a^2 r_b^2\Big]$$
$$+ \sum_{a=1}^{4}\left[\left(2\lambda^0 T_2 R\alpha\gamma\right)^2(\partial_\xi r_a\partial_\eta \mu_a - \partial_\eta r_a\partial_\xi \mu_a)^2 - (\beta\partial_\xi\mu_a + \delta\partial_\eta\mu_a + \omega_a)^2\right] r_a^2$$
$$- \sum_{a=1}^{4}(\beta\partial_\xi r_a + \delta\partial_\eta r_a)^2 + (\kappa/2)^2\Bigg\} + \Lambda\left(\sum_{a=1}^{4} r_a^2 - 1\right).$$

We have found a set of sufficient conditions, which reduce the above Lagrangian to the NR one. First of all, two of the angles φ_a should be set to zero. The corresponding r_a coordinates must depend only on η in a specific way. The remaining variables r_a and μ_a can depend only on ξ[16]. In principle, there are six such possibilities. How they are realized for the $R_t \times S^7/Z_k$ background, we will discuss in the next section. Here, we will work out the following example

$$\begin{aligned} &r_1 = r_1(\xi), \quad r_2 = r_2(\xi), \quad \mu_1 = \mu_1(\xi), \quad \mu_2 = \mu_2(\xi), \\ &r_3 = r_3(\eta) = r_0\sin\eta, \quad r_4 = r_4(\eta) = r_0\cos\eta, \quad r_0 < 1, \\ &\varphi_3 = \varphi_4 = 0. \end{aligned} \tag{3.331}$$

For this choice, we receive (prime is used for $d/d\xi$)

$$\mathcal{L} = -\frac{R^2}{4\lambda^0}\Bigg\{\sum_{a=1}^{2}\left[(\tilde{A}^2 - \beta^2)r_a'^2 + (\tilde{A}^2 - \beta^2)r_a^2\left(\mu_a' - \frac{\beta\omega_a}{\tilde{A}^2 - \beta^2}\right)^2 - \frac{\tilde{A}^2}{\tilde{A}^2 - \beta^2}\omega_a^2 r_a^2\right]$$
$$+ (\kappa/2)^2 - r_0^2\delta^2\Bigg\} + \Lambda\left[\sum_{a=1}^{2} r_a^2 - (1 - r_0^2)\right],$$

where $\tilde{A}^2 \equiv \left(2\lambda^0 T_2 R\alpha\gamma r_0\right)^2$. Now we can integrate once the equations of motion for μ_a following from the above Lagrangian to get

$$\mu_a' = \frac{1}{\tilde{A}^2 - \beta^2}\left(\frac{C_a}{r_a^2} + \beta\omega_a\right), \tag{3.332}$$

where C_a are arbitrary constants. By using (3.332) in the equations of motion for $r_a(\xi)$, one finds that they can be obtained from the effective Lagrangian

$$L_{NR} = \sum_{a=1}^{2}\left[(\tilde{A}^2 - \beta^2)r_a'^2 - \frac{1}{\tilde{A}^2 - \beta^2}\left(\frac{C_a^2}{r_a^2} + \tilde{A}^2\omega_a^2 r_a^2\right)\right] + \Lambda_M\left[\sum_{a=1}^{2} r_a^2 - (1 - r_0^2)\right].$$

This Lagrangian, in full analogy with the string considerations, corresponds to particular case of the NR integrable system. For $C_a = 0$ one obtains the Neumann integrable model, which in the case at hand describes two-dimensional harmonic oscillator, constrained to a circle of radius $\sqrt{1 - r_0^2}$.

[16] Of course, the roles of ξ and η can be interchanged in this context.

Let us consider the three constraints (3.321), (3.322) for the present case. For more close correspondence with the string case, we want the third one, $G_{02} = 0$, to be identically satisfied. To this end, since $G_{02} \sim r_0^2\gamma\delta$, we set $\delta = 0$, i.e. $\eta = \gamma\sigma_2$. Then, the first two constraints give the conserved Hamiltonian H_{NR} and a relation between the parameters involved:

$$H_{NR} = \sum_{a=1}^{2}\left[(\tilde{A}^2 - \beta^2)r_a'^2 + \frac{1}{\tilde{A}^2 - \beta^2}\left(\frac{C_a^2}{r_a^2} + \tilde{A}^2\omega_a^2 r_a^2\right)\right] = \frac{\tilde{A}^2 + \beta^2}{\tilde{A}^2 - \beta^2}(\kappa/2)^2,$$
$$\sum_{a=1}^{2}\omega_a C_a + \beta(\kappa/2)^2 = 0. \tag{3.333}$$

For closed membranes, r_a and μ_a must satisfy the following periodicity conditions

$$r_a(\xi + 2\pi\alpha, \eta + 2\pi\gamma) = r_a(\xi, \eta), \qquad \mu_a(\xi + 2\pi\alpha, \eta + 2\pi\gamma) = \mu_a(\xi, \eta) + 2\pi n_a, \tag{3.334}$$

where n_a are integer winding numbers. In particular, γ is a non-zero integer.

Since the background metric does not depend on t and φ_a, the corresponding conserved quantities are the membrane energy E and four angular momenta J_a, defined by

$$E = -\int d^2\sigma \frac{\partial\mathcal{L}}{\partial(\partial_0 t)}, \qquad J_a = \int d^2\sigma \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_a)}, \qquad a = 1, 2, 3, 4.$$

For our ansatz (3.331) $J_3 = J_4 = 0$. The energy and the other two angular momenta are given by

$$E = \frac{\pi R^2\kappa}{4\lambda^0\alpha}\int d\xi, \qquad J_a = \frac{\pi R^2}{\lambda^0\alpha(\tilde{A}^2 - \beta^2)}\int d\xi\left(\beta C_a + \tilde{A}^2\omega_a r_a^2\right), \qquad a = 1, 2. \tag{3.335}$$

From here, by using the constraints (3.333), one obtains the energy-charge relation

$$\frac{4}{\tilde{A}^2 - \beta^2}\left[\tilde{A}^2(1 - r_0^2) + \beta\sum_{a=1}^{2}\frac{C_a}{\omega_a}\right]\frac{E}{\kappa} = \sum_{a=1}^{2}\frac{J_a}{\omega_a}.$$

As usual, it is linear with respect to E and J_a before taking the semiclassical limit.

To identically satisfy the embedding condition

$$\sum_{a=1}^{2} r_a^2 - (1 - r_0^2) = 0,$$

we set

$$r_1(\xi) = \sqrt{1 - r_0^2}\sin\theta(\xi), \qquad r_2(\xi) = \sqrt{1 - r_0^2}\cos\theta(\xi).$$

Then from the conservation of the NR Hamiltonian (3.333) one finds

$$\theta' = \frac{\pm 1}{\tilde{A}^2 - \beta^2}\left[(\tilde{A}^2 + \beta^2)\tilde{\kappa}^2 - \frac{\tilde{C}_1^2}{\sin^2\theta} - \frac{\tilde{C}_2^2}{\cos^2\theta} - \tilde{A}^2\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right)\right]^{1/2}, \tag{3.336}$$
$$\sum_{a=1}^{2}\omega_a\tilde{C}_a + \beta\tilde{\kappa}^2 = 0, \qquad \tilde{\kappa}^2 = \frac{(\kappa/2)^2}{1 - r_0^2}, \qquad \tilde{C}_a^2 = \frac{C_a^2}{(1 - r_0^2)^2}.$$

By replacing the solution for $\theta(\xi)$ received from (3.336) into (3.332), one obtains the solutions for μ_a:

$$\mu_1 = \frac{1}{\tilde{A}^2-\beta^2}\left(\tilde{C}_1\int\frac{d\xi}{\sin^2\theta}+\beta\omega_1\xi\right), \quad \mu_2 = \frac{1}{\tilde{A}^2-\beta^2}\left(\tilde{C}_2\int\frac{d\xi}{\cos^2\theta}+\beta\omega_2\xi\right). \tag{3.337}$$

The above analysis shows that the NR integrable models for membranes on $R_t\times S^7$ and $R_t\times S^7/Z_k$ are the same [15]. Therefore, we can use the results obtained in [15] for the present case. For convenience, the corresponding solutions and dispersion relations are given in Appendix.

M2-brane solutions on $R_t\times S^7/Z_k$ and dispersion relations

For our membrane embedding in $R_t\times S^7/Z_k$, the angular variables φ_a are related to the corresponding background coordinates as follows

$$\begin{aligned}
\varphi_1 &= \frac{\varphi}{k}+\frac{1}{2}\left(\phi_1+\phi_3\right), \quad \varphi_2 = \frac{\varphi}{k}-\frac{1}{2}\left(\phi_1-\phi_3\right),\\
\varphi_3 &= \frac{\varphi}{k}+\frac{1}{2}\left(\phi_2-\phi_3\right), \quad \varphi_4 = \frac{\varphi}{k}-\frac{1}{2}\left(\phi_2+\phi_3\right).
\end{aligned}$$

As a consequence, for the angular momenta we have

$$\begin{aligned}
J_{\varphi_1} &= \frac{J_\varphi}{k}+\frac{1}{2}\left(J_{\phi_1}+J_{\phi_3}\right), \quad J_{\varphi_2} = \frac{J_\varphi}{k}-\frac{1}{2}\left(J_{\phi_1}-J_{\phi_3}\right),\\
J_{\varphi_3} &= \frac{J_\varphi}{k}+\frac{1}{2}\left(J_{\phi_2}-J_{\phi_3}\right), \quad J_{\varphi_4} = \frac{J_\varphi}{k}-\frac{1}{2}\left(J_{\phi_2}+J_{\phi_3}\right).
\end{aligned}$$

φ_a and J_{φ_a} satisfy the equalities

$$\sum_{a=1}^{4}\varphi_a=\frac{4}{k}\varphi, \quad \sum_{a=1}^{4}J_{\varphi_a}=\frac{4}{k}J_\varphi.$$

One of the conditions for the existence of NR description of the M2-brane dynamics is that two of the angles φ_a must be zero, which means that two of the four angular momenta J_{φ_a} vanish. The six possible cases are

$$\begin{aligned}
&\bullet\ \varphi_1=\phi_3+\frac{\phi_1}{2}=\frac{2}{k}\varphi+\frac{\phi_1}{2}, \quad \varphi_2=\phi_3-\frac{\phi_1}{2}=\frac{2}{k}\varphi-\frac{\phi_1}{2}, \quad \varphi_3=0, \quad \varphi_4=0;\\
&\bullet\ \varphi_1=\phi_1=\frac{2}{k}\varphi+\phi_3, \quad \varphi_3=\phi_2=\frac{2}{k}\varphi-\phi_3, \quad \varphi_2=0, \quad \varphi_4=0;\\
&\bullet\ \varphi_1=\phi_1=\frac{2}{k}\varphi+\phi_3, \quad \varphi_4=-\phi_2=\frac{2}{k}\varphi-\phi_3, \quad \varphi_2=0, \quad \varphi_3=0;\\
&\bullet\ \varphi_2=-\phi_1=\frac{2}{k}\varphi+\phi_3, \quad \varphi_3=\phi_2=\frac{2}{k}\varphi-\phi_3, \quad \varphi_1=0, \quad \varphi_4=0;\\
&\bullet\ \varphi_2=-\phi_1=\frac{2}{k}\varphi+\phi_3, \quad \varphi_4=-\phi_2=\frac{2}{k}\varphi-\phi_3, \quad \varphi_1=0, \quad \varphi_3=0;\\
&\bullet\ \varphi_3=-\phi_3+\frac{\phi_2}{2}=\frac{2}{k}\varphi+\frac{\phi_2}{2}, \quad \varphi_4=-\phi_3-\frac{\phi_2}{2}=\frac{2}{k}\varphi-\frac{\phi_2}{2}, \quad \varphi_1=0, \quad \varphi_2=0.
\end{aligned} \tag{3.338}$$

Here, ϕ_1 and ϕ_2 are the isometry angles on the two two-spheres inside $\mathbb{CP}^3$, while ϕ_3 is isometry angle on the $U(1)$ fiber over $S^2\times S^2$, as can be seen from (3.327).

From (3.338) it is clear that we have two alternative descriptions for φ_a. One is only in terms of the isometry angles on $\mathbb{CP}^3$, and the other includes the eleventh coordinate φ. This is a consequence of our restriction to M2-brane configurations, which can be described by the NR integrable system.

The six cases above can be divided into two classes. The first one contains the first and last possibilities, and the other one - the remaining ones. The cases belonging to the first class are related to each other by the exchange of ϕ_1 and ϕ_2. This corresponds to exchanging the two S^2 inside $\mathbb{CP}^3$. Since these spheres enter symmetrically, the two cases are equivalent. In terms of (φ,ϕ_3), the four cases from the second class are actually identical. That is why, all of them can be described simultaneously by choosing one representative from the class.

Let us first give the M2-brane solutions for cases in the first class. Since they correspond to our example in the previous section, the membrane configuration reads

$$\begin{aligned}
&W_1 = Rr_1(\xi)\exp\{i\varphi_1(\tau,\xi)\} = R\sqrt{1-r_0^2}\sin\theta(\xi)\exp\left\{i\left[\frac{2}{k}\varphi(\tau,\xi)+\frac{\phi(\tau,\xi)}{2}\right]\right\},\\
&W_2 = Rr_2(\xi)\exp\{i\varphi_2(\tau,\xi)\} = R\sqrt{1-r_0^2}\cos\theta(\xi)\exp\left\{i\left[\frac{2}{k}\varphi(\tau,\xi)-\frac{\phi(\tau,\xi)}{2}\right]\right\},\\
&W_3 = Rr_0\sin(\gamma\sigma_2), \qquad W_4 = Rr_0\cos(\gamma\sigma_2),
\end{aligned}$$

where ϕ is equal to ϕ_1 or ϕ_2.

From the NR system viewpoint, the membrane solutions for the second class configurations differ from the ones just given by the exchange of W_2, W_3, and by the replacement $\phi/2 \to \phi_3$. In other words, we have

$$\begin{aligned}
&W_1 = Rr_1(\xi)\exp\{i\varphi_1(\tau,\xi)\} = R\sqrt{1-r_0^2}\sin\theta(\xi)\exp\left\{i\left[\frac{2}{k}\varphi(\tau,\xi)+\phi_3(\tau,\xi)\right]\right\},\\
&W_2 = Rr_0\sin(\gamma\sigma_2),\\
&W_3 = Rr_3(\xi)\exp\{i\varphi_3(\tau,\xi)\} = R\sqrt{1-r_0^2}\cos\theta(\xi)\exp\left\{i\left[\frac{2}{k}\varphi(\tau,\xi)-\phi_3(\tau,\xi)\right]\right\},\\
&W_4 = Rr_0\cos(\gamma\sigma_2),
\end{aligned}$$

The explicit solutions for $\theta(\xi)$ and $\varphi_{1,2,3}(\tau,\xi)$, of the M2-brane GM and SS, along with the energy-charge relations for the infinite and finite sizes are given in the Appendix E (see [16]). Here, we will present them in terms of φ and $\phi_{1,2,3}$.

In accordance with (A.27), we have for the M2-brane GM with two angular momenta the following dispersion relation

$$\sqrt{1-r_0^2}E-\frac{1}{2}\left(\frac{2}{k}J_\varphi+J_\phi\right)=\sqrt{\frac{1}{4}\left(\frac{2}{k}J_\varphi-J_\phi\right)^2+8\lambda k^2\left[r_0(1-r_0^2)\gamma\right]^2\sin^2\frac{p}{2}}, \tag{3.339}$$

where J_ϕ can be equal to $J_{\phi_1}/2$, $J_{\phi_2}/2$ or J_{ϕ_3}. In writing (3.339), we have used that

$$R = l_p\left(2^5\pi^2kN\right)^{1/6}, \qquad T_2 = \frac{1}{(2\pi)^2 l_p^3},$$

and the 't Hooft coupling is defined by $\lambda = N/k$.

If we introduce the notations

$$\mathcal{E} = a\sqrt{1-r_0^2}E, \quad \mathcal{J}_\varphi = a\frac{J_\varphi}{2}, \quad \mathcal{J}_\phi = a\frac{J_\phi}{2}, \quad a = \frac{1}{\sqrt{2\lambda}kr_0(1-r_0^2)\gamma}, \tag{3.340}$$

the above energy-charge relation takes the form

$$\mathcal{E} - \mathcal{J}_1(k) = \sqrt{\mathcal{J}_2^2(k) + 4\sin^2\frac{p}{2}},$$

where

$$\mathcal{J}_1(k) = \frac{2}{k}\mathcal{J}_\varphi + \mathcal{J}_\phi, \quad \mathcal{J}_2(k) = \frac{2}{k}\mathcal{J}_\varphi - \mathcal{J}_\phi. \tag{3.341}$$

By using (3.340), (3.341) and (A.35), we can write down the dispersion relation for the dyonic GM, including the leading finite-size correction as

$$\mathcal{E} - \mathcal{J}_1(k) = \sqrt{\mathcal{J}_2^2(k) + 4\sin^2\frac{p}{2}} - \frac{16\sin^4\frac{p}{2}}{\sqrt{\mathcal{J}_2^2(k) + 4\sin^2\frac{p}{2}}}$$
$$\exp\left[-\frac{2\sin^2\frac{p}{2}\left(\mathcal{J}_1(k) + \sqrt{\mathcal{J}_2^2(k) + 4\sin^2\frac{p}{2}}\right)\sqrt{\mathcal{J}_2^2(k) + 4\sin^2\frac{p}{2}}}{\mathcal{J}_2^2(k) + 4\sin^4\frac{p}{2}}\right].$$

The reason to introduce $\mathcal{E}$, $\mathcal{J}_\varphi$ and $\mathcal{J}_\phi$ namely in this way is the following. For GM on the $R_t \times S^3$ subspace of $AdS_5 \times S^5$, in terms of

$$\mathcal{E} = \frac{2\pi}{\sqrt{\lambda}}E, \quad \mathcal{J}_1 = \frac{2\pi}{\sqrt{\lambda}}J_1, \quad \mathcal{J}_2 = \frac{2\pi}{\sqrt{\lambda}}J_2,$$

we have

$$\mathcal{E} - \mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4\sin^2\frac{p}{2}}.$$

The same result can be obtained for the GM on the $R_t \times \mathbb{CP}^3$ subspace of $AdS_4 \times \mathbb{CP}^3$, if we use the identification [67]

$$\mathcal{E} = \frac{E}{\sqrt{2\lambda}}, \quad \mathcal{J}_1 = \frac{J_1}{\sqrt{2\lambda}}, \quad \mathcal{J}_2 = \frac{J_2}{\sqrt{2\lambda}}.$$

In the all three cases, the second term under the square root is the same. In this description it is universal - for different backgrounds and for different extended objects.

Analogously, for the SS case one can find (see (A.37))

$$\mathcal{E} - \Delta\varphi_1 = p + 8\sin^2\frac{p}{2}\tan\frac{p}{2}\exp\left(-\frac{(\Delta\varphi_1 + p)\tan\frac{p}{2}}{\mathcal{J}_2^2(k)\csc^2 p + \tan^2\frac{p}{2}}\right)$$
$$\mathcal{J}_1(k) = \sqrt{\mathcal{J}_2^2(k) + 4\sin^2\frac{p}{2}}.$$

Let us point out that for $k = 1$ the above dispersion relations coincide with the ones obtained earlier in [15]. We can also reproduce the energy-charge relations for dyonic GM and SS strings on $R_t \times \mathbb{CP}^3$ by taking an appropriate limit. To show this, let us consider the second case in (3.338), for which

$$J_{\phi_1} = \frac{2}{k} J_\varphi + J_{\phi_3}, \qquad J_{\phi_2} = \frac{2}{k} J_\varphi - J_{\phi_3}.$$

In accordance with our membrane embedding, the following identification should be made

$$J_1^{str} = \frac{J_{\phi_1}}{2}, \qquad J_2^{str} = \frac{J_{\phi_2}}{2}.$$

Then in the limit $k \to \infty$, $r_0 \to 0$, such that $k r_0 \gamma = 1$, we obtain from (3.339)

$$E - J_1^{str} = \sqrt{(J_2^{str})^2 + 8\lambda \sin^2 \frac{p}{2}}.$$

This is exactly what we have derived in [67] for dyonic GM strings on $R_t \times \mathbb{CP}^3$. Obviously, this also applies for the leading finite-size correction. In the same way, the SS string dispersion relation for $R_t \times \mathbb{CP}^3$ background can be reproduced.

4 Three-point correlation functions

The AdS/CFT conjecture [33] implies that the correlation functions in the dual (boundary) quantum field theory can be computed alternatively in string theory, i.e., essentially by the methods of a two - dimensional theory. The first computations however were mostly performed in the supergravity approximation, representing the correlators in terms of integrals over the target (bulk) coordinates [90, 91].

On the string side one may start with a semiclassical approach, when the string path integral for the correlation functions is evaluated in the saddle-point approximation with large 't Hooft coupling $\lambda >> 1$. In this calculation one has to identify the correct vertex operators [92, 93] and to find the corresponding classical solutions, which provide the appropriate saddle-point approximation.

It is known that the correlation functions of any conformal field theory can be determined in principle in terms of the basic conformal data $\{\Delta_i, C_{ijk}\}$, where Δ_i are the conformal dimensions defined by the two-point correlation functions

$$\left\langle \mathcal{O}_i^\dagger(x_1)\mathcal{O}_j(x_2)\right\rangle = \frac{C_{12}\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}}$$

and C_{ijk} are the structure constants in the operator product expansion

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}.$$

Therefore, the determination of the initial conformal data for a given conformal field theory is the most important step in the conformal bootstrap approach.

The three-point functions of two "heavy" operators and a "light" operator can be approximated by a supergravity vertex operator evaluated at the "heavy" classical string configuration [99, 100]:

$$\langle V_H(x_1)V_H(x_2)V_L(x_3)\rangle = V_L(x_3)_{\text{classical}}.$$

For $|x_1| = |x_2| = 1$, $x_3 = 0$, the correlation function reduces to

$$\langle V_H(x_1)V_H(x_2)V_L(0)\rangle = \frac{C_{123}}{|x_1 - x_2|^{2\Delta_H}}.$$

Then, the normalized structure constants

$$\mathcal{C} = \frac{C_{123}}{C_{12}}$$

can be found from

$$\mathcal{C} = c_\Delta V_L(0)_{\text{classical}}, \tag{4.1}$$

were c_Δ is the normalized constant of the corresponding "light" vertex operator.

4.1 On the semiclassical three-point correlation function in AdS_3

In [24] we reconsider the problem of determining the semiclassical 3-point function in the Euclidean AdS_3 model. Exploiting the affine symmetry of the model we use solutions of the classical Knizhnik-Zamolodchikov (KZ) equation to compute the saddle point of the action in the presence of three vertex operators. This alternative derivation reproduces the "heavy charge" classical limit of the quantum 3-point correlator. Our result is different from the proposed expression obtained by generalised Pohlmeyer reduction in AdS_2 [94]. The latter is interpreted as the AdS_2 part of the correlator of three heavy strings propagating in $AdS_2 \times S^5$ model, and is assumed to be universal for heavy (scalar) AdS_{d+1} operators. This motivated us to reconsider the problem and compute directly the semiclassical constant, generalising the method in [95], which was proposed originally as a semiclassical check of the quantum Liouville 3-point constant.

While the quantum AdS_3 theory and its applications to the superstring on $AdS_3 \times S^3 \times M^4$ in the NS-NS background (see [96, 97] and references therein) is well studied, and, thus, this is no more than a toy model in the semiclassical context under consideration, the elaboration of 2d CFT techniques is important for the analogous unsolved problems in more realistic and less known cases.

Our derivation is based on the affine algebra symmetry of the model generated by a current and (in the euclidean version) its complex conjugate. This leads to a chiral equation, a classical version of the KZ equation [98]. The equation determines the classical fundamental vertex V of isospin $j = 1/2$ as a function of the coordinates of the three vertex sources. The solution is then used, analogously to [95], to evaluate the contribution of the sources to the saddle point of the action and thus to compute the semiclassical 3-point function, confirming the direct classical limit of the quantum correlator.

At the saddle point of $S^{(cl)} = S^{(cl)}[\phi, \gamma, \bar{\gamma}]$, i.e., on a solution of the classical equations, one obtains

$$\frac{\partial}{\partial \eta_i} \hat{S}^{(cl)} = -X_i, \qquad i = 1, 2, 3, \tag{4.2}$$

where X_i are the string coordinates on AdS_3,

$$\hat{S}^{(cl)} = S^{(cl)} + S^{(src)},$$

and the terms

$$\begin{aligned} S^{(src)} &= -\int d^2 z \sum_i \delta^2(z - z_i, \bar{z} - \bar{z}_i) \left(2\eta_i \phi^{(-)}(z, \bar{z}) + 2\eta_i^2 \log \mid z - z_i \mid^2 \right) \\ &= \sum_i \eta_i X_i \end{aligned} \tag{4.3}$$

account for the three vertex insertions.

The set of equations (4.2) integrates to

$$\begin{aligned}\frac{\hat{S}^{(cl)}}{b^2} &= \sum_{i\neq j\neq k\neq i}\left(\delta_{ij}^k \log \mid z_{ij}\mid^2 - j_{ij}^k \log\mid x_{ij}\mid^2\right) \\ &+ \frac{1}{b^2}\left[F(\eta_{123}) + \sum_i F(\eta_{123}-2\eta_i) - \sum_i F(2\eta_i) - F(0)\right],\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}&\delta_{ik}^l = \delta_i + \delta_k - \delta_l,\ \delta_i = -\frac{\eta_i^2}{b^2},\ j_{ik}^l = j_i + j_k - j_l,\ j_i = -\frac{\eta_i}{b^2}, \\ &\frac{1}{b^2}F(\eta) := \frac{1}{b^2}\int_{1/2}^{\eta} dx \log\gamma(x).\end{aligned}$$

From the r.h.s. of (4.4) one obtains the 3-point function

$$e^{-\frac{\hat{S}^{(cl)}}{b^2}} = C^{(cl)}(\eta_1,\eta_2,\eta_3)\ \frac{\mid x_{13}\mid^{2j_{13}^2}\mid x_{12}\mid^{2j_{12}^3}\mid x_{23}\mid^{2j_{23}^1}}{\mid z_{13}\mid^{2\delta_{13}^2}\mid z_{12}\mid^{2\delta_{12}^3}\mid z_{23}\mid^{2\delta_{23}^1}}. \tag{4.5}$$

Since $F(x) = F(1-x)$, the choice $F(0) = F(1)$ for the arbitrary integration constant ensures that $C^{(cl)} = 1$ for $\sum_i \eta_i = 1$, a charge conservation condition, corresponding to absence of screening charges in the quantum case.

4.2 Semiclassical three-point correlation functions in $AdS_5 \times S^5$

In [19] we computed holographic three-point correlation functions or structure constants of a zero-momentum dilaton operator and two (dyonic) giant magnon string states with a finite-size length in the semiclassical approximation. We show that the semiclassical structure constants match exactly with the three-point functions between two $su(2)$ magnon single trace operators with finite size and the Lagrangian in the large 't Hooft coupling constant limit. A special limit $J \gg \sqrt{\lambda}$ of our result is compared with the relevant result based on the Lüscher corrections.

[19] is the first paper where the finite-size effects on the semiclassical three-point correlation functions have been taken into account.

In [22, 23, 25] we extended the results found in [19] to the following three cases:

1. Dilaton operator with non-zero momentum: $V_L = V_j^d$
2. Primary scalar operators: $V_L = V_j^{pr}$
3. Singlet scalar operators on higher string levels: $V_L = V^q$

4.2.1 Two GM states and dilaton with zero momentum

Let us start with the case of two GM and the zero-momentum dilaton operator, namely the Lagrangian whose vertex operator is given by

$$V^d = (Y_4 + Y_5)^{-4} \left[z^{-2} \left(\partial_+ x_m \partial_- x^m + \partial_+ z \partial_- z \right) + \partial_+ X_k \partial_- X_k \right], \tag{4.6}$$

where

$$Y_4 = \frac{1}{2z} \left(x^m x_m + z^2 - 1 \right), \qquad Y_5 = \frac{1}{2z} \left(x^m x_m + z^2 + 1 \right),$$

and x_m, z are coordinates on AdS_5, while X_k are the coordinates on S^5.

Giant magnons with finite size

The finite-size giant magnon solution can be represented as $(i\tau = \tau_e)$

$$\begin{aligned}
& x_{0e} = \tanh(\kappa\tau_e), \qquad x_i = 0, \qquad z = \frac{1}{\cosh(\kappa\tau_e)}, \\
& \cos\theta = \sqrt{1 - v^2\kappa^2}\ \mathbf{dn} \left(\frac{\sqrt{1 - v^2\kappa^2}}{1 - v^2} (\sigma - v\tau) \Big| 1 - \epsilon \right), \\
& \phi = \frac{\tau - v\sigma}{1 - v^2} + \frac{1}{v\sqrt{1 - v^2\kappa^2}} \times \\
& \Pi \left(am \left(\frac{\sqrt{1 - v^2\kappa^2}}{1 - v^2} (\sigma - v\tau) \right), -\frac{1 - v^2\kappa^2}{v^2\kappa^2} (1 - \epsilon), \Big| 1 - \epsilon \right),
\end{aligned} \tag{4.7}$$

where

$$\epsilon = \frac{1 - \kappa^2}{1 - v^2\kappa^2}.$$

To find the finite-size effect on the three-point correlator, we will use (4.1) and (4.6), which computed on (4.7) gives

$$\mathcal{C}^d = c^d_\Delta \int_{-\infty}^{\infty} \frac{d\tau_e}{\cosh^4(\kappa\tau_e)} \int_{-L}^{L} d\sigma \left[\kappa^2 + \partial_+ X_k \partial_- X_k \right], \tag{4.8}$$

where

$$\begin{aligned}
\partial_+ X_k \partial_- X_k \quad = \quad & -\frac{1}{(1 - v^2)\sin^2\theta} \left\{ 2 - (1 + v^2)\kappa^2 \right. \\
& \left. - \cos^2\theta \left[4 - (1 + v^2)\kappa^2 - 2\cos^2\theta \right] \right\}.
\end{aligned}$$

Performing the integrations in (4.8), one finds

$$\mathcal{C}^d = \frac{16}{3} c^d_\Delta \sqrt{\frac{1 - v^2}{1 - \epsilon}} \left[\mathbf{E}(1 - \epsilon) - \epsilon\ \mathbf{K}(1 - \epsilon) \right]. \tag{4.9}$$

Let us point out that the parameter L in (4.9) is given by

$$L = \frac{1-v^2}{\sqrt{1-v^2\kappa^2}}\mathbf{K}(1-\epsilon).$$

This is our *exact* result for the normalized coefficient $\mathcal{C}^d$ in the semiclassical three-point correlation function, corresponding to the case when the "heavy" vertex operators are *finite-size* giant magnons, and the light vertex is taken to be the zero-momentum dilaton operator.

For the case of this dilaton operator, the three-point function of the SYM can be easily related to the conformal dimension of the heavy operators. This corresponds to shift 't Hooft coupling constant which is the overall coefficient of the Lagrangian [101]. This gives an important relation between the structure constant and the conformal dimension as follows:

$$\mathcal{C}_3^d = \frac{32\pi}{3} c_\Delta^d \sqrt{\lambda}\partial_\lambda \Delta. \tag{4.10}$$

We want to show that this relation is correct for the case of the giant magnons with arbitrary finite size.

In the context of the AdS/CFT correspondence, it is now well-established that the conformal dimension of a single trace operator with one magnon state is the same as $E-J$ in the strong coupling limit. For an exact relation from the gauge theory side, one should solve the thermodynamic Bethe equations. It has been shown that finite-size corrections to the conformal dimensions of the SYM (dyonic) giant magnon operators computed by the Lüscher formula for $J \gg \sqrt{\lambda}$ match exactly with $E-J$ of corresponding string state configurations. Based on these results, we can assume that the conformal dimensions Δ of the SYM operators are the same as $E-J$ of corresponding string states.

The *exact* classical expression for finite-size giant magnon energy-charge relation is given by

$$E - J \equiv \Delta = \frac{\sqrt{\lambda}}{\pi}\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{E}(1-\epsilon) - \left(1-\sqrt{(1-v^2\epsilon)(1-\epsilon)}\right)\mathbf{K}(1-\epsilon)\right]. \tag{4.11}$$

The corresponding expressions for J and p are

$$J = \frac{\sqrt{\lambda}}{\pi}\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{K}(1-\epsilon) - \mathbf{E}(1-\epsilon)\right],$$
$$p = 2v\sqrt{\frac{1-v^2\epsilon}{1-v^2}}\left[\frac{1}{v^2}\mathbf{\Pi}\left(1-\frac{1}{v^2}\middle|1-\epsilon\right) - \mathbf{K}(1-\epsilon)\right],$$

where J is the angular momentum of the string, and p is the magnon momentum. One can obtain $E-J$ in terms of J and p by eliminating v, ϵ from these expressions.

To take λ-derivative on Δ, we need know λ dependence of v and ϵ. Our strategy is to find $v'(\lambda)$ and $\epsilon'(\lambda)$ from the conditions that J and p are independent variables of λ, namely,

$$\frac{dJ}{d\lambda} = \frac{dp}{d\lambda} = 0. \tag{4.12}$$

Solving these conditions, we find the derivatives of the functions $v(\lambda)$ and $\epsilon(\lambda)$

$$\frac{dv}{d\lambda} = -\frac{v(1-v^2)\epsilon\left[\mathbf{E}(1-\epsilon)-\mathbf{K}(1-\epsilon)\right]^2}{2\lambda(1-\epsilon)\left[\mathbf{E}(1-\epsilon)^2-v^2\epsilon\mathbf{K}(1-\epsilon)^2\right]},$$
$$\frac{d\epsilon}{d\lambda} = -\frac{\epsilon\left[\mathbf{E}(1-\epsilon)-\mathbf{K}(1-\epsilon)\right]\left[\mathbf{E}(1-\epsilon)-v^2\epsilon\mathbf{K}(1-\epsilon)\right]}{\lambda\left[\mathbf{E}(1-\epsilon)^2-v^2\epsilon\mathbf{K}(1-\epsilon)^2\right]}. \tag{4.13}$$

Replacing (4.13) into the derivative of (4.11), one finds

$$\lambda\partial_\lambda\Delta = \frac{\sqrt{\lambda}}{2\pi}\sqrt{\frac{1-v^2}{1-\epsilon}}\left[\mathbf{E}(1-\epsilon)-\epsilon\mathbf{K}(1-\epsilon)\right]. \tag{4.14}$$

Comparing (4.9) and (4.14), we conclude that the equality (4.10) holds.

Next, we would like to compare (4.9) with the known leading finite-size correction to the giant magnon dispersion relation [64]. To this end, we have to consider the limit $\epsilon \to 0$ in (4.9). Taking into account the behavior of the elliptic integrals in the $\epsilon \to 0$ limit, we can use the ansatz

$$v(\epsilon) = v_0 + v_1\epsilon + v_2\epsilon\log(\epsilon). \tag{4.15}$$

Actually, all parameters in (4.15) are already known and are given by

$$v_0 = \cos(p/2), \qquad v_1 = \frac{1}{4}\sin^2(p/2)\cos(p/2)(1-\log(16)),$$
$$v_2 = \frac{1}{4}\sin^2(p/2)\cos(p/2), \qquad \epsilon = 16\exp\left(-\frac{2\pi J}{\sqrt{\lambda}\sin(p/2)}-2\right). \tag{4.16}$$

Expanding (4.9) in ϵ and using (4.15), (4.16), we obtain

$$\mathcal{C}^d = \frac{16}{3}c^d_\Delta\sin(p/2)\left[1-4\sin(p/2)\left(\sin(p/2)+\frac{2\pi J}{\sqrt{\lambda}}\right)\exp\left(-\frac{2\pi J}{\sqrt{\lambda}\sin(p/2)}-2\right)\right]. \tag{4.17}$$

On the other hand, from the giant magnon dispersion relation, including the leading finite-size effect,

$$\Delta = \frac{\sqrt{\lambda}}{\pi}\sin(p/2)\left[1-4\sin^2(p/2)\exp\left(-\frac{2\pi J}{\sqrt{\lambda}\sin(p/2)}-2\right)\right],$$

one finds

$$\lambda\partial_\lambda\Delta = \frac{\sqrt{\lambda}}{2\pi}\sin(p/2)\left[1-4\sin(p/2)\left(\sin(p/2)+\frac{2\pi J}{\sqrt{\lambda}}\right)\exp\left(-\frac{2\pi J}{\sqrt{\lambda}\sin(p/2)}-2\right)\right]. \tag{4.18}$$

This confirms explicitly that the relation (4.10) holds in the small ϵ i.e. $J \gg \sqrt{\lambda}$ limit.

Dyonic giant magnons with finite size

The dyonic finite-size giant magnon solution is given by

$$\begin{aligned}
&x_{0e} = \tanh(\kappa\tau_e), \quad x_i = 0, \quad z = \frac{1}{\cosh(\kappa\tau_e)}, \\
&\cos\theta = z_+ \mathbf{dn}\left(\frac{\sqrt{1-u^2}}{1-v^2} z_+(\sigma - v\tau)\Big|1-\epsilon\right), \\
&\phi_1 = \frac{\tau - v\sigma}{1-v^2} + \frac{vW}{\sqrt{1-u^2} z_+(1-z_+^2)} \times \\
&\Pi\left(am\left(\frac{\sqrt{1-u^2}}{1-v^2} z_+(\sigma - v\tau)\right), -\frac{z_+^2}{1-z_+^2}(1-\epsilon), \Big|1-\epsilon\right) \\
&\phi_2 = u\frac{\tau - v\sigma}{1-v^2},
\end{aligned} \tag{4.19}$$

where

$$\epsilon = \frac{z_-^2}{z_+^2}, \quad W = \kappa^2.$$

$z_\pm^2$ can be written as

$$\begin{aligned}
&z_\pm^2 = \frac{1}{2(1-u^2)}\left\{q_1 + q_2 - u^2 \pm \sqrt{(q_1-q_2)^2 - \left[2\left(q_1+q_2-2q_1q_2\right) - u^2\right]u^2}\right\}, \\
&q_1 = 1 - W, \quad q_2 = 1 - v^2 W.
\end{aligned}$$

Now, we have to replace into (4.8) the following expression obtained from the above solution

$$\begin{aligned}
\partial_+ X_k \partial_- X_k \;=\; & -\frac{1}{(1-v^2)\sin^2\theta}\left\{1 - v^2W^2 + (1-u^2)z_+^4\epsilon + 2(1-u^2)\cos^4\theta\right. \\
& \left. - \cos^2\theta\left[2 + z_+^2(1+\epsilon) - u^2\left(1 + z_+^2(1+\epsilon)\right)\right]\right\}.
\end{aligned}$$

Computing the integrals in (4.8), we find

$$\begin{aligned}
&\mathcal{C}^d = \frac{8}{3} c_\Delta^d \frac{1}{\sqrt{(1-u^2)W\chi_p}(1-\chi_p)}\Bigg\{(1-\chi_p)\left[2(1-u^2)\chi_p \mathbf{E}(1-\epsilon)\right. \\
&- \left.\left(u^2 - \left(1-v^2\right)W + (1-u^2)(1+\epsilon)\chi_p\right)\mathbf{K}(1-\epsilon)\right] \\
&- \left(1 - v^2W^2 - \chi_p - (1-\chi_p)\left(\epsilon\chi_p + u^2(1-\epsilon\chi_p)\right)\right) \times \\
&\mathbf{\Pi}\left(-\frac{\chi_p}{1-\chi_p}(1-\epsilon)\Big|1-\epsilon\right)\Bigg\},
\end{aligned} \tag{4.20}$$

where we introduced the notations

$$\chi_p = z_+^2, \quad \chi_m = z_-^2, \quad \Rightarrow \epsilon = \frac{\chi_m}{\chi_p}.$$

This is our *exact* result for the normalized coefficient $\mathcal{C}^d$ in the three-point correlation function, corresponding to the case when the "heavy" vertex operators are *finite-size* dyonic giant magnons.

To check the relation (4.10), we need to know Δ. As GM case, we claim that this is given by $E - J_1$. The explicit results are given by [15]

$$\begin{aligned}
\mathcal{E} &= \frac{2\sqrt{W}(1-v^2)}{\sqrt{1-u^2}\sqrt{\chi_p}}\mathbf{K}(1-\epsilon), \\
\mathcal{J}_1 &= \frac{2\sqrt{\chi_p}}{\sqrt{1-u^2}}\left[\frac{1-v^2W}{\chi_p}\mathbf{K}(1-\epsilon) - \mathbf{E}(1-\epsilon)\right], \qquad (4.21) \\
\mathcal{J}_2 &= \frac{2u\sqrt{\chi_p}}{\sqrt{1-u^2}}\mathbf{E}(1-\epsilon) \\
p &= \frac{2v}{\sqrt{1-u^2}\sqrt{\chi_p}}\left[\frac{W}{1-\chi_p}\mathbf{\Pi}\left(-\frac{\chi_p}{1-\chi_p}(1-\epsilon)\middle|1-\epsilon\right) - \mathbf{K}(1-\epsilon)\right], \qquad (4.22)
\end{aligned}$$

and

$$\mathcal{E} = \frac{2\pi E}{\sqrt{\lambda}}, \qquad \mathcal{J}_{1,2} = \frac{2\pi J_{1,2}}{\sqrt{\lambda}}.$$

In this case, we need to obtain $v'(\lambda), \epsilon'(\lambda), u'(\lambda)$ from the condition that J_1, J_2, p be independent of λ. It turns out that the exact calculations for these are too complicated. Instead, we will just focus on the $\epsilon \to 0$ limit of (4.20) and λ derivative of Δ from the Lüscher formula to check (4.10). To this end, we will use the expansions

$$\begin{aligned}
&\chi_p = \chi_{p0} + (\chi_{p1} + \chi_{p2}\log(\epsilon))\,\epsilon, \qquad \chi_m = \chi_{m1}\epsilon, \\
&v = v_0 + (v_1 + v_2\log(\epsilon))\,\epsilon, \qquad u = u_0 + (u_1 + u_2\log(\epsilon))\,\epsilon, \qquad (4.23) \\
&W = 1 + W_1\epsilon.
\end{aligned}$$

First note that χ_p and χ_m satisfy the following relations

$$\begin{aligned}
\chi_p + \chi_m &= \frac{2-(1+v^2)W-u^2}{1-u^2} \qquad (4.24) \\
\chi_p\chi_m &= \frac{1-(1+v^2)W-v^2W^2}{1-u^2}.
\end{aligned}$$

Expanding (4.24) and using the definition of ϵ, we arrive at

$$\begin{aligned}
\chi_{p0} &= 1 - \frac{v_0^2}{1-u_0^2}, \qquad (4.25) \\
\chi_{p1} &= \frac{v_0}{\left(1-v_0^2\right)\left(1-u_0^2\right)^2}\Big\{v_0\left[(1-v_0^2)^2 - 3(1-v_0^2)u_0^2 + 2u_0^4 - 2(1-v_0^2)u_0u_1\right] \\
&\quad -2\left(1-v_0^2\right)\left(1-u_0^2\right)v_1\Big\}, \\
\chi_{p2} &= -2v_0\frac{v_2 + (v_0u_2 - u_0v_2)u_0}{\left(1-u_0^2\right)^2} \\
\chi_{m1} &= 1 - \frac{v_0^2}{1-u_0^2}, \\
W_1 &= -\frac{(1-u_0^2-v_0^2)^2}{(1-u_0^2)(1-v_0^2)}.
\end{aligned}$$

The coefficients in the expansions of v and u, we take from [102], where for the case under consideration we have to set $K_1 = \chi_{n1} = 0$, or equivalently $\Phi = 0$. This gives

$$
\begin{aligned}
v_0 &= \frac{\sin(p)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}, \quad u_0 = \frac{\mathcal{J}_2}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}} \\
v_1 &= \frac{v_0(1-v_0^2-u_0^2)}{4(1-u_0^2)(1-v_0^2)} \left[(1-v_0^2)(1-\log(16)) - u_0^2\left(5 - v_0^2(1+\log(16)) - \log(4096)\right)\right] \\
v_2 &= \frac{v_0(1-v_0^2-u_0^2)}{4(1-u_0^2)(1-v_0^2)} \left[1 - v_0^2 - u_0^2(3+v_0^2)\right] \\
u_1 &= \frac{u_0(1-v_0^2-u_0^2)}{4(1-v_0^2)} \left[1 - \log(16) - v_0^2(1+\log(16))\right] \\
u_2 &= \frac{u_0(1-v_0^2-u_0^2)}{4(1-v_0^2)}(1+v_0^2).
\end{aligned}
\tag{4.26}
$$

We need also the expression for ϵ. To the leading order, it can be written as [102]

$$
\epsilon = 16\exp\left(-\frac{2 - \frac{2v_0^2}{1-u_0^2} + \mathcal{J}_1\sqrt{1-v_0^2-u_0^2}}{1-v_0^2}\right). \tag{4.27}
$$

By using (4.25), (4.26) and (4.27) in the ϵ-expansion of (4.20), we derive

$$
\begin{aligned}
\mathcal{C}^d = \frac{16}{3}c_\Delta^d \Bigg\{ &\frac{\mathcal{J}_2^2 + 4\sin^2(p/2) - 16\sin^4(p/2)\exp(f)}{2\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}} \\
&+ \frac{1}{\left(\mathcal{J}_2^2 + 4\sin^2(p/2)\right)\left(\mathcal{J}_2^2 + 4\sin^4(p/2)\right)^2} \Bigg[32\exp(f)\Bigg(2\mathcal{J}_2^2\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} - 3\mathcal{J}_1 \\
&+ 2\left(\mathcal{J}_1\left(2+\mathcal{J}_2^2\right) + \mathcal{J}_2^2\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)\cos(p) - \mathcal{J}_1\cos(2p)\Bigg)\sin^8(p/2)\Bigg] \\
&- \frac{\mathcal{J}_2^2}{2\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}} - \frac{8\mathcal{J}_2^2\sin^4(p/2)}{\left(\mathcal{J}_2^2 + 4\sin^2(p/2)\right)^{3/2}}\exp(f)\Bigg\},
\end{aligned}
\tag{4.28}
$$

where

$$
f = -\frac{2\left(\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\sin^2(p/2)}{\mathcal{J}_2^2 + 4\sin^4(p/2)}.
$$

On the other hand, from the dyonic giant magnon dispersion relation, including the leading finite-size correction,

$$
\Delta_{dyonic} = \frac{\sqrt{\lambda}}{2\pi}\left[\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} - \frac{16\sin^4(p/2)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}\exp(f)\right], \tag{4.29}
$$

one obtains

$$\lambda\partial_\lambda \Delta_{dyonic} = \frac{\sqrt{\lambda}}{2\pi}\Bigg\{ \frac{\mathcal{J}_2^2 + 4\sin^2(p/2) - 16\sin^4(p/2)\exp(f)}{2\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}} \tag{4.30}$$

$$+\frac{1}{\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)^2}\Bigg[32\exp(f)\left(2\mathcal{J}_2^2\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}-3\mathcal{J}_1\right.$$

$$\left.+2\left(\mathcal{J}_1\left(2+\mathcal{J}_2^2\right)+\mathcal{J}_2^2\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\right)\cos(p)-\mathcal{J}_1\cos(2p)\right)\sin^8(p/2)\Bigg]$$

$$-\frac{\mathcal{J}_2^2}{2\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}-\frac{8\mathcal{J}_2^2\sin^4(p/2)}{\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}}\exp(f)\Bigg\}.$$

Comparing (4.28) and (4.30), we see that the relation (4.10) is also valid for finite-size dyonic giant magnons, as it should be.

4.2.2 Two GM states and dilaton with non-zero momentum

For the dilaton vertex we have [99]

$$V^d = (Y_4+Y_5)^{-\Delta_d}(X_1+iX_2)^j\left[z^{-2}\left(\partial_+x_m\partial_-x^m+\partial_+z\partial_-z\right)+\partial_+X_k\partial_-X_k\right], \tag{4.31}$$

where the scaling dimension $\Delta_d = 4+j$ to the leading order in the large $\sqrt{\lambda}$ expansion. The corresponding operator in the dual gauge theory is proportional to $Tr\left(F_{\mu\nu}^2\, Z^j+\ldots\right)$, or for $j=0$, just to the SYM Lagrangian.

The normalized structure constant (4.1) can be computed by using (4.31), applied for the case of giant magnons, to be [22]

$$\mathcal{C}_j^d = 2\pi^{3/2}c_\Delta^d\frac{\Gamma\left(\frac{4+j}{2}\right)}{\Gamma\left(\frac{5+j}{2}\right)}\frac{\chi_p^{\frac{j-1}{2}}}{\sqrt{(1-u^2)W}} \tag{4.32}$$

$$\Bigg[(1-u^2)\chi_p\ {}_2F_1\left(\frac{1}{2},-\frac{1}{2}-\frac{j}{2};1;1-\frac{\chi_m}{\chi_p}\right)$$

$$-(1-W)\ {}_2F_1\left(\frac{1}{2},\frac{1}{2}-\frac{j}{2};1;1-\frac{\chi_m}{\chi_p}\right)\Bigg].$$

A few comments are in order. The structure constant in (4.32) corresponds to finite-size dyonic giant magnons, i.e. with two angular momenta. The case of finite-size giant magnons with one angular momentum nonzero can be obtained by setting $u=0$ (χ_p, χ_m also depend on u according to (4.24)). The infinite size case [100] is reproduced for $W=1$, $\chi_m=0$. For $j=0$, (4.32) reduces to the result of [19].

Leading finite-size effect

Expanding (4.32) in ϵ one finds [25]

$j = 1$:

$$\mathcal{C}_1^d \approx \frac{3}{4}\pi^2 c_5^d \sin^3(p/2)\Bigg\{\frac{1}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}$$
$$-\frac{1}{128\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)^2}\Big[\Big(840+826\mathcal{J}_2^2+258\mathcal{J}_2^4-24\mathcal{J}_2^6$$
$$-2\left(744+707\mathcal{J}_2^2+244\mathcal{J}_2^4+72\mathcal{J}_2^6\right)\cos(p)$$
$$+4\left(255+218\mathcal{J}_2^2+62\mathcal{J}_2^4-6\mathcal{J}_2^6\right)\cos(2p)-\left(520+367\mathcal{J}_2^2+24\mathcal{J}_2^4\right)\cos(3p)$$
$$+2\left(92+47\mathcal{J}_2^2+3\mathcal{J}_2^4\right)\cos(4p)-\left(40+11\mathcal{J}_2^2\right)\cos(5p)+4\cos(6p)\Big)$$
$$+8\mathcal{J}_1\sin^2(p/2)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\Big(\left(8+19\mathcal{J}_2^2+12\mathcal{J}_2^4\right)\cos(p)+\left(8-16\mathcal{J}_2^2\right)\cos(2p)$$
$$-\left(8+3\mathcal{J}_2^2\right)\cos(3p)-2\left(5+5\mathcal{J}_2^2-2\mathcal{J}_2^4-\cos(4p)\right)\Big)\Big]\epsilon\Bigg\},$$

$j = 2$:

$$\mathcal{C}_2^d \approx \frac{2^8}{3^2 5} c_6^d \sin^4(p/2)\Bigg\{\frac{1}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}$$
$$-\frac{1}{128\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)^2}\Big[\Big(210+8\mathcal{J}_2^2\left(6-\mathcal{J}_2^2\right)\left(7+4\mathcal{J}_2^2\right)$$
$$-8\left(63+84\mathcal{J}_2^2+38\mathcal{J}_2^4+16\mathcal{J}_2^6\right)\cos(p)$$
$$+\left(585+576\mathcal{J}_2^2+176\mathcal{J}_2^4-32\mathcal{J}_2^6\right)\cos(2p)-4\left(115+84\mathcal{J}_2^2+4\mathcal{J}_2^4\right)\cos(3p)$$
$$+2\left(111+56\mathcal{J}_2^2+4\mathcal{J}_2^4\right)\cos(4p)-4\left(15+4\mathcal{J}_2^2\right)\cos(5p)+7\cos(6p)\Big)$$
$$-8\mathcal{J}_1\sin^2(p/2)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\Big(15+8\mathcal{J}_2^2-8\mathcal{J}_2^4-4\left(3+5\mathcal{J}_2^2+4\mathcal{J}_2^4\right)\cos(p)$$
$$-\left(12-8\mathcal{J}_2^2\right)\cos(2p)+4\left(3+\mathcal{J}_2^2\right)\cos(3p)-3\cos(4p)\Big)\Big]\epsilon\Bigg\},$$

$j = 3$:

$$\mathcal{C}_3^d \approx \frac{3.5}{2^5}\pi^2 c_7^d \sin^5(p/2)\Bigg\{\frac{1}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}$$
$$+\frac{1}{960\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)^2}\Big[20\Big(256\,(13+15\cos(p))\sin^{10}(p/2)$$
$$+288\mathcal{J}_2^2\,(5+7\cos(p))\sin^8(p/2)+\mathcal{J}_2^4\,(54+241\cos(p)+10\cos(2p)$$
$$+15\,\cos(3p))\sin^2(p/2)+10\mathcal{J}_2^6\cos(p)\,(5+3\cos(p))\,\Big)$$
$$+60\mathcal{J}_1\sin^2(p/2)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\Big(20+6\mathcal{J}_2^2-12\mathcal{J}_2^4$$
$$-\left(16+21\mathcal{J}_2^2+20\mathcal{J}_2^4\right)\cos(p)-2\left(8-5\mathcal{J}_2^2\right)\cos(2p)$$
$$+\left(16+5\mathcal{J}_2^2\right)\cos(3p)-4\cos(4p)\Big)\Big]\,\epsilon\Bigg\},$$

$j = 4$:

$$\mathcal{C}_4^d \approx \frac{2^{11}}{3.5^2.7}c_8^d \sin^6(p/2)\Bigg\{\frac{1}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}$$
$$+\frac{1}{8192\left(\mathcal{J}_2^2+4\sin^2(p/2)\right)^{3/2}\left(\mathcal{J}_2^2+4\sin^4(p/2)\right)^2}\Big[64\Big(294+14\mathcal{J}_2^2-60\mathcal{J}_2^4+48\mathcal{J}_2^6$$
$$-4\left(51-49\mathcal{J}_2^2-53\mathcal{J}_2^4-36\mathcal{J}_2^6\right)\cos(p)$$
$$-\left(435+8\mathcal{J}_2^2\left(61+19\mathcal{J}_2^2-6\mathcal{J}_2^4\right)\right)\cos(2p)+2\left(305+209\mathcal{J}_2^2+6\mathcal{J}_2^4\right)\cos(3p)$$
$$-2\left(179+83\mathcal{J}_2^2+6\mathcal{J}_2^4\right)\cos(4p)+2\left(53+13\mathcal{J}_2^2\right)\cos(5p)-13\cos(6p)\Big)$$
$$+512\mathcal{J}_1\sin^2(p/2)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\Big(25+4\mathcal{J}_2^2-16\mathcal{J}_2^4-\left(20+22\mathcal{J}_2^2+24\mathcal{J}_2^4\right)\cos(p)$$
$$-4\left(5-3\mathcal{J}_2^2\right)\cos(2p)+2\left(10+3\mathcal{J}_2^2\right)\cos(3p)-5\cos(4p)\Big)\Big]\,\epsilon\Bigg\}.$$

In the four formulas above ϵ is given by

$$\epsilon = 16\exp\left[-\frac{2\left(\mathcal{J}_1+\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\,\sin^2(p/2)}{\mathcal{J}_2^2+4\sin^4(p/2)}\right]. \qquad (4.33)$$

Actually, we computed the normalized coefficients in the three-point correlators up to $j = 10$. However, since the expressions for them are too complicated, we give here only the results for the first two odd and two even values of j. Knowing these expressions, the conclusion is that they have the same structure for any j in the small ϵ limit[17]. Namely

[17]The only difference in that sense is that for j odd an additional overall factor of π^2 appears, as can be seen from the formulas above.

$$\mathcal{C}_j^d \approx A_j c_{j+4}^d \sin^{j+2}\left(\frac{p}{2}\right)\left\{\frac{1}{\sqrt{\mathcal{J}_2^2+4\sin^2\left(\frac{p}{2}\right)}}+\frac{a_j}{\left(\mathcal{J}_2^2+4\sin^2\left(\frac{p}{2}\right)\right)^{3/2}\left(\mathcal{J}_2^2+4\sin^4\left(\frac{p}{2}\right)\right)^2}\right.$$

$$\left.\left[P_j^3(\mathcal{J}_2^2)+\mathcal{J}_1\sin^2\left(\frac{p}{2}\right)\sqrt{\mathcal{J}_2^2+4\sin^2\left(\frac{p}{2}\right)}\,Q_j^2(\mathcal{J}_2^2)\right]\epsilon\right\}. \tag{4.34}$$

where ϵ is given in (4.33), A_j and a_j are numerical coefficients, while $P_j^3(\mathcal{J}_2^2)$ and $Q_j^2(\mathcal{J}_2^2)$ are polynomials of third and second order respectively, with coefficients depending on p in a trigonometric way.

Now, let us restrict ourselves to the simpler case when $\mathcal{J}_2 = 0$, i.e. giant magnon string states with one (large) angular momentum $\mathcal{J}_1 \neq 0$. Knowing the above results for $1 \leq j \leq 10$, one can conclude that the normalized structure constants in the three-point correlators for any $j \geq 1$ in the small ϵ limit look like[18]

$$\mathcal{C}_{j0}^d \approx \frac{A_j}{2} c_{j+4}^d \sin^j\left(\frac{p}{2}\right)\left[\sin\left(\frac{p}{2}\right)\right. \tag{4.35}$$
$$\left.+\left(B_{j0}\sin\left(\frac{p}{2}\right)+C_{j0}\sin\left(\frac{3p}{2}\right)+D_{j0}\left(1+\cos(p)\right)\mathcal{J}_1\right)e^{-2-\frac{\mathcal{J}_1}{\sin\frac{p}{2}}}\right],$$

where

$$B_{j0} = (-2^2, 3, \frac{2.11}{3}, 11, \frac{2^3 3^2}{5}, \frac{53}{3}, \frac{2.73}{7}, 2^3 3, \ldots) \quad \text{for} \quad j = (1, \ldots, 8, \ldots),$$
$$C_{j0} = 1 + 3j, \quad D_{j0} = 2(j+1).$$

4.2.3 Two GM states and primary scalar operators

The primary scalar vertex is [103, 104, 99]

$$V^{pr} = (Y_4+Y_5)^{-\Delta_{pr}}(X_1+iX_2)^j\left[z^{-2}\left(\partial_+ x_m\partial_- x^m-\partial_+ z\partial_- z\right)-\partial_+ X_k\partial_- X_k\right], \tag{4.36}$$

where now the scaling dimension is $\Delta_{pr} = j$. The corresponding operator in the dual gauge theory is $Tr\left(Z^j\right)$.

For giant magnons we have [100]

$$z^{-2}\left(\partial_+ x_m\partial_- x^m-\partial_+ z\partial_- z\right)=\kappa^2\left(\frac{2}{\cosh^2(\kappa\tau_e)}-1\right).$$

Then the light vertex operator becomes

$$V^{pr}=\frac{\cos^j\theta}{\cosh^j(\kappa\tau_e)}\left[\kappa^2\left(\frac{2}{\cosh^2(\kappa\tau_e)}-1\right)-\mathcal{L}_{S^3}^{gm}\right],$$

[18] $\mathcal{C}_{j0}^d$ is used for $\mathcal{C}_j^d$ computed for $\mathcal{J}_2 = 0$ case.

where the infinite-size case was considered in [100], while for the finite-size giant magnons $\mathcal{L}^{gm}_{S^3}$ should be taken from

$$\mathcal{L}^{gm}_{S^3} = -\frac{1}{1-v^2}\left[2-(1+v^2)W-2\left(1-u^2\right)\chi\right]. \tag{4.37}$$

Let us also note that the first integral for χ is given by

$$\chi' = \frac{2\sqrt{1-u^2}}{1-v^2}\sqrt{\chi(\chi_p-\chi)(\chi-\chi_m)}. \tag{4.38}$$

As a consequence, the normalized structure constant in the corresponding three-point function, for the case under consideration, takes the form:

$$\begin{aligned}\mathcal{C}^{pr}_j &= c^{pr}_\Delta\left[\int_{-\infty}^{\infty}d\tau_e\frac{W}{\cosh^j(\sqrt{W}\tau_e)}\left(\frac{2}{\cosh^2(\sqrt{W}\tau_e)}-1\right)\int_{-L}^{L}d\sigma\chi^{\frac{j}{2}}\right.\\ &\left.-\int_{-\infty}^{\infty}\frac{d\tau_e}{\cosh^j(\sqrt{W}\tau_e)}\int_{-L}^{L}d\sigma\chi^{\frac{j}{2}}\mathcal{L}^{gm}_{S^3}\right].\end{aligned} \tag{4.39}$$

Performing the integrations in (4.39) one finally finds [22]

$$\begin{aligned}\mathcal{C}^{pr}_j &= \pi^{3/2}c^{pr}_\Delta\frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{3+j}{2}\right)}\frac{\chi_p^{\frac{j-1}{2}}}{\sqrt{(1-u^2)W}}\\ &\left[\left(1-W+j(1-v^2W)\right)\ {}_2F_1\left(\frac{1}{2},\frac{1}{2}-\frac{j}{2};1;1-\epsilon\right)\right.\\ &\left.-(1+j)\left(1-u^2\right)\chi_p\ {}_2F_1\left(\frac{1}{2},-\frac{1}{2}-\frac{j}{2};1;1-\epsilon\right)\right].\end{aligned} \tag{4.40}$$

Leading finite-size effect

Let us start with the simpler case when $J_2=0$, or equivalently $u=0$. Expanding (4.40) in ϵ one finds [25]

$$\begin{aligned}&\mathcal{C}^{pr}_{10}\approx 0,\quad \mathcal{C}^{pr}_{20}\approx\frac{4}{3}c^{pr}_2\mathcal{J}_1\sin^2(p/2)\ \epsilon,\\ &\mathcal{C}^{pr}_{j0}\approx c^{pr}_j a_j\sin(p/2)^{j+1}\ \epsilon,\quad j=3,...,10,\end{aligned} \tag{4.41}$$

where

$$\epsilon = 16\exp[-2-\mathcal{J}_1\csc(p/2)], \tag{4.42}$$

for the case under consideration. The numerical coefficients a_j are given by

$$a_j=\left(\frac{1}{4}\pi^2,\frac{2^4}{3.5},\frac{1}{16}\pi^2,\frac{2^7}{3^2.5.7},\frac{3.5}{2^9}\pi^2,\frac{2^{10}}{3^3.5^2.7},\frac{5.7}{2^{11}}\pi^2,\frac{2^{14}}{3^2.5^2.7^2.11}\right).$$

A few comments are in order. From (4.41) one can conclude that the $\mathcal{C}^{pr}_{10}$ and $\mathcal{C}^{pr}_{20}$ cases are exceptional, while $\mathcal{C}^{pr}_{j0}$ have the same structure for $j\geq 3$. $\mathcal{C}^{pr}_{10}\approx 0$ means that the small ϵ -

contribution to the three point correlator is zero to the leading order in ϵ. $\mathcal{C}_{20}^{pr}$ is the only one normalized structure constant of this type proportional to $\mathcal{J}_1$. It is still exponentially suppressed by ϵ. The common feature of $\mathcal{C}_{j0}^{pr}$ in (4.41) is that they all vanish in the *infinite size* case, i.e., for $\epsilon = 0$. This property was established in [100]. Here, we obtained the leading finite-size corrections to it.

Now, let us turn to the dyonic case, i.e. $J_2 \neq 0$. Working in the same way, but with $u \neq 0$, we derive

$j = 1$:

$$\begin{aligned}
&\mathcal{C}_1^{pr} \approx c_1^{pr} \frac{\pi^2}{16} \frac{\mathcal{J}_2^2 \csc(p/2)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^{3/2}[\mathcal{J}_2^2 + 4\sin^4(p/2)]} \times \\
&\big\{8[\mathcal{J}_2^2 + 4\sin^2(p/2)][\mathcal{J}_2^2 + 4\sin^4(p/2)] \\
&+ \sin^2(p/2)\big[40 + 17\mathcal{J}_2^2 + 2\mathcal{J}_2^4 - 20(3+\mathcal{J}_2^2)\cos(p) \\
&+ 3(8+\mathcal{J}_2^2)\cos(2p) - 4\cos(3p) - 4\frac{\mathcal{J}_2^2 + 8\sin^2(p/2)}{\mathcal{J}_2^2 + 4\sin^4(p/2)} \times \\
&\left(\mathcal{J}_1\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} + \mathcal{J}_2^2 + 4\sin^2(p/2)\right) \times \\
&\big(\mathcal{J}_2^2 + 4\sin^4(p/2) + 2\sin^2(p)\big)\sin^2(p/2)\big]\,\epsilon\big\},
\end{aligned} \tag{4.43}$$

$j = 2$:

$$\begin{aligned}
&\mathcal{C}_2^{pr} \approx \frac{4}{3} c_2^{pr} \frac{1}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^{3/2}[\mathcal{J}_2^2 + 4\sin^4(p/2)]} \times \\
&\big\{2\mathcal{J}_2^2[\mathcal{J}_2^2 + 4\sin^2(p/2)]\left[\mathcal{J}_2^2 + 4\sin^4(p/2)\right] - \sin^4(p/2)\times \\
&\big[20 + 3\mathcal{J}_2^2 - 2\mathcal{J}_2^4 - 2(15 + 2\mathcal{J}_2^2)\cos(p) + (12+\mathcal{J}_2^2)\cos(2p) - 2\cos(3p) \\
&+ \frac{8}{\mathcal{J}_2^2 + 4\sin^4(p/2)}\left(\mathcal{J}_1\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} + \mathcal{J}_2^2 + 4\sin^2(p/2)\right) \times \\
&\big(-3 + 2(2+\mathcal{J}_2^2)\cos(p) - \cos(2p)\big)\sin^4(p/2)\big]\,\epsilon\big\},
\end{aligned} \tag{4.44}$$

$j = 3$:

$$\mathcal{C}_3^{pr} \approx c_3^{pr} \frac{\pi^2}{256} \csc(p/2) \frac{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^{5/2}}{\mathcal{J}_2^2 + 4\sin^4(p/2)} \times \tag{4.45}$$
$$\Bigg\{ 48\mathcal{J}_2^2 \sin^2(p/2) \frac{\mathcal{J}_2^2 + 4\sin^4(p/2)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^3} - \Bigg[\frac{25\mathcal{J}_2^4}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^2}$$
$$-\mathcal{J}_2^2 \frac{\mathcal{J}_2^2 + 4\sin^4(p/2)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^3} \left(21 - 16\cos(p) - 5\cos(2p) + 8\mathcal{J}_2^2\right)$$
$$-\frac{3}{2}\mathcal{J}_2^6 \frac{11 - 12\cos(p) + \cos(2p) + 6\mathcal{J}_2^2}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^4}$$
$$+ \left(3\mathcal{J}_2^2 \left(\mathcal{J}_2^2 + 4\sin^2(p/2) + \mathcal{J}_1\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right) \times\right.$$
$$\left(80 + 42\mathcal{J}_2^2 + 12\mathcal{J}_2^4 - \left(120 + 47\mathcal{J}_2^2 - 4\mathcal{J}_2^4\right)\cos(p)\right.$$
$$\left.\left. + \left(8 + \mathcal{J}_2^2\right)\left(6\cos(2p) - \cos(3p)\right)\right) \sin^4(p/2)\right) \frac{1}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^4[\mathcal{J}_2^2 + 4\sin^4(p/2)]}$$
$$- \frac{20\mathcal{J}_2^4\sin^2(p)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^3} + \frac{3\mathcal{J}_2^4\sin^4(p)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^4} - 8\left(\frac{\mathcal{J}_2^2 + 4\sin^4(p/2)}{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)^2 \Bigg] \epsilon \Bigg\},$$

$j = 4$:

$$\mathcal{C}_4^{pr} \approx \frac{2}{45} c_4^{pr} \frac{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^{5/2}}{\mathcal{J}_2^2 + 4\sin^4(p/2)} \tag{4.46}$$
$$\Bigg\{ \frac{32\mathcal{J}_2^2[\mathcal{J}_2^2 + 4\sin^4(p/2)]\sin^2(p/2)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^3} - \Bigg[\frac{17\mathcal{J}_2^4}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^2}$$
$$-\frac{1}{2}\mathcal{J}_2^2 \frac{\mathcal{J}_2^2 + 4\sin^4(p/2)}{\mathcal{J}_2^2 + 4\sin^2(p/2)]^3} \left(39 - 32\cos(p) - 7\cos(2p) + 16\mathcal{J}_2^2\right)$$
$$-\mathcal{J}_2^6 \frac{11 - 12\cos(p) + \cos(2p) + 6\mathcal{J}_2^2}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^4}$$
$$+ \left(2\mathcal{J}_2^2 \left(\mathcal{J}_2^2 + 4\sin^2(p/2) + \mathcal{J}_1\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right) \times\right.$$
$$\left(75 + 44\mathcal{J}_2^2 + 16\mathcal{J}_2^4 - 2\left(58 + 23\mathcal{J}_2^2 - 4\mathcal{J}_2^4\right)\cos(p)\right.$$
$$\left.\left. +4\left(13 + \mathcal{J}_2^2\right)\cos(2p) - 2\left(6 + \mathcal{J}_2^2\right)\cos(3p) + \cos(4p)\right)\sin^4(p/2)\right) \times$$
$$\frac{1}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^4[\mathcal{J}_2^2 + 4\sin^4(p/2)]}$$
$$- \frac{13\mathcal{J}_2^4\sin^2(p)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^3} + \frac{2\mathcal{J}_2^4\sin^4(p)}{[\mathcal{J}_2^2 + 4\sin^2(p/2)]^4} - 3\left(\frac{\mathcal{J}_2^2 + 4\sin^4(p/2)}{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)^2 \Bigg] \epsilon \Bigg\}.$$

In the four formulas above ϵ is given by (4.33).

4.2.4 Two GM states and singlet scalar operators on higher string levels

As explained in [99], there exist special massive string states vertex operators with finite quantum numbers for which the leading-order bosonic part is known explicitly and thus they can be used as candidates for "light" vertex operators in the semiclassical computation of the correlation functions. These are singlet operators which do not mix with other operators to leading nontrivial order in $\frac{1}{\sqrt{\lambda}}$ [93, 107]. An example of such scalar operator carrying no spins is [99]

$$V^q = (Y_4 + Y_5)^{-\Delta}\Big[(\partial X_k \bar{\partial} X_k)^q + ...\Big]. \tag{4.47}$$

The marginality condition for this operator is [99]: $2(1-q) + \frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta-4) + 2q(q-1)\Big]$ $+ \frac{1}{(\sqrt{\lambda})^2}\left[\frac{2}{3}q(q-1)(q-\frac{7}{2}) + 4q\right] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^3}\right) = 0$. This operator corresponds to a scalar string state at level $n = q-1$ so that the fermionic contributions should make the $q=1$ state massless (BPS), with $\Delta = 4$ following from the marginality condition. The $q=2$ choice corresponds to a scalar state on the first excited string level. In that case, we have [107]

$$\Delta(\Delta-4) = 4(\sqrt{\lambda}-1) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right),$$

with solution

$$\Delta = 2(\lambda^{1/4}+1) + \frac{0}{\lambda^{1/4}} + \mathcal{O}\left(\frac{1}{\lambda^{3/4}}\right).$$

However, the subleading terms here should not be trusted as far as the fermions are expected to change the Δ-independent terms in the 1-loop anomalous dimension. For arbitrary string level n, the solution of the marginality condition with respect to Δ, to leading order in $\frac{1}{\sqrt{\lambda}}$, is given by

$$\Delta_q = 2\left(\sqrt{(q-1)\sqrt{\lambda} + 1 - \frac{1}{2}q(q-1)} + 1\right). \tag{4.48}$$

Let us also point out that the number q of $\partial X_k \bar{\partial} X_k$ factors in an operator never increases due to renormalization [93]. That is why, it can be used as a quantum number to characterize the leading term in the corresponding operator [105].

The normalized structure constant can be computed by using (4.47), applied for the case of giant magnons, to be [23]

$$\mathcal{C}^q = c_{\Delta_q}\pi^{3/2}\frac{\Gamma\left(\frac{\Delta_q}{2}\right)}{\Gamma\left(\frac{\Delta_q+1}{2}\right)}\frac{(-1)^q\left[2-(1+v^2)W\right]^q}{(1-v^2)^{q-1}\sqrt{(1-u^2)W\chi_p}} \tag{4.49}$$

$$\sum_{k=0}^{q}\frac{q!}{k!(q-k)!}\left[-\frac{1-u^2}{1-\frac{1}{2}(1+v^2)W}\right]^k \chi_p^k\ {}_2F_1\left(\frac{1}{2},\frac{1}{2}-k;1;1-\frac{\chi_m}{\chi_p}\right),$$

where

$$\chi_p = \frac{1}{2(1-u^2)} \left\{ q_1 + q_2 - u^2 + \sqrt{(q_1-q_2)^2 - \left[2\left(q_1+q_2-2q_1q_2\right) - u^2\right] u^2} \right\},$$
$$\chi_m = \frac{1}{2(1-u^2)} \left\{ q_1 + q_2 - u^2 - \sqrt{(q_1-q_2)^2 - \left[2\left(q_1+q_2-2q_1q_2\right) - u^2\right] u^2} \right\},$$
$$q_1 = 1-W, \quad q_2 = 1 - v^2 W. \tag{4.50}$$

This is our general result corresponding to finite-size giant magnons with two angular momenta and to arbitrary string level $n = q-1 = 0, 1, 2, ...$ Now, let us give some particular examples contained in (4.49).

Giant magnons with one angular momentum

The case of finite-size giant magnons with one angular momentum $J_1 \neq 0$ corresponds to $u = 0$. This can be seen from the explicit expression for the second angular momentum J_2:

$$\mathcal{J}_2 \equiv \frac{2\pi J_2}{\sqrt{\lambda}} = \frac{2u\sqrt{\chi_p}}{\sqrt{1-u^2}} \, \mathbf{E}\left(1 - \frac{\chi_m}{\chi_p}\right).$$

Then from (4.50) one obtains the following simplified expressions for χ_p, χ_m:

$$\chi_p = 1 - v^2 W, \quad \chi_m = 1 - W.$$

Taking this into account, and using (4.49), one can find that the normalized structure constants for the first three string levels, for the case at hand, are given by:

$q = 1$ (level $n = 0$)

$$\mathcal{C}^1 = 2c_{\Delta_1}\pi^{1/2}\frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{\Delta_1+1}{2}\right)}\frac{1}{\sqrt{W(1-v^2W)}}$$
$$\left[2(1-v^2W)\,\mathbf{E}\left(1-\frac{1-W}{1-v^2W}\right) - \left(2-(1+v^2)W\right)\mathbf{K}\left(1-\frac{1-W}{1-v^2W}\right)\right].$$

$q = 2$ (level $n = 1$)

$$\mathcal{C}^2 = 2c_{\Delta_2}\pi^{1/2}\frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta_2+1}{2}\right)}\frac{1}{(1-v^2)\sqrt{W(1-v^2W)}}$$
$$\left[\left(2-(1+v^2)W\right)^2\mathbf{K}\left(1-\frac{1-W}{1-v^2W}\right)\right.$$
$$-4\left(2-(1+v^2)W\right)(1-v^2W)\,\mathbf{E}\left(1-\frac{1-W}{1-v^2W}\right)$$
$$\left.+\,2\pi(1-v^2W)^2\,{}_2F_1\left(\frac{1}{2},-\frac{3}{2};1;1-\frac{1-W}{1-v^2W}\right)\right].$$

$q = 3$ (level $n = 2$)

$$\mathcal{C}^3 = -2c_{\Delta_3}\pi^{1/2}\frac{\Gamma\left(\frac{\Delta_3}{2}\right)}{\Gamma\left(\frac{\Delta_3+1}{2}\right)}\frac{\big(2-(1+v^2)W\big)^3}{(1-v^2)^2\sqrt{W(1-v^2W)}}$$
$$\Bigg[\mathbf{K}\left(1-\frac{1-W}{1-v^2W}\right)-\frac{6(1-v^2W)}{2-(1+v^2)W}\ \mathbf{E}\left(1-\frac{1-W}{1-v^2W}\right)$$
$$+\frac{6\pi(1-v^2W)^2}{\big(2-(1+v^2)W\big)^2}\ {}_2F_1\left(\frac{1}{2},-\frac{3}{2};1;1-\frac{1-W}{1-v^2W}\right)$$
$$-\frac{4\pi(1-v^2W)^3}{\big(2-(1+v^2)W\big)^3}\ {}_2F_1\left(\frac{1}{2},-\frac{5}{2};1;1-\frac{1-W}{1-v^2W}\right)\Bigg].$$

Giant magnons with two angular momenta

$q = 1$ (level $n = 0$):

$$\mathcal{C}^1 = 2c_{\Delta_1}\pi^{1/2}\frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{\Delta_1+1}{2}\right)}\frac{1}{\sqrt{(1-u^2)W\chi_p}}$$
$$\left[2(1-u^2)\chi_p\ \mathbf{E}\left(1-\frac{\chi_m}{\chi_p}\right)-\big(2-(1+v^2)W\big)\,\mathbf{K}\left(1-\frac{\chi_m}{\chi_p}\right)\right].$$

$q = 2$ (level $n = 1$):

$$\mathcal{C}^2 = 2c_{\Delta_2}\pi^{1/2}\frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta_2+1}{2}\right)}\frac{1}{(1-v^2)\sqrt{(1-u^2)W\chi_p}}$$
$$\Bigg[\big(2-(1+v^2)W\big)^2\,\mathbf{K}\left(1-\frac{\chi_m}{\chi_p}\right)-4(1-u^2)\,\big(2-(1+v^2)W\big)\,\chi_p\ \mathbf{E}\left(1-\frac{\chi_m}{\chi_p}\right)$$
$$+2\pi(1-u^2)^2\chi_p^2\ {}_2F_1\left(\frac{1}{2},-\frac{3}{2};1;1-\frac{\chi_m}{\chi_p}\right)\Bigg].$$

$q = 3$ (level $n = 2$):

$$\mathcal{C}^3 = -2c_{\Delta_3}\pi^{1/2}\frac{\Gamma\left(\frac{\Delta_3}{2}\right)}{\Gamma\left(\frac{\Delta_3+1}{2}\right)}\frac{\big(2-(1+v^2)W\big)^3}{(1-v^2)^2\sqrt{(1-u^2)W\chi_p}}$$
$$\Bigg[\mathbf{K}\left(1-\frac{\chi_m}{\chi_p}\right)-\frac{6(1-u^2)\chi_p}{2-(1+v^2)W}\ \mathbf{E}\left(1-\frac{\chi_m}{\chi_p}\right)$$
$$+\frac{6\pi(1-u^2)^2\chi_p^2}{\big(2-(1+v^2)W\big)^2}\ {}_2F_1\left(\frac{1}{2},-\frac{3}{2};1;1-\frac{\chi_m}{\chi_p}\right)$$
$$-\frac{4\pi(1-u^2)^3\chi_p^3}{\big(2-(1+v^2)W\big)^3}\ {}_2F_1\left(\frac{1}{2},-\frac{5}{2};1;1-\frac{\chi_m}{\chi_p}\right)\Bigg].$$

4.3 Semiclassical three-point correlation functions in TsT-deformed $AdS_5 \times S^5$

4.3.1 Two GM states and dilaton with zero momentum

Working as in the undeformed case and taking into account the deformation of the sphere, one finds that the normalized structure constant is given by [20]

$$\begin{aligned}\mathcal{C}^d_{\tilde{\gamma}} = & \frac{16}{3} c^d_\Delta \frac{1}{\sqrt{(1-u^2)W(\chi_p-\chi_n)}} \times \\ & \left[\left((1-u^2)(1-\tilde{\gamma}K) - \tilde{\gamma}uvW\right)\sqrt{\chi_p-\chi_n}\mathbf{E}(1-\epsilon)\right. \\ & \left. + \left(\left(W(1-\tilde{\gamma}uv\chi_n) - (1-\tilde{\gamma}K)\left(1-(1-u^2)\chi_n\right)\right)\mathbf{K}(1-\epsilon)\right)\right],\end{aligned} \tag{4.51}$$

where

$$\epsilon = \frac{\chi_m - \chi_n}{\chi_p - \chi_n}, \tag{4.52}$$

and

$$\begin{aligned}\chi_p + \chi_m + \chi_n &= \frac{2-(1+v^2)W-u^2}{1-u^2}, \\ \chi_p\chi_m + \chi_p\chi_n + \chi_m\chi_n &= \frac{1-(1+v^2)W+(vW-uK)^2-K^2}{1-u^2}, \\ \chi_p\chi_m\chi_n &= -\frac{K^2}{1-u^2}.\end{aligned} \tag{4.53}$$

The case of dyonic finite-size giant magnons we are interested in, corresponds to

$$0<u<1, \quad 0<v<1, \quad 0<W<1, \quad 0<\chi_m<\chi<\chi_p<1, \quad \chi_n<0.$$

This is our *exact* semiclassical result for the normalized coefficient $\mathcal{C}^d_{\tilde{\gamma}}$ in the three-point correlation function, corresponding to the case when the heavy vertex operators are *finite-size* dyonic giant magnons living on the γ-deformed three-sphere.

Leading finite-size effect

For the case of the dilaton operator, the three-point function of the SYM can be easily related to the conformal dimension of the heavy operators. This corresponds to shift 't Hooft coupling constant which is the overall coefficient of the Lagrangian [101]. This gives an important relation between the structure constant and the conformal dimension as follows:

$$\mathcal{C}^d_{\tilde{\gamma}} = \frac{32\pi}{3} c^d_\Delta \sqrt{\lambda}\partial_\lambda \Delta. \tag{4.54}$$

We want to show here that this relation holds for the case of finite-size giant magnons ($J_2 = 0$), assuming that $\Delta = E - J_1$, and considering the limit $\epsilon \to 0$. To this end, we introduce the expansions

$$\begin{aligned}
\chi_p &= \chi_{p0} + (\chi_{p1} + \chi_{p2}\log(\epsilon))\,\epsilon,\\
\chi_m &= \chi_{m0} + (\chi_{m1} + \chi_{m2}\log(\epsilon))\,\epsilon,\\
\chi_n &= \chi_{n0} + (\chi_{n1} + \chi_{n2}\log(\epsilon))\,\epsilon,\\
v &= v_0 + (v_1 + v_2\log(\epsilon))\,\epsilon,\\
u &= u_0 + (u_1 + u_2\log(\epsilon))\,\epsilon,\\
W &= W_0 + (W_1 + W_2\log(\epsilon))\,\epsilon,\\
K &= K_0 + (K_1 + K_2\log(\epsilon))\,\epsilon.
\end{aligned} \tag{4.55}$$

A few comments are in order. To be able to reproduce the dispersion relation for the infinite-size giant magnons, we set

$$\chi_{m0} = \chi_{n0} = K_0 = 0, \qquad W_0 = 1. \tag{4.56}$$

In addition, one can check that if we keep the coefficients χ_{m2}, χ_{n2}, W_2 and K_2 nonzero, the known leading correction to the giant magnon energy-charge relation [65] will be modified by a term proportional to $\mathcal{J}_1^2$. That is why we choose

$$\chi_{m2} = \chi_{n2} = W_2 = K_2 = 0. \tag{4.57}$$

Finally, since we are considering for simplicity giant magnons with one angular momentum, we also set

$$u_0 = 0, \tag{4.58}$$

because the leading term in the ϵ-expansion of $\mathcal{J}_2$ is proportional to u_0.

By replacing (4.55) in (4.52) and (4.53), and taking into account (4.56), (4.57), (4.58), we obtain

$$\begin{aligned}
\chi_{p0} &= 1 - v_0^2, \\
\chi_{p1} &= \frac{v_0}{1-v_0^2}\left[v_0\sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)} - 2(1-v_0^2)v_1\right],\\
\chi_{p2} &= -2v_0v_2,\\
\chi_{m1} &= \frac{(1-v_0^2)^2 + \sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}}{2(1-v_0^2)},\\
\chi_{n1} &= -\frac{(1-v_0^2)^2 - \sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}}{2(1-v_0^2)},\\
W_1 &= -\frac{\sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}}{1-v_0^2}.
\end{aligned} \tag{4.59}$$

The other parameters in (4.55) and (4.59) can be found in the following way. First, we impose the conditions $J_2 = 0$ and p_1 to be independent of ϵ. This leads to four equations with solution

$$\begin{aligned} v_1 &= \frac{v_0\sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}\,(1-\log 16)}{4(1-v_0^2)}, \\ v_2 &= \frac{v_0\sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}}{4(1-v_0^2)}, \\ u_1 &= \frac{K_1 v_0 \log 4}{1-v_0^2}, \\ u_2 &= -\frac{K_1 v_0}{2(1-v_0^2)}, \end{aligned} \tag{4.60}$$

where

$$v_0 = \cos\frac{p_1}{2}. \tag{4.61}$$

Next, to the leading order, the expansions for $\mathcal{J}_1$ and $p_2 = 2\pi n_2$ ($n_2 \in \mathbb{Z}$) give

$$\epsilon = 16\exp\left(-2 - \frac{\mathcal{J}_1}{\sin\frac{p_1}{2}}\right), \qquad K_1 = \frac{1}{2}\sin^3\frac{p_1}{2}\sin\Phi, \qquad \Phi = 2\pi\left(n_2 - \frac{\tilde{\gamma}}{\sqrt{\lambda}}J_1\right). \tag{4.62}$$

Now, we consider the limit $\epsilon \to 0$ in the expression (4.51) for the structure constant in the 3-point correlation function, by using (4.55), (4.56), (4.57), (4.58), (4.59), (4.60), and obtain

$$\begin{aligned} \mathcal{C}^d_{\tilde{\gamma}} \approx\ & \frac{4}{3}c^d_{\Delta}\frac{1}{(1-v_0^2)^{3/2}}\Bigg[4 + 4v_0^4\Big(1 - \tilde{\gamma}K_1(1-\log 4)\ \epsilon\Big) \\ & -v_0^2\left(8 + \left(\sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}\ (1-\log 16) - 8\tilde{\gamma}K_1(1-\log 4)\right)\epsilon\right) \\ & -\left(4\tilde{\gamma}K_1(1-\log 4) - \sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}\ (1-\log 256)\right)\epsilon \\ & -\left(v_0^2\sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)} + 2\tilde{\gamma}K_1(1-v_0^2)^2\right)\epsilon\log\epsilon \\ & +\sqrt{(1-v_0^2)^4 - 4K_1^2(1-v_0^2)}\ \epsilon\log(16\ \epsilon)\Bigg]. \end{aligned} \tag{4.63}$$

According to (4.61), (4.62), the above expression for $\mathcal{C}^d_{\tilde{\gamma}}$ can be rewritten in terms of p_1, $\mathcal{J}_1$, as

$$\mathcal{C}^d_{\tilde{\gamma}} \approx \frac{16}{3}c^d_{\Delta}\sin\frac{p_1}{2}\left[1 - 4\sin^2\frac{p_1}{2}\left(\cos\Phi + \mathcal{J}_1\csc\frac{p_1}{2}\cos\Phi - \tilde{\gamma}\mathcal{J}_1\sin\Phi\right)e^{-2-\frac{\mathcal{J}_1}{\sin\frac{p_1}{2}}}\right]. \tag{4.64}$$

In order to check if the equality (4.54) holds for the present case, let us now consider the dispersion relation of giant magnons on TsT-transformed $AdS_5 \times S^5$, including the leading finite-size correction, which is known to be [80, 18]

$$E - J_1 = \frac{\sqrt{\lambda}}{\pi}\sin(p/2)\left[1 - 4\sin^2(p/2)\cos\Phi\exp\left(-2 - \frac{2\pi J_1}{\sqrt{\lambda}\sin(p/2)}\right)\right]. \tag{4.65}$$

Taking the λ derivative of (4.65), one finds

$$\lambda\partial_\lambda\Delta = \frac{\sqrt{\lambda}}{2\pi}\sin\frac{p}{2}\left[1 - 4\sin^2\frac{p}{2}\left(\cos\Phi + \mathcal{J}_1\csc\frac{p}{2}\cos\Phi - \tilde{\gamma}\mathcal{J}_1\sin\Phi\right)e^{-2-\frac{\mathcal{J}_1}{\sin\frac{p}{2}}}\right]. \tag{4.66}$$

Identifying $p \equiv p_1$, and comparing (4.64) with (4.66), we see that the equality (4.54) is also valid for the γ-deformed case.

4.3.2 Two GM states and dilaton with non-zero momentum

The normalized structure constant for the case at hand is given by [22]

$$\begin{aligned}\mathcal{C}^d_{j\tilde{\gamma}} &= 2\pi^{3/2}c^d_\Delta\frac{\Gamma\left(\frac{4+j}{2}\right)}{\Gamma\left(\frac{5+j}{2}\right)}\frac{\chi_p^{j/2}}{\sqrt{(1-u^2)W\left(\chi_p-\chi_n\right)}} \\ &\Big\{\left[1-\tilde{\gamma}K - u\left(u+\tilde{\gamma}(vW-uK)\right)\right]\chi_p F_1\left(1/2,1/2,-1-j/2;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right) \\ &-\left(1-W-\tilde{\gamma}K\right)F_1\left(1/2,1/2,-j/2;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right)\Big\}.\end{aligned} \tag{4.67}$$

The small ϵ limit corresponds to considering the leading finite-size effect, while $\epsilon = 0$, $\chi_m = 0$, $\chi_n = 0$, $K = 0$, $W = 1$, describes the infinite-size case.

Leading finite-size effect

Here, we restrict ourselves to the case $\mathcal{J}_2 = 0$, $\mathcal{J}_1 \equiv \mathcal{J}$ large but finite, i.e. $J_1 \equiv J \gg \sqrt{\lambda}$.

Expanding (4.67) for this case to the leading order in ϵ, one finds [26] ($j \geq 1$)

$$\begin{aligned}\mathcal{C}^d_{j\tilde{\gamma}} &\approx c^d_{4+j}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{4+j}{2}\right)}{\Gamma\left(\frac{5+j}{2}\right)}\chi_{p0}^{\frac{1}{2}(j-1)}\Big\{\epsilon\left[4W_1 + (2+j)(2\chi_{m1}-\chi_{p0}) + 4\tilde{\gamma}K_1\right] \\ &\times\log\frac{16}{\epsilon}\ {}_1F_0\left(-\frac{j}{2},1\right) \\ &+\sqrt{\pi}\frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{3+j}{2}\right)}\Big[2j\chi_{p0} + \left(2\chi_{m1}-\chi_{p0}+W_1\left(2+j(2-\chi_{p0}\right)\right) \\ &+ j(\chi_{m1}+\chi_{n1}+(1+j)\chi_{p1}) + 2\tilde{\gamma}(K_1 + j(K_1-(K_1+v_0u_1)\chi_{p0})))\ \epsilon \\ &+(j(1+j)\chi_{p2} - 2\tilde{\gamma}jv_0u_2\chi_{p0})\ \epsilon\log\epsilon\Big]\Big\}.\end{aligned} \tag{4.68}$$

This can be rewritten as

$$\begin{aligned}\mathcal{C}^d_{j\tilde{\gamma}} &\approx c^d_{4+j}\pi\frac{\Gamma\left(\frac{j}{2}\right)\Gamma\left(\frac{4+j}{2}\right)}{\Gamma\left(\frac{3+j}{2}\right)\Gamma\left(\frac{5+j}{2}\right)}\sin^{1+j}(p/2)\Big\{j - \frac{1}{8}\Big[\left(4 - j(1+3j)(1+\cos p)\right. \\ &\left.- j(1+j)(1+\cos p)\csc(p/2)\right)\mathcal{J}\cos\Phi \\ &-\tilde{\gamma}\left(4\sin(p/2) - j(1+\cos p)\mathcal{J}\right)\sin\Phi\Big]\epsilon\Big\}.\end{aligned}$$

4.3.3 Two GM states and primary scalar operators

According to [22] the normalized structure constant for this case is given by

$$\mathcal{C}^{pr}_{j\tilde{\gamma}} = \pi^{3/2} c^{pr}_{\Delta} \frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{1+j}{2}\right)} \frac{(1-v^2)\chi_p^{j/2}}{\sqrt{(1-u^2)\left(\chi_p-\chi_n\right)}} \tag{4.69}$$
$$\left\{\left[\sqrt{W}\frac{j-1}{j+1} + \frac{1}{\sqrt{W}(1-v^2)}\left(2-(1+v^2)W-2\tilde{\gamma}K\right)\right]\right.$$
$$\times F_1\left(1/2, 1/2, -j/2; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right)$$
$$-\frac{2}{\sqrt{W}(1-v^2)}\left[1-\tilde{\gamma}K - u\left(u-\tilde{\gamma}uK+\tilde{\gamma}vW\right)\right]\chi_p$$
$$\left.\times F_1\left(1/2, 1/2, -1-j/2; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right)\right\}.$$

It can be shown that (4.69) reduces to the undeformed case if we fix

$$\tilde{\gamma} = K = \chi_n = 0 \qquad \Rightarrow \epsilon = \frac{\chi_m}{\chi_p}.$$

This can be done by using the following property of the hypergeometric function F_1

$$F_1(a, b_1, b_2; c; z, z) = {}_2F_1\left(a, b_1+b_2; c; z\right).$$

For the infinite-size case, (4.69) gives

$$\mathcal{C}^{pr}_{j\tilde{\gamma}\infty} = \pi c^{pr}_{\Delta} \frac{\Gamma\left(\frac{j}{2}\right)\Gamma\left(1+\frac{j}{2}\right)}{\Gamma\left(\frac{3+j}{2}\right)\Gamma\left(\frac{1+j}{2}\right)} \mathcal{J}_2 \frac{\sin^{j-2}(p/2)}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}}\left[\mathcal{J}_2+\tilde{\gamma}\sin^2(p/2)\sin(p)\right]. \tag{4.70}$$

Leading finite-size effect

Again, we will consider here the particular case $\mathcal{J}_2 = 0$, $\mathcal{J}_1 \equiv \mathcal{J}$ large but finite, i.e. $J_1 \equiv J \gg \sqrt{\lambda}$.

As was pointed out in [25], where the undeformed case has been considered, $j = 1$ and $j = 2$ are special values. That is why we will start with these two cases first.

The case $j = 1$

Expanding the coefficients in $\mathcal{C}^{pr}_{1\tilde{\gamma}}$, one can rewrite it in the following form

$$\begin{aligned}
\mathcal{C}^{pr}_{1\tilde{\gamma}} &\approx c^{pr}_1 \frac{\pi^2}{2}\Bigg\{\frac{1}{\chi_{p0}} F_1\left(1/2, 1/2, -1/2; 1; 1-\epsilon, 1-\frac{\chi_{m1}}{\chi_{p0}}\ \epsilon\right)\Big[\left(1-v_0^2\right)\chi_{n1}\ \epsilon \\
&+ \left(2-(4v_0v_1+3W_1+4\tilde{\gamma}K_1)\ \epsilon - v_0^2(2+W_1\ \epsilon)\right)\chi_{p0} - 4v_0v_2\chi_{p0}\ \epsilon\log(\epsilon)\Big] \\
&- F_1\left(1/2, 1/2, -3/2; 1; 1-\epsilon, 1-\frac{\chi_{m1}}{\chi_{p0}}\ \epsilon\right) \\
&\times\Big[4\chi_{p0} + 2\left(\chi_{n1} - (W_1+2\tilde{\gamma}(K_1+v_0u_1))\chi_{p0} + 2\chi_{p1}\right)\ \epsilon \\
&+ 4(\chi_{p2} - \tilde{\gamma}v_0u_2\chi_{p0})\ \epsilon\log(\epsilon)\Big]\Bigg\}.
\end{aligned} \tag{4.71}$$

$\mathcal{C}^{pr}_{1\tilde{\gamma}}$ can be represented as a function of $\mathcal{J}$, p and Φ in the following way [26]

$$\begin{aligned}
\mathcal{C}^{pr}_{1\tilde{\gamma}} &\approx -c^{pr}_1 \frac{\pi^2}{4}\sin^2(p/2)\Bigg\{8F_1\left(1/2, 1/2, -3/2; 1; 1-\epsilon, 1-\frac{1}{2}(1+\cos\Phi)\ \epsilon\right) \\
&- 4F_1\left(1/2, 1/2, -1/2; 1; 1-\epsilon, 1-\frac{1}{2}(1+\cos\Phi)\ \epsilon\right) \\
&+ \Bigg[F_1\left(1/2, 1/2, -1/2; 1; 1-\epsilon, 1-\frac{1}{2}(1+\cos\Phi)\ \epsilon\right) \\
&\times\left(1-\cos\Phi\left(9+2\cos p+\mathcal{J}(1+\cos p)\csc(p/2)\right) + 4\tilde{\gamma}\sin(p/2)\sin\Phi\right) \\
&- F_1\left(1/2, 1/2, -3/2; 1; 1-\epsilon, 1-\frac{1}{2}(1+\cos\Phi)\ \epsilon\right) \\
&\times\left(2-2\cos\Phi\left(5+2\cos p+\mathcal{J}(1+\cos p)\csc(p/2)\right)\right. \\
&\left.+\ \tilde{\gamma}\left(\mathcal{J}(1+\cos p)+4\sin(p/2)\right)\sin\Phi\right)\Bigg]\epsilon\Bigg\}.
\end{aligned} \tag{4.72}$$

For the undeformed case, when $\tilde{\gamma}=0$, $\Phi=0$, (4.72) simplifies to

$$\mathcal{C}^{pr}_1 \approx -c^{pr}_1 \frac{\pi^2}{4}\sin(p/2)\left[3\sin(p/2)+\sin(3p/2)+\mathcal{J}(1+\cos p)\right]\epsilon^2. \tag{4.73}$$

This is in accordance with the result $\mathcal{C}^{pr}_1 \approx 0$ found in [25], where only the leading order in ϵ was taken into account.

The case $j=2$

Now we have

$$\begin{aligned}
\mathcal{C}^{pr}_{2\tilde{\gamma}} &= -\frac{8}{3}c^{pr}_2 \frac{1}{(1-\epsilon)^2\sqrt{(1-u^2)W(\chi_p-\chi_n)}}\Big\{\Big[3-(1+2v^2)W-3\tilde{\gamma}K\Big](1-\epsilon) \\
&\times\Big[(\chi_m-\chi_p)\,\mathbf{E}(1-\epsilon) - (\chi_m-\chi_p\ \epsilon)\mathbf{K}(1-\epsilon) + (1-u(u-\tilde{\gamma}(Ku-vW))-\tilde{\gamma}K) \\
&\times(2(\chi_p-\chi_m)((2-\epsilon)\chi_m+(1-2\epsilon)\chi_p)\mathbf{E}(1-\epsilon) + ((3-\epsilon)\chi_m^2 - 4\chi_m\chi_p\ \epsilon \\
&- \chi_p^2(1-3\epsilon)\ \epsilon)\mathbf{K}(1-\epsilon)\Big]\Big\}.
\end{aligned} \tag{4.74}$$

Expanding (4.74) in ϵ, one finds

$$\mathcal{C}_{2\tilde{\gamma}}^{pr} \approx \frac{2}{3} c_2^{pr} \sin^2(p/2) \left[2\mathcal{J}\cos\Phi - \tilde{\gamma}\left(2\sin(p/2) - \mathcal{J}(1+\cos p)\right)\sin(p/2)\sin\Phi\right]\epsilon. \quad (4.75)$$

Obviously, the result for the undeformed case is properly reproduced by the above formula.

Now, we will deal with $j \geq 3$, when we can use the following representation of $F_1(a, b_1, b_2; c; z_1, z_2)$ [108]:

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{k=0}^{\infty} \frac{(a)_k (b_2)_k}{(c)_k} \ {}_2F_1\left(a+k, b_1; c+k; z_1\right) \frac{z_2^k}{k!}. \quad (4.76)$$

Then, expending ${}_2F_1\left(\frac{1}{2}+k, \frac{1}{2}; 1+k; 1-\epsilon\right)(1-\chi_m/\chi_p)^k$ around $\epsilon = 0$, one finds

$$\begin{aligned} &{}_2F_1\left(\frac{1}{2}+k, \frac{1}{2}; 1+k; 1-\epsilon\right)\left(1-\frac{\chi_m}{\chi_p}\right)^k \approx \frac{\Gamma(1+k)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+k)} \Big\{ \log(4) - H_{k-\frac{1}{2}} \\ &- \frac{1}{4\chi_{p0}}\Big[2\chi_{p0} + (4k\chi_{m1} - (1+2k)\chi_{p0})\Big(\log(4) - H_{k-\frac{1}{2}}\Big)\Big]\epsilon - \log(\epsilon) \\ &- \frac{\chi_{p0} + 2k(\chi_{p0} - 2\chi_{m1})}{4\chi_{p0}}\,\epsilon\log(\epsilon)\Big\}, \end{aligned} \quad (4.77)$$

where H_z is defined as [108]

$$H_z = \psi(z+1) + \gamma.$$

The replacement of (4.77) in (4.76), taking into account that

$$a = \frac{1}{2}, \quad b_1 = \frac{1}{2}, \quad c = 1, \quad z_1 = 1-\epsilon, \quad z_2 = 1 - \frac{\chi_m}{\chi_p},$$

gives

$$F_1\left(\frac{1}{2}, \frac{1}{2}, b_2; 1; 1-\epsilon, 1-\frac{\chi_m}{\chi_p}\right) \approx C_0 + C_1\,\epsilon + C_2\,\epsilon\log(\epsilon) + C_3\log(\epsilon), \quad (4.78)$$

where

$$\begin{aligned} C_0 &= \frac{\Gamma(-b_2)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-b_2\right)} + \frac{\log(16)}{\pi}\ {}_1F_0(b_2, 1), \quad (4.79) \\ C_1 &= \frac{1}{4\pi}\Big\{\frac{1}{\chi_{p0}}\Big[-\frac{\sqrt{\pi}\,\Gamma(-1-b_2)}{\Gamma\left(\frac{1}{2}-b_2\right)}\ (\chi_{p0} + 2b_2\chi_{m1}) \\ &+ 8\log(2)\ b_2(\chi_{p0} - 2\chi_{m1})\ {}_1F_0(1+b_2, 1)\Big] - 2(1-\log(4))\ {}_1F_0(b_2, 1)\Big\}, \\ C_2 &= -\frac{1}{4\pi\chi_{p0}}\Big[\chi_{p0}\ {}_1F_0(b_2, 1) + 2b_2(\chi_{p0} - 2\chi_{m1})\ {}_1F_0(1+b_2, 1)\Big], \\ C_3 &= -\frac{1}{\pi}\ {}_1F_0(b_2, 1). \end{aligned}$$

In the normalized structure constants (4.69), there are two hypergeometric functions $F_1\left(\frac{1}{2},\frac{1}{2},b_2;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right)$ with $b_2=-j/2$ and $b_2=-1-j/2$.

By using (4.78), (4.79) in (4.69) and expanding it about $\epsilon=0$, we can write down the following approximate equality for $j\geq 3$

$$\mathcal{C}^{pr}_{j\tilde{\gamma}} \approx A_0 + A_1\epsilon + A_2\epsilon\log(\epsilon), \tag{4.80}$$

where the coefficients are given by

$$A_0 = c^{pr}_j \pi \frac{\Gamma(\frac{j}{2})^2}{\Gamma(\frac{1+j}{2})\Gamma(\frac{3+j}{2})}\ j\ \chi_{p0}^{\frac{1}{2}(j-1)}(1-v_0^2-\chi_{p0}), \tag{4.81}$$
$$\begin{aligned}
A_1 &= c^{pr}_j \frac{\pi}{4}\frac{\Gamma(\frac{j}{2})\Gamma(\frac{j}{2}-1)}{\Gamma(\frac{1+j}{2})\Gamma(\frac{3+j}{2})}\ \chi_{p0}^{\frac{1}{2}(j-3)}\Big\{4(W_1+\chi_{m1})\chi_{p0}-2\chi_{p0}^2\\
&-\left[2\chi_{n1}(1-v_0^2-\chi_{p0})+\chi_{p0}(1-v_0(v_0+8v_1+2v_0W_1)\right.\\
&\left.-\chi_{p0}(1-2W_1))-2(1-v_0^2+\chi_{p0})\chi_{p1}\right]j\\
&+\left[\chi_{n1}-4v_0v_1\chi_{p0}+\chi_{m1}(1-v_0^2-\chi_{p0})-v_0^2(\chi_{n1}+W_1\chi_{p0}-3\chi_{p1})-3\chi_{p1}\right.\\
&\left.+\chi_{p0}\left(-\chi_{n1}+W_1(-1+\chi_{p0})+\chi_{p1}\right)\right]j^2\\
&+(1-v_0^2-\chi_{p0})\chi_{p1}\ j^3\\
&+\tilde{\gamma}\left[4K_1\chi_{p0}+\left(2\chi_{p0}\left(K_1-2(K_1+v_0u_1)\chi_{p0}\right)\right)j+\left(2\chi_{p0}\left(v_0u_1\chi_{p0}-K_1(1-\chi_{p0})\right)\right)j^2\right]\Big\},\\
A_2 &= -c^{pr}_j\frac{\pi}{2}\frac{\Gamma(\frac{j}{2})^2}{\Gamma(\frac{1+j}{2})\Gamma(\frac{3+j}{2})}\ j\ \chi_{p0}^{\frac{1}{2}(j-3)}\left[4v_0v_2\chi_{p0}+(1-v_0^2+\chi_{p0})\chi_{p2}\right.\\
&\left.-(1-v_0^2-\chi_{p0})\chi_{p2}\ j-2\tilde{\gamma}v_0u_2\chi_{p0}^2\right].
\end{aligned}$$

Now, our goal is to express (4.80) in terms of $\mathcal{J}$, p, and Φ. The result is given by [26]

$$\begin{aligned}
\mathcal{C}^{pr}_{j\tilde{\gamma}} &\approx c^{pr}_j\frac{\pi}{8}\frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{1+j}{2}\right)\Gamma\left(\frac{3+j}{2}\right)}\sin\left(\frac{p}{2}\right)^{1+j}\left[4(j-1)\Gamma\left(\frac{j}{2}-1\right)\cos(\Phi)\right.\\
&\left.-\tilde{\gamma}\Gamma\left(\frac{j}{2}\right)\left(4\sin\left(\frac{p}{2}\right)-j\left(1+\cos\left(p\right)\right)\mathcal{J}\right)\sin(\Phi)\right]\epsilon.
\end{aligned} \tag{4.82}$$

Let us point out that (4.82) reduces exactly to the result found for the undeformed case in [25], when $\tilde{\gamma}=0$, $\Phi=0$. Moreover, it generalizes it for any $j\geq 3$.

4.3.4 Two GM states and singlet scalar operators on higher string levels

According to [23], the normalized structure constant for the present case is given by

$$\begin{aligned}
\mathcal{C}^q_{\tilde{\gamma}} &= c_{\Delta_q}\pi^{3/2}\frac{\Gamma\left(\frac{\Delta_q}{2}\right)}{\Gamma\left(\frac{\Delta_q+1}{2}\right)}\frac{(-2A)^q}{(1-v^2)^{q-1}\sqrt{(1-u^2)W(\chi_p-\chi_n)}}\\
&\sum_{k=0}^{q}\frac{q!}{k!(q-k)!}\left(-\frac{B}{A}\right)^k\chi_p^k\ F_1\left(\frac{1}{2},\frac{1}{2},-k;1;1-\epsilon,1-\frac{\chi_m}{\chi_p}\right),
\end{aligned} \tag{4.83}$$

where

$$A = 1 - \frac{1}{2}(1+v^2)W - \tilde{\gamma}K, \qquad B = 1 - \tilde{\gamma}K - u\left[u - \tilde{\gamma}(Ku - vW)\right], \tag{4.84}$$
$$\epsilon = \frac{\chi_m - \chi_n}{\chi_p - \chi_n},$$

Now, let us write down what the general formula (4.83) for the normalized structure constant in the γ-deformed case gives for the first two string levels.

$q = 1$ (level $n = 0$):

$$\mathcal{C}^1_{\tilde{\gamma}} = 2c_{\Delta_1}\pi^{3/2}\frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{\Delta_1+1}{2}\right)}\frac{1-\frac{1}{2}(1+v^2)W-\tilde{\gamma}K}{\sqrt{(1-u^2)W(\chi_p-\chi_n)}}$$
$$\left[\frac{1-u^2-\tilde{\gamma}\left(uvW+(1-u^2)K\right)}{1-\frac{1}{2}(1+v^2)W-\tilde{\gamma}K}\chi_p\ F_1\left(1/2,1/2,-1;1;1-\epsilon,1-\chi_m/\chi_p\right)\right.$$
$$\left.-\frac{2}{\pi}\mathbf{K}\left(1-\epsilon\right)\right].$$

$q = 2$ (level $n = 1$):

$$\mathcal{C}^2_{\tilde{\gamma}} = 4c_{\Delta_2}\pi^{3/2}\frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta_2+1}{2}\right)}\frac{\left(1-\frac{1}{2}(1+v^2)W-\tilde{\gamma}K\right)^2}{(1-v^2)\sqrt{(1-u^2)W(\chi_p-\chi_n)}}$$
$$\left[\frac{2}{\pi}\mathbf{K}\left(1-\epsilon\right)-2\frac{1-u^2-\tilde{\gamma}\left(uvW+(1-u^2)K\right)}{1-\frac{1}{2}(1+v^2)W-\tilde{\gamma}K}\chi_p\ F_1\left(1/2,1/2,-1;1;1-\epsilon,1-\chi_m/\chi_p\right)\right.$$
$$\left.+\left(\frac{1-u^2-\tilde{\gamma}\left(uvW+(1-u^2)K\right)}{1-\frac{1}{2}(1+v^2)W-\tilde{\gamma}K}\right)^2\chi_p^2\ F_1\left(1/2,1/2,-2;1;1-\epsilon,1-\chi_m/\chi_p\right)\right].$$

Leading finite-size effect

Here, we restrict ourselves to the case $\mathcal{J}_2 = 0$, $\mathcal{J}_1 = \mathcal{J}$ large but finite, i.e. $J_1 \gg \sqrt{\lambda}$.

For this case, we were not able to obtain a general formula for the leading finite-size corrections to the three-point correlation functions in terms of $\mathcal{J}$, p, and Φ, for any $q \geq 1$. That is why, we are going to present here the results for $q = 1, ..., 5$ (string levels $n = 0, 1, 2, 3, 4$).

Here, we are interested in the case of small ϵ (or, equivalently, large $\mathcal{J}$) limit. So, we will expand everything in ϵ. Since the computations are similar to the previously considered cases, we will write down the final results only. They are given by the following approximate equalities:

$$\mathcal{C}^1_{\tilde{\gamma}} \approx c_{\Delta_1}\frac{\sqrt{\pi}}{8}\frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{1+\Delta_1}{2}\right)}\sin(p/2)\Big\{16 - 8\mathcal{J}\csc(p/2) + \big[4 - \big(2\left(1-\cos p + \mathcal{J}^2\cot^2(p/2)\right)$$
$$+\ \mathcal{J}(5-\cos p)\csc(p/2)\big)\cos\Phi + 8\tilde{\gamma}\mathcal{J}\sin^2(p/2)\sin\Phi\big]\,\epsilon\Big\},$$

$$\mathcal{C}^2_{\tilde{\gamma}} \approx -c_{\Delta_2}\frac{\sqrt{\pi}}{24}\frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{1+\Delta_2}{2}\right)}\Big\{8(2\sin(p/2)-3\mathcal{J})+\Big[12\sin(p/2)$$
$$+(2(27+5\cos p)\sin(p/2)-\mathcal{J}(31+13\cos p+3\mathcal{J}(1+\cos p)\csc(p/2)))\cos\Phi$$
$$-8\tilde{\gamma}\sin(p/2)(8\sin(p/2)-\mathcal{J}(7+\cos p))\sin\Phi\Big]\epsilon\Big\},$$

$$\mathcal{C}^3_{\tilde{\gamma}} \approx c_{\Delta_3}\frac{\sqrt{\pi}}{120}\frac{\Gamma\left(\frac{\Delta_3}{2}\right)}{\Gamma\left(\frac{1+\Delta_3}{2}\right)}\Big\{8(38\sin(p/2)-15\mathcal{J})+\Big[60\sin(p/2)$$
$$+(18(13+19\cos p)\sin(p/2)-\mathcal{J}(187+97\cos p+15\mathcal{J}(1+\cos p)\csc(p/2)))\cos\Phi$$
$$-12\tilde{\gamma}\sin(p/2)(48\sin(p/2)-\mathcal{J}(23-7\cos p))\sin\Phi\Big]\epsilon\Big\},$$

$$\mathcal{C}^4_{\tilde{\gamma}} \approx -c_{\Delta_4}\frac{\sqrt{\pi}}{840}\frac{\Gamma\left(\frac{\Delta_4}{2}\right)}{\Gamma\left(\frac{1+\Delta_4}{2}\right)}\Big\{1264\sin(p/2)-840\mathcal{J}+\Big[\sin(p/2)\,(420$$
$$+\big(4730+2054\cos p-\mathcal{J}(1837+1207\cos p)\csc(p/2)-210\mathcal{J}^2\cot^2(p/2)\big)\cos\Phi$$
$$-16\;\tilde{\gamma}\,(424\sin(p/2)-3\mathcal{J}(79+9\cos p))\sin\Phi)\Big]\epsilon\Big\},$$

$$\mathcal{C}^5_{\tilde{\gamma}} \approx c_{\Delta_5}\frac{\sqrt{\pi}}{2520}\frac{\Gamma\left(\frac{\Delta_5}{2}\right)}{\Gamma\left(\frac{1+\Delta_5}{2}\right)}\Big\{8(902\sin(p/2)-315\mathcal{J})+\Big[1260\sin(p/2)$$
$$+(2(6093+7667\cos p)\sin(p/2)-\mathcal{J}\,(6343+4453\cos p$$
$$+\;315\mathcal{J}(1+\cos p)\csc(p/2)))\cos\Phi$$
$$-20\tilde{\gamma}\sin(p/2)(1376\sin(p/2)-\mathcal{J}(523-107\cos p))\sin\Phi\Big]\epsilon\Big\}.$$

4.4 Semiclassical three-point correlation functions in η-deformed $AdS_5\times S^5$

4.4.1 Two GM states and dilaton with zero momentum

We derive the 3-point correlation function between two giant magnons heavy string states and the light dilaton operator with zero momentum in the η-deformed $AdS_5\times S^5$ valid for any J_1 and η in the semiclassical limit. We show that this result satisfies a consistency relation between the 3-point correlation function and the conformal dimension of the giant magnon. We also provide a leading finite J_1 correction explicitly [29].

The normalized structure constant in the 3-point correlation function for the case under consideration can be written as follows

$$C_{\tilde{\eta}}^{d} = \frac{16c_{\Delta}^{d}}{3\tilde{\eta}} \frac{\chi_m}{\sqrt{\chi_p(1-\chi_m)(\chi_\eta - \chi_m)}} \left[\mathbf{\Pi}\left(1 - \frac{\chi_m}{\chi_p}, 1-\epsilon\right) - \mathbf{K}\left(1-\epsilon\right) \right], \tag{4.85}$$

where

$$\epsilon = \frac{\chi_m(\chi_\eta - \chi_p)}{\chi_p(\chi_\eta - \chi_m)}. \tag{4.86}$$

Eq.(4.85) is the main result of this paper, which is an *exact* semiclassical result for the normalized structure constant $C_{\tilde{\eta}}^{d}$ valid for any value of $\tilde{\eta}$ and J_1. Here, χ_p and χ_m are determined by the angular momentum J_1 and world-sheet momentum p from the following equations: [19]

$$\begin{aligned} J_1 &= \frac{2T}{\tilde{\eta}} \frac{1}{\sqrt{\chi_p(\chi_\eta - \chi_m)}} \left[\chi_p \mathbf{K}\left(1-\epsilon\right) - \chi_m \mathbf{\Pi}\left(1 - \frac{\chi_m}{\chi_p}, 1-\epsilon\right) \right], & (4.87) \\ p &= \frac{2\chi_m}{\tilde{\eta}} \sqrt{\frac{1-\chi_p}{\chi_p(1-\chi_m)(\chi_\eta - \chi_m)}} \left[\mathbf{K}\left(1-\epsilon\right) - \mathbf{\Pi}\left(\frac{\chi_p - \chi_m}{\chi_p(1-\chi_m)}, 1-\epsilon\right) \right]. & (4.88) \end{aligned}$$

The world-sheet energy of the giant magnon is given by

$$E = \frac{2T}{\tilde{\eta}} \frac{\chi_p - \chi_m}{\sqrt{\chi_p(1-\chi_m)(\chi_\eta - \chi_m)}} \mathbf{K}\left(1-\epsilon\right). \tag{4.89}$$

One of nontrivial check is that the g derivative of $\Delta = E - J_1$ should be proportional to the normalized structure constant $C_{\tilde{\eta}}^{d}$ since the g derivative of the two-point function inserts the dilaton (Lagrangian) operator into the two-point function of the heavy operators [101]. This can be expressed by

$$C_{\tilde{\eta}}^{d} = \frac{8c_{\Delta}^{d}}{3\sqrt{1+\tilde{\eta}^2}} \frac{\partial \Delta}{\partial g}. \tag{4.90}$$

To check that Eqs.(4.85), (4.87)-(4.89) satisfy Eq.(4.90), we use the fact that

$$\frac{\partial J_1}{\partial g} = \frac{\partial p}{\partial g} = 0 \tag{4.91}$$

as noticed in [19] for the case of undeformed giant magnon. From these, we can obtain the expressions for $\partial\chi_p/\partial g$ and $\partial\chi_m/\partial g$ which can be inserted to $\partial\Delta/\partial g$. The η-deformed case involves much more complicated expressions which can be dealt with the Mathematica. In the Appendix of [29], we provided our Mathematica code which confirms that the structure constant $C_{\tilde{\eta}}^{d}$ in Eq.(4.85) do sastisfy the consistency condition (4.90) exactly.

In the limit $\tilde{\eta} \to 0$ with $\tilde{\eta}^2\chi_\eta \to 1$, Eq.(4.85) becomes

$$C_3^0 = \frac{16c_{\Delta}^{d}}{3} \sqrt{\frac{\chi_p}{1-\chi_m}} \left[\mathbf{E}\left(1-\epsilon\right) - \epsilon \mathbf{K}\left(1-\epsilon\right) \right], \qquad \epsilon = \frac{\chi_m}{\chi_p} \tag{4.92}$$

[19] We express J_1 and p in terms of different but equivalent combinations of elliptic functions compared with Eqs. (3.23) and (3.25) in [28].

where we used the identity $(1-a)\mathbf{\Pi}(a,a) = \mathbf{E}(a)$. This is the structure constant of the undeformed theory derived in [19].

Leading finite-size effect

It is straightforward to compute the leading finite-size effect on $C^d_{\tilde\eta}$ for $J_1 \gg g$ by taking the limit $\epsilon \to 0$ in (4.85).

First we expand the parameters χ_p, W and v for small ϵ as follows:

$$\begin{aligned} \chi_p &= \chi_{p0} + (\chi_{p1} + \chi_{p2}\log\epsilon)\epsilon, \\ W &= 1 + W_1\epsilon, \\ v &= v_0 + (v_1 + v_2\log\epsilon)\epsilon. \end{aligned} \tag{4.93}$$

Inserting into Eq.(4.85), we obtain

$$\begin{aligned} C^d_{\tilde\eta} \approx & \frac{16 c^d_\Delta}{3\tilde\eta^2\sqrt{\left(1+\frac{1}{\tilde\eta^2}\right)\chi_{p0}}}\Bigg\{\sqrt{(1+\tilde\eta^2)\chi_{p0}}\ \mathrm{arctanh}\frac{\tilde\eta\sqrt{\chi_{p0}}}{\sqrt{1+\tilde\eta^2}} \\ & -\Bigg[\frac{W_1}{2}\sqrt{(1+\tilde\eta^2)\chi_{p0}}\ \mathrm{arctanh}\frac{\tilde\eta\sqrt{\chi_{p0}}}{\sqrt{1+\tilde\eta^2}} + \frac{\tilde\eta}{4\left(1+\tilde\eta^2(1-\chi_{p0})\right)}\times \\ & \left((1+\tilde\eta^2)(\chi_{p0}-2\chi_{p1}) - 4\left((1+\tilde\eta^2)\chi_{p0} + 2W_1\left(1+\tilde\eta^2(1-\chi_{p0})\right)\right)\log 2\right)\Bigg]\epsilon \\ & -\frac{\tilde\eta}{4\left(1+\tilde\eta^2(1-\chi_{p0})\right)}\left(\left((1+\tilde\eta^2)(\chi_{p0}-2\chi_{p2}) + 2W_1\left(1+\tilde\eta^2(1-\chi_{p0})\right)\right)\right)\epsilon\log\epsilon\Bigg\}. \end{aligned} \tag{4.94}$$

In view of the equations

$$\chi_m = 1 - W, \qquad \chi_p = 1 - v^2 W, \qquad \chi_\eta = 1 + \frac{1}{\tilde\eta^2} \tag{4.95}$$

and (4.86), we can express all the auxiliary parameters in terms of v (or its coefficients v_0, v_1, and v_2):

$$\begin{aligned} &\chi_{p0} = 1 - v_0^2, \qquad \chi_{p1} = 1 - v_0^2 - 2v_0v_1 - \frac{(1-v_0^2)^2}{1+\tilde\eta^2 v_0^2}, \qquad \chi_{p2} = -2v_0v_2, \\ &W_1 = -\frac{(1+\tilde\eta^2)(1-v_0^2)}{1+\tilde\eta^2 v_0^2}. \end{aligned} \tag{4.96}$$

This leads to

$$C^d_{\tilde\eta}\approx\frac{16c^d_\Delta}{3\tilde\eta}\left\{\text{arctanh}\frac{\tilde\eta\sqrt{1-v_0^2}}{\sqrt{1+\tilde\eta^2}}+\frac{1}{4\sqrt{(1+\tilde\eta^2)(1-v_0^2)}\left(1+\tilde\eta^2v_0^2\right)^2}\times\right.\tag{4.97}$$

$$\left[(1+\tilde\eta^2)\left((1-v_0^2)\left(1+\tilde\eta^2v_0^2\right)\left(2\sqrt{(1+\tilde\eta^2)\left((1-v_0^2\right)}\text{arctanh}\frac{\tilde\eta\sqrt{1-v_0^2}}{\sqrt{1+\tilde\eta^2}}-\tilde\eta\log 16\right)\right.\right.$$

$$\left.\left.-\tilde\eta\left(1-v_0(3v_0-2v_0^3-4v_1+v_0(1-v_0^2-4v_0v_1)\tilde\eta^2)\right)\right)\right]\epsilon$$

$$\left.+\frac{\tilde\eta(1+\tilde\eta^2)(1-v_0^2-4v_0v_2)}{4\sqrt{(1+\tilde\eta^2)(1-v_0^2)}(1+\tilde\eta^2v_0^2)}\,\epsilon\log\epsilon\right\}.$$

To fix v_0, v_1, and v_2, one can use the small ϵ expansion of the angular difference

$$\Delta\phi_1=\phi_1(\tau,L)-\phi_1(\tau,-L)\equiv p,$$

where we identified the angular difference $\Delta\phi_1$ with the magnon momentum p on the dual spin chain. The result is [28]

$$v_0=\frac{\cot\frac{p}{2}}{\sqrt{\tilde\eta^2+\csc^2\frac{p}{2}}},\tag{4.98}$$

and

$$v_1=\frac{v_0(1-v_0^2)\left[1-\log 16+\tilde\eta^2\left(2-v_0^2(1+\log 16)\right)\right]}{4(1+\tilde\eta^2v_0^2)},\qquad v_2=\frac{1}{4}v_0(1-v_0^2).\tag{4.99}$$

By using (4.98), (4.99) in (4.97), one finds

$$C_3^{\tilde\eta}\approx\frac{16c^d_\Delta}{3\tilde\eta}\left\{\text{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right)+\frac{(1+\tilde\eta^2)\sin^2\frac{p}{2}}{4\sqrt{\tilde\eta^2+\csc^2\frac{p}{2}}}\times\right.\tag{4.100}$$

$$\left.\left[\left(2\sqrt{\tilde\eta^2+\csc^2\frac{p}{2}}\text{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right)-\tilde\eta(1+\log 16)\right)\epsilon+\tilde\eta\epsilon\log\epsilon\right]\right\}.$$

The expansion parameter ϵ in the leading order is given by [28]

$$\epsilon=16\ \exp\left[-\left(\frac{J_1}{g}+\frac{2\sqrt{1+\tilde\eta^2}}{\tilde\eta}\text{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde\eta^2\sin^2\frac{p}{2}}{(1+\tilde\eta^2)\sin^2\frac{p}{2}}}\right].\tag{4.101}$$

Here we used Eq.(3.194) for the string tension T.

The final expression for the normalized structure costant is given by

$$C^d_{\tilde\eta}\approx\frac{16c^d_\Delta}{3\tilde\eta}\left\{\text{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right)-4\frac{\tilde\eta(1+\tilde\eta^2)\sin^3\frac{p}{2}}{\sqrt{1+\tilde\eta^2\sin^2\frac{p}{2}}}\left[1+\frac{J_1}{g}\sqrt{\frac{\tilde\eta^2+\csc^2\frac{p}{2}}{1+\tilde\eta^2}}\right]\right.\tag{4.102}$$

$$\left.\times\exp\left[-\left(\frac{J_1}{g}+\frac{2\sqrt{1+\tilde\eta^2}}{\tilde\eta}\text{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde\eta^2\sin^2\frac{p}{2}}{(1+\tilde\eta^2)\sin^2\frac{p}{2}}}\right]\right\}.$$

Let us point out that in the limit $\tilde{\eta} \to 0$, (4.102) reduces to

$$C_3 \approx \frac{16}{3} c_\Delta^d \sin\frac{p}{2}\left[1 - 4\sin\frac{p}{2}\left(\sin\frac{p}{2} + \frac{J_1}{g}\right)\exp\left(-\frac{J_1}{g\sin\frac{p}{2}} - 2\right)\right],$$

which reproduces the result for the undeformed case found in [19]. Another check is that this satisfies Eq.(4.90) with Δ computed in [28]

$$\Delta \equiv E - J_1 \approx 2g\sqrt{1+\tilde{\eta}^2}\Bigg\{\frac{1}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right) - 4\frac{(1+\tilde{\eta}^2)\sin^3\frac{p}{2}}{\sqrt{1+\tilde{\eta}^2\sin^2\frac{p}{2}}} \times \tag{4.103}$$
$$\exp\left[-\left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde{\eta}^2\sin^2\frac{p}{2}}{(1+\tilde{\eta}^2)\sin^2\frac{p}{2}}}\right]\Bigg\}.$$

4.4.2 Two GM states and dilaton with non-zero momentum

Here we will be interested in the case when the dilaton momentum $j > 0$. According to [30] the semiclassical normalized structure constants for the case under consideration is given by

$$\mathcal{C}_{\tilde{\eta}}^{d,j} = \frac{2\pi^{\frac{3}{2}} c_\Delta^{d,j}\Gamma\left(2+\frac{j}{2}\right)(1-v^2\kappa^2)^{\frac{j-1}{2}}}{\Gamma\left(\frac{5+j}{2}\right)\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)}} \times \tag{4.104}$$
$$\Bigg[(1-v^2\kappa^2)F_1\left(\frac{1}{2}, \frac{2+j}{2}, -\frac{1+j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}, \frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right)$$
$$-(1-\kappa^2)F_1\left(\frac{1}{2}, \frac{j}{2}, \frac{1-j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}, \frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right)\Bigg].$$

Now we take the limit $\tilde{\eta} \to 0$ in (4.104) and obtain

$$\mathcal{C}^{d,j} = \frac{2\pi^{\frac{3}{2}} c_\Delta^{d,j}\Gamma\left(2+\frac{j}{2}\right)(1-v^2\kappa^2)^{\frac{j-1}{2}}}{\kappa\Gamma\left(\frac{5+j}{2}\right)} \times \tag{4.105}$$
$$\Bigg[(1-v^2\kappa^2)\,{}_2F_1\left(\frac{1}{2}, -\frac{1+j}{2}; 1; \frac{(1-v^2)\kappa^2)}{1-v^2\kappa^2}\right)$$
$$-(1-\kappa^2)\,{}_2F_1\left(\frac{1}{2}, \frac{1-j}{2}; 1; \frac{(1-v^2)\kappa^2)}{1-v^2\kappa^2}\right)\Bigg].$$

This is exactly what was found in [22] for $u = 0$, as it should be.

Let us also say that in the particular case when $j = 1$, (4.104) simplifies to

$$\mathcal{C}_\eta^{d,1} = \frac{3\pi c_\Delta^{d,1}\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)}}{2\tilde{\eta}^2\kappa^2}\left[\mathbf{K}\left(\tilde{\eta}^2\frac{(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}\right) - \mathbf{E}\left(\tilde{\eta}^2\frac{(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}\right)\right].$$

In the limit $\tilde{\eta} \to 0$, $\mathcal{C}_\eta^{d,1}$ becomes

$$\mathcal{C}^{d,1} = \frac{3}{8}\pi^2 c_\Delta^{d,1} \kappa (1 - v^2).$$

4.4.3 Two GM states and primary scalar operators

It was proven in [30] that the normalized structure constant for this case can be represented as

$$\mathcal{C}_\eta^{pr,j} = \frac{2c_\Delta^{pr,j}\sqrt{\pi}}{\tilde{\eta}\kappa} \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1+j}{2})} \left[\frac{1 - \kappa^2 + j(1 - v^2\kappa^2)}{1 + j} J_j - J_{jp} \right], \tag{4.106}$$

where

$$J_j = \int_{\chi_m}^{\chi_p} \frac{\chi^{\frac{j}{2}}}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}} \, d\chi, \tag{4.107}$$

$$J_{jp} = \int_{\chi_m}^{\chi_p} \frac{\chi^{\frac{j}{2}+1}}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}} \, d\chi. \tag{4.108}$$

To compute the above two integrals, we introduce the variable

$$x = \frac{\chi - \chi_m}{\chi_p - \chi_m} \in (0, 1).$$

Then J_j becomes

$$J_j = \chi_m^{\frac{j-1}{2}} (\chi_\eta - \chi_m)^{-\frac{1}{2}} \int_0^1 x^{-\frac{1}{2}} (1 - x)^{-\frac{1}{2}} \left(1 - \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m} x\right)^{-\frac{1}{2}} \left(1 + \frac{\chi_p - \chi_m}{\chi_m} x\right)^{\frac{j-1}{2}} dx. \tag{4.109}$$

Comparing the above expression with the integral representation for the hypergeometric function of two variables $F_1(a, b_1, b_2; c; z_1, z_2)$ [51]

$$F_1(a, b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 x^{a-1} (1 - x)^{c-a-1} (1 - z_1 x)^{-b_1} (1 - z_2 x)^{-b_2},$$
$$Re(a) > 0, \quad Re(c - a) > 0,$$

one finds

$$J_j = \pi \chi_m^{\frac{j-1}{2}} (\chi_\eta - \chi_m)^{-\frac{1}{2}} F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{j-1}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, -\frac{\chi_p - \chi_m}{\chi_m} \right). \tag{4.110}$$

In order to compute J_{jp}, we have to replace j with $j + 2$. Doing this, we obtain

$$J_{jp} = \pi \chi_m^{\frac{j+1}{2}} (\chi_\eta - \chi_m)^{-\frac{1}{2}} F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{j+1}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, -\frac{\chi_p - \chi_m}{\chi_m} \right). \tag{4.111}$$

The replacement of (4.110) and (4.111) into (4.106) gives

$$\begin{aligned}\mathcal{C}_{\eta}^{pr,j} &= \frac{2\pi^{\frac{3}{2}}c_{\Delta}^{pr,j}}{\tilde{\eta}\kappa}\frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1+j}{2})}\frac{\chi_m^{\frac{j-1}{2}}}{\sqrt{\chi_\eta-\chi_m}}\Bigg\{\left[1-\frac{(1+jv^2)\kappa^2}{1+j}\right]\times \\ &F_1\left(\frac{1}{2},\frac{1}{2},\frac{1-j}{2};1;\frac{\chi_p-\chi_m}{\chi_\eta-\chi_m},-\frac{\chi_p-\chi_m}{\chi_m}\right) \\ &-\chi_m F_1\left(\frac{1}{2},\frac{1}{2},-\frac{1+j}{2};1;\frac{\chi_p-\chi_m}{\chi_\eta-\chi_m},-\frac{\chi_p-\chi_m}{\chi_m}\right)\Bigg\}.\end{aligned} \tag{4.112}$$

Knowing that

$$\chi_\eta = 1+\frac{1}{\tilde{\eta}^2} \qquad \chi_p = 1-v^2\kappa^2, \qquad \chi_m = 1-\kappa^2,$$

and using the relation [51]

$$F_1(a,b_1,b_2;c;z_1,z_2) = (1-z_1)^{c-a-b_1}(1-z_2)^{-b_2}F_1\left(c-a,c-b_1-b_2,b_2;c;z_1,\frac{z_1-z_2}{1-z_2}\right),$$

we can rewrite (4.112) in the following form

$$\begin{aligned}\mathcal{C}_{\eta}^{pr,j} &= \frac{2\pi^{\frac{3}{2}}c_{\Delta}^{pr,j}\Gamma(\frac{j}{2})(1-v^2\kappa^2)^{\frac{j-1}{2}}}{\Gamma(\frac{1+j}{2})\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)}}\Bigg\{\left[1-\frac{(1+jv^2)\kappa^2}{1+j}\right]\times \\ &F_1\left(\frac{1}{2},\frac{j}{2},\frac{1-j}{2};1;\frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2},\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right) \\ &-(1-v^2\kappa^2)F_1\left(\frac{1}{2},\frac{2+j}{2},-\frac{1+j}{2};1;\frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2},\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right)\Bigg\}.\end{aligned} \tag{4.113}$$

This is our final exact semiclassical result for this type of three-point correlation functions.

Next, we would like to compare (4.113) with the known expression for the undeformed case [22]. To this end, we take the limit $\tilde{\eta}\to 0$ and by using that [51]

$$F_1\left(a,b_1,b_2;c;0,z_2\right) = {}_2F_1(a,b_2;c;z_2),$$

we find

$$\begin{aligned}\mathcal{C}^{pr,j} &= \frac{2\pi^{\frac{3}{2}}c_{\Delta}^{pr,j}\Gamma\left(2+\frac{j}{2}\right)(1-v^2\kappa^2)^{\frac{j-1}{2}}}{\kappa\Gamma\left(\frac{5+j}{2}\right)}\Bigg[(1-v^2\kappa^2){}_2F_1\left(\frac{1}{2},-\frac{1+j}{2};1;,\frac{(1-v^2)\kappa^2}{1-v^2\kappa^2}\right) \\ &-(1-\kappa^2){}_2F_1\left(\frac{1}{2},\frac{1-j}{2};1;\frac{(1-v^2)\kappa^2)}{1-v^2\kappa^2}\right)\Bigg].\end{aligned}$$

This is exactly the same result found in [22] for $u=0$ (finite-size giant magnons with one nonzero angular momentum) as it should be.

Let us also give an example for the simplest case when $j=1$. In that case (4.113) reduces to

$$\mathcal{C}_{\eta}^{pr,1} = \frac{2\pi c_{\Delta}^{pr,1}}{\tilde{\eta}^2\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)}}\Bigg[2(1+\tilde{\eta}^2\kappa^2)\mathbf{E}\left(\tilde{\eta}^2\frac{(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}\right) -(2+(1+v^2)\tilde{\eta}^2\kappa^2)\mathbf{K}\left(\tilde{\eta}^2\frac{(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}\right)\Bigg].$$

In the limit $\tilde{\eta}\to 0$, $\mathcal{C}_{\eta}^{pr,1}\to 0$.

4.4.4 Two GM states and singlet scalar operators on higher string levels

It was found in [30] that the normalized structure constants for the case at hand are given by

$$\begin{aligned}
\mathcal{C}_{\tilde{\eta}}^{q} &= c_{\Delta}^{q}\frac{\sqrt{\pi}}{\kappa}\frac{\Gamma\left(\frac{\Delta_q^{\eta}}{2}\right)}{\Gamma\left(\frac{\Delta_q^{\eta}+1}{2}\right)}\frac{(-1)^q}{\tilde{\eta}(1-v^2)^{q-1}}\int_{\chi_m}^{\chi_p}d\chi\frac{\left[2-(1+v^2)\kappa^2-2\chi\right]^q}{\sqrt{(\chi_\eta-\chi)(\chi_p-\chi)(\chi-\chi_m)\chi}} \qquad (4.114)\\
&= c_{\Delta}^{q}\frac{\pi^{\frac{3}{2}}}{\kappa}\frac{\Gamma\left(\frac{\Delta_q^{\eta}}{2}\right)}{\Gamma\left(\frac{\Delta_q^{\eta}+1}{2}\right)}\frac{(-1)^q\left[2-(1+v^2)\kappa^2\right]^q}{\tilde{\eta}(1-v^2)^{q-1}\sqrt{\chi_\eta-\chi_m}}\times\\
&\sum_{k=0}^{q}\frac{q!}{k!(q-k)!}\left[-\frac{1}{1-\frac{1}{2}(1+v^2)\kappa^2}\right]^k\chi_m^{k-\frac{1}{2}}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}-k;1;\frac{\chi_p-\chi_m}{\chi_\eta-\chi_m},-\frac{\chi_p-\chi_m}{\chi_m}\right)\\
&= c_{\Delta}^{q}\pi^{\frac{3}{2}}\frac{\Gamma\left(\frac{\Delta_q^{\eta}}{2}\right)}{\Gamma\left(\frac{\Delta_q^{\eta}+1}{2}\right)}\frac{(-1)^q\left[2-(1+v^2)\kappa^2\right]^q}{(1-v^2)^{q-1}\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}}\sum_{k=0}^{q}\frac{q!}{k!(q-k)!}\\
&\times\left[-\frac{1-v^2\kappa^2}{1-\frac{1}{2}(1+v^2)\kappa^2}\right]^k F_1\left(\frac{1}{2},k,\frac{1}{2}-k;1;\frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2},\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right),
\end{aligned}$$

where

$$\Delta_q^{\eta}=2\left(1+\sqrt{2\pi g\sqrt{1+\tilde{\eta}^2}(q-1)+1-\frac{1}{2}q(q-1)}\right). \qquad (4.115)$$

In order to compare with the undeformed case, we take the limit $\tilde{\eta}\to 0$ in (4.114) and obtain

$$\mathcal{C}^{q} = c_{\Delta}^{q}\pi^{\frac{3}{2}}\frac{\Gamma\left(\frac{\Delta_q}{2}\right)}{\Gamma\left(\frac{\Delta_q+1}{2}\right)}\frac{(-1)^q\left[2-(1+v^2)\kappa^2\right]^q}{(1-v^2)^{q-1}\sqrt{\kappa^2(1-v^2\kappa^2)}}\sum_{k=0}^{q}\frac{q!}{k!(q-k)!}\times\left[-\frac{1-v^2\kappa^2}{1-\frac{1}{2}(1+v^2)\kappa^2}\right]^k {}_2F_1\left(\frac{1}{2},\frac{1}{2}-k;1;\frac{(1-v^2)\kappa^2}{1-v^2\kappa^2}\right).$$

This is exactly what was found in [23] for finite-size giant magnons with one nonzero angular momentum.

Let us consider two particular cases. From (4.114) it follows that the normalized structure constants for the first two string levels, for the case at hand, are given by

$q = 1$ (level $n = 0$)

$$\begin{aligned} C^1_{\tilde{\eta}} &= 2c^1_\Delta \pi^{\frac{1}{2}} \frac{\Gamma\left(\frac{\Delta^\eta_1}{2}\right)}{\Gamma\left(\frac{\Delta^\eta_1+1}{2}\right)} \frac{1}{\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}} \times \\ & \left[\pi(1-v^2\kappa^2)F_1\left(\frac{1}{2},1,-\frac{1}{2};1;\frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2},\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right)\right. \\ & \left. -\left(2-(1+v^2)\kappa^2\right)\mathbf{K}\left(\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right)\right]. \end{aligned}$$

$q = 2$ (level $n = 1$)

$$\begin{aligned} \mathcal{C}^2_{\tilde{\eta}} &= 2c^2_\Delta \pi^{\frac{3}{2}} \frac{\Gamma\left(\frac{\Delta^\eta_2}{2}\right)}{\Gamma\left(\frac{\Delta^\eta_2+1}{2}\right)} \frac{\left(2-(1+v^2)\kappa^2\right)^2}{(1-v^2)\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}} \times \\ & \left\{\frac{1}{\pi}\mathbf{K}\left(\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right) - \frac{2(1-v^2\kappa^2)}{(2-(1+v^2)\kappa^2)^2} \times \right. \\ & \left[\left(2-(1+v^2)\kappa^2\right)F_1\left(\frac{1}{2},1,-\frac{1}{2};1;\frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2},\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right)\right. \\ & \left.\left. -(1-v^2\kappa^2)F_1\left(\frac{1}{2},2,-\frac{3}{2};1;\frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2},\frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right)\right]\right\}. \end{aligned}$$

In the limit $\tilde{\eta} \to 0$, the above two expressions simplify to

$$\begin{aligned} \mathcal{C}^1 &= 2c^1_\Delta \pi^{\frac{1}{2}} \frac{\Gamma\left(\frac{\Delta_1}{2}\right)}{\Gamma\left(\frac{\Delta_1+1}{2}\right)} \frac{1}{\sqrt{\kappa^2(1-v^2\kappa^2)}} \times \\ & \left[2(1-v^2\kappa^2)\mathbf{E}\left(\frac{(1-v^2)\kappa^2}{1-v^2\kappa^2}\right)\right. \\ & \left. -\left(2-(1+v^2)\kappa^2\right)\mathbf{K}\left(\frac{(1-v^2)\kappa^2}{(1-v^2\kappa^2)}\right)\right], \end{aligned}$$

and

$$
\begin{aligned}
\mathcal{C}^2 &= 2c_\Delta^2\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta_2+1}{2}\right)}\frac{1}{(1-v^2)\sqrt{\kappa^2(1-v^2\kappa^2)}}\times \\
& \left[\left(2-(1+v^2)\kappa^2\right)^2\mathbf{K}\left(\frac{(1-v^2)\kappa^2}{(1-v^2\kappa^2)}\right)\right. \\
& -4\left(2-(1+v^2)\kappa^2\right)(1-v^2\kappa^2)\mathbf{E}\left(\frac{(1-v^2)\kappa^2}{1-v^2\kappa^2}\right) \\
& \left.+2\pi(1-v^2\kappa^2)^2{}_2F_1\left(\frac{1}{2},-\frac{3}{2};1;\frac{(1-v^2)\kappa^2}{1-v^2\kappa^2}\right)\right]
\end{aligned}
$$

respectively.

Appendices

A Notations for the special functions

Euler gamma function - $\Gamma(z)$

Pochhammer symbol - $(a)_n$

Jacobi elliptic functions - $\mathbf{sn}(z|m)$, $\mathbf{cn}(z|m)$, $\mathbf{dn}(z|m)$

Jacobi amplitude - am(z)

Incomplete elliptic integrals of the first, second and third kind - $F\,(z|m)$, $E\,(z|m)$, $\Pi\,(n,z|m)$

Complete elliptic integrals of the first, second and third kind - $\mathbf{K}\,(m)$, $\mathbf{E}\,(m)$, $\mathbf{\Pi}\,(n|m)$

Hypergeometric functions of one variable - ${}_1F_0(a;z)$, ${}_2F_1(a,b;c;z)$

Hypergeometric functions of two variables - $F_1(a,b_1,b_2;c;z_1,z_2)$, $F_2(a,b_1,b_2;c_1,c_2;z_1,z_2)$

Hypergeometric functions of n variables - $F_D^{(n)}(a,b_1,...,b_n;c;z_1,...,z_n)$

B Relation between the NR system and CSG

B.1 Explicit Relations between the Parameters

In the general case, the relation between the parameters in the solutions of the NR and CSG integrable systems is given by

$$K^2 = R^2M^2, \quad C_\phi = \frac{2}{\alpha^2-\beta^2}\left\{3M^2 - 2\left[\kappa^2 + \frac{(\kappa^2-\omega_1^2)-\omega_2^2}{1-\beta^2/\alpha^2}\right]\right\},$$

$$\begin{aligned}
&\frac{1}{4}M^4(\alpha^2-\beta^2)\frac{A^2}{\beta^2} = M^4\left(M^2-\kappa^2+\frac{\omega_2^2}{1-\beta^2/\alpha^2}\right)\\
&-\left(\frac{\kappa^2-\omega_1^2}{1-\beta^2/\alpha^2}\right)\left\{M^4+\left[M^2-\left(\frac{\kappa^2-\omega_1^2}{1-\beta^2/\alpha^2}\right)\right]\left(\frac{\kappa^2-\omega_1^2}{1-\beta^2/\alpha^2}\right)\right.\\
&\left.-\left[2M^2-\left(\frac{\kappa^2-\omega_1^2}{1-\beta^2/\alpha^2}\right)\right]\left(\kappa^2-\frac{\omega_2^2}{1-\beta^2/\alpha^2}\right)\right\}\\
&-\frac{(\omega_1^2-\omega_2^2)}{\omega_1^2\left(1-\beta^2/\alpha^2\right)^3}\left\{\left[M^2\left(1-\beta^2/\alpha^2\right)-\kappa^2\right](\omega_1^2-\omega_2^2)\check{C}_2^2\right.\\
&\left.-\left[M^2\left(1-\beta^2/\alpha^2\right)-(\kappa^2-\omega_1^2)\right]\left[2\frac{\beta}{\alpha}\omega_2\kappa^2\check{C}_2+\left(\kappa^2-\omega_1^2\right)\left(\frac{\beta^2}{\alpha^2}\kappa^2-\omega_1^2\right)\right]\right\},
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&\frac{1}{4}M^4(\alpha^2-\beta^2)C_\chi^2 = -\left(\frac{\kappa^2-\omega_1^2}{1-\beta^2/\alpha^2}\right)^2\left[\kappa^2-\frac{(\kappa^2-\omega_1^2)+\omega_2^2}{1-\beta^2/\alpha^2}\right]\\
&+\frac{(\omega_1^2-\omega_2^2)}{\omega_1^2\left(1-\beta^2/\alpha^2\right)^3}\left\{\kappa^2(\omega_1^2-\omega_2^2)\check{C}_2^2-\left(\kappa^2-\omega_1^2\right)\left[2\frac{\beta}{\alpha}\omega_2\kappa^2\check{C}_2+\left(\kappa^2-\omega_1^2\right)\left(\frac{\beta^2}{\alpha^2}\kappa^2-\omega_1^2\right)\right]\right\},
\end{aligned}$$

where $\check{C}_2 = C_2/\alpha$. Thus, we have expressed the CSG parameters C_ϕ, A and C_χ through the NR parameters α, β, κ, ω_1, ω_2, C_2. The mass parameter M remains free.

Let us consider several examples, which illustrate the established NR - CSG correspondence. We are interested in the GM and SS configurations on $R_t \times S^2$ and $R_t \times S^3$. From the NR-system viewpoint, we have to set $C_2 = 0$ in (3.45), (3.47) and (3.48) for the GM and SS string solutions. This condition is to require one of the turning points, where $\theta' = 0$, to lay on the equator of the sphere, i.e. $\theta = \pi/2$ [56].

B.2 On $R_t \times S^2$

We begin with the $R_t \times S^2$ case, when $C_2 = \omega_2 = 0$ and θ' in (3.45) takes the form

$$\theta' = \frac{\pm\alpha\omega_1}{(\alpha^2-\beta^2)\sin\theta}\sqrt{\left(\frac{\beta^2\kappa^2}{\alpha^2\omega_1^2}-\sin^2\theta\right)\left(\sin^2\theta-\frac{\kappa^2}{\omega_1^2}\right)}. \tag{A.2}$$

B.2.1 The Giant Magnon

The GM solution corresponds to $\kappa^2 = \omega_1^2$ with $\alpha^2 > \beta^2$, which is given by

$$\cos\theta = \frac{\sqrt{1-\beta^2/\alpha^2}}{\cosh\left(\omega_1 \frac{\sigma+\tau\beta/\alpha}{\sqrt{1-\beta^2/\alpha^2}}\right)}.$$

From 3.47), one finds $f_2 = 0$ and

$$f_1 = \arctan\left[\frac{\alpha}{\beta}\sqrt{1-\beta^2/\alpha^2}\tanh\left(\omega_1 \frac{\sigma+\tau\beta/\alpha}{\sqrt{1-\beta^2/\alpha^2}}\right)\right].$$

For $R^2 = M^2 = \omega_1^2 = 1$, $\beta/\alpha = -\sin\theta_0$, this string solution coincides with the Hofman-Maldacena solution [60], and is equivalent to the solution in [58] for $R_t \times S^2$ after the identification $W_1 = Z_1 \exp(i\pi/2)$, $W_2 = Z_2$. Now, the parameters in (3.58) take the values

$$K^2 = R^2\omega_1^2 = 1, \quad M^2 = \omega_1^2 = 1,$$
$$C_\phi = \frac{2\omega_1^2}{\alpha^2-\beta^2} = \frac{2}{\alpha^2\cos^2\theta_0}, \quad A = C_\chi = 0,$$

and from Eqs.(3.55),(3.56) and (3.57)) the corresponding SG solution becomes

$$\sin(\phi/2) = \frac{1}{\cosh\left(\frac{\sigma-\tau\sin\theta_0}{\cos\theta_0} - \eta_0\right)}.$$

This can be also obtained from (3.59) by setting $\omega_2 = 0$.

However, for $M^2 > \omega_1^2 = \kappa^2$, we have

$$A = 2\beta\sqrt{\frac{M^2-\omega_1^2}{\alpha^2-\beta^2}} \neq 0.$$

This case is related to the CSG system instead of the SG one. It is interesting to find the CSG solution associated with it. Using (3.55) again, we find

$$\sin(\phi/2) = \frac{\omega_1}{M\cosh\left(\omega_1\frac{\sigma+\tau\beta/\alpha}{\sqrt{1-\beta^2/\alpha^2}} - \eta_0\right)}, \quad \chi = 2\sqrt{\frac{M^2-\omega_1^2}{1-\beta^2/\alpha^2}}\left(\frac{\beta}{\alpha}\sigma + \tau\right).$$

B.2.2 The Single Spike

The SS solution corresponds to $\beta^2\kappa^2 = \alpha^2\omega_1^2$. In this case, the expressions for θ and f_1 are

$$\cos\theta = \frac{\sqrt{1-\alpha^2/\beta^2}}{\cosh(C\xi)}, \quad f_1 = -\omega_1(\sigma\alpha/\beta + \tau) + \arctan\left[\frac{\beta}{\alpha}\sqrt{1-\alpha^2/\beta^2}\tanh(C\xi)\right],$$

and the corresponding string solution is

$$W_1 = R\sqrt{1-\frac{1-\alpha^2/\beta^2}{\cosh^2(C\xi)}}\exp\left\{-i\omega_1\sigma\alpha/\beta + i\arctan\left[\frac{\beta}{\alpha}\sqrt{1-\alpha^2/\beta^2}\tanh(C\xi)\right]\right\},$$
$$W_2 = \frac{R\sqrt{1-\alpha^2/\beta^2}}{\cosh(C\xi)}, \qquad Z_0 = R\exp\left(i\frac{\alpha}{\beta}\omega_1\tau\right),$$

where we used a short notation

$$C\xi \equiv \omega_1\frac{\alpha}{\beta}\frac{\sigma\alpha/\beta+\tau}{\sqrt{1-\alpha^2/\beta^2}}.$$

The "dual" SG solution can be obtained from (3.62) by setting $\omega_2 = 0$. If we choose $R = 1$, $\alpha/\beta = \sin\theta_1$, $\omega_1 = -\cot\theta_1$, $\beta = 1$, the SS solution on $R_t \times S^2$ in [61] is reproduced.

B.3 On $R_t \times S^3$

B.3.1 The Giant Magnon

Let us continue with the $R_t \times S^3$ case, when $C_2 = 0$, $\omega_2 \neq 0$. First, we would like to establish the correspondence between the dyonic GM string solution [56] ($\kappa^2 = \omega_1^2$) to those found in [58]

$$Z_1 = \frac{1}{\sqrt{1+k^2}}\left\{\tanh\left[\cos\alpha^D\left(\sigma\sqrt{1+k^2\cos^2\alpha^D} - k\tau\cos\alpha^D\right)\right] - ik\right\}\exp(i\tau),$$
$$Z_2 = \frac{1}{\sqrt{1+k^2}}\frac{\exp\left[i\sin\alpha^D\left(\tau\sqrt{1+k^2\cos^2\alpha^D} - k\sigma\cos\alpha^D\right)\right]}{\cosh\left[\cos\alpha^D\left(\sigma\sqrt{1+k^2\cos^2\alpha^D} - k\tau\cos\alpha^D\right)\right]},$$

where the parameter k is related to the soliton rapidity $\hat{\theta}$ through the equality

$$k = \frac{\sinh\hat{\theta}}{\cos\alpha^D},$$

and α^D determines the $U(1)$ charge carried by the CSG soliton [58].

The solutions of Eqs.(3.45), (3.47) and (3.48) are given by

$$\cos\theta = \frac{\cos\theta_0}{\cosh(C\xi)}, \qquad f_1 = \arctan\left[\cot\theta_0\tanh(C\xi)\right], \qquad f_2 = \frac{\beta\omega_2}{\alpha^2-\beta^2}\xi,$$
$$\sin^2\theta_0 \equiv \frac{\beta^2\omega_1^2}{\alpha^2(\omega_1^2-\omega_2^2)}, \qquad C \equiv \frac{\alpha\sqrt{\omega_1^2-\omega_2^2}}{\alpha^2-\beta^2}\cos\theta_0.$$

Then, the comparison shows that the two solutions are equivalent if

$$Z_1\exp(i\pi/2) = W_1 = R\sin\theta\exp\left[i(\omega_1\tau + f_1)\right],$$
$$Z_2 = W_2 = R\cos\theta\exp\left[i(\omega_2\tau + f_2)\right],$$
$$R = \kappa = \omega_1 = 1, \qquad \alpha = \cos\alpha^D\sqrt{1+k^2\cos^2\alpha^D},$$
$$\beta = -k\cos^2\alpha^D, \qquad \omega_2 = \frac{\sin\alpha^D}{\sqrt{1+k^2\cos^2\alpha^D}}.$$

As a consequence, the CSG parameters in (3.58) reduce to

$$C_\phi = \frac{2}{\cos^2\alpha^D}\left(1+2\sin^2\alpha^D\right)], \qquad A = k\sin(2\alpha^D), \qquad C_\chi = 0, \qquad K^2 = 1.$$

B.3.2 The Single Spike

Now, let us turn to the SS solutions on $R_t \times S^3$ as described by the NR integrable system [62]. By using the SS-condition $\beta^2\kappa^2 = \alpha^2\omega_1^2$ in (3.45) one derives

$$\theta' = \frac{\alpha\sqrt{\omega_1^2-\omega_2^2}}{\alpha^2-\beta^2}\frac{\cos\theta}{\sin\theta}\sqrt{\sin^2\theta - \frac{\alpha^2\omega_1^2}{\beta^2(\omega_1^2-\omega_2^2)}},$$

whose solution is given by

$$\cos\theta = \frac{\sqrt{(1-\alpha^2/\beta^2)\,\omega_1^2-\omega_2^2}}{\sqrt{\omega_1^2-\omega_2^2}\cosh(C\xi)}, \qquad C\xi \equiv \sqrt{\omega_1^2 - \frac{\omega_2^2}{1-\alpha^2/\beta^2}}\,\frac{\alpha(\sigma\alpha/\beta+\tau)}{\sqrt{\beta^2-\alpha^2}}.$$

By using (3.47), (3.48), one finds the following expressions for the string embedding coordinates $\varphi_j = \omega_j\tau + f_j$

$$\begin{aligned}
\varphi_1 &= -\omega_1\sigma\alpha/\beta + \arctan\left\{\frac{\beta}{\alpha\omega_1}\sqrt{\left(1-\frac{\alpha^2}{\beta^2}\right)\left(\omega_1^2-\frac{\omega_2^2}{1-\alpha^2/\beta^2}\right)}\tanh(C\xi)\right\}\\
\varphi_2 &= -\omega_2\frac{\alpha(\sigma+\tau\alpha/\beta)}{\beta(1-\alpha^2/\beta^2)}.
\end{aligned}$$

Comparing the above results with the SS string solution given in (4.1) - (4.7) of [63], we see that the two solutions coincide for

$$R = 1, \qquad \sin\theta_1 = -\frac{1}{\sqrt{\omega_1^2-\omega_2^2}}, \qquad \sin\gamma_1 = \frac{\omega_2}{\omega_1}, \qquad \omega_1 = -\frac{\beta}{\alpha}. \tag{A.3}$$

From (3.60), the CSG parameters are

$$\begin{aligned}
C_\phi &= \frac{2}{\beta^2\left(1-\sin^2\theta_1\cos^2\gamma_1\right)}\left[4-3M^2+\frac{2\cos^4\gamma_1}{\sin^2\gamma_1\left(1-\sin^2\theta_1\cos^2\gamma_1\right)}\right], \qquad K^2 = M^2,\\
A &= \frac{M^2-1}{M^2\sqrt{1-\sin^2\theta_1\cos^2\gamma_1}}\sqrt{\frac{\cos^4\gamma_1}{\sin^2\gamma_1\left(1-\sin^2\theta_1\cos^2\gamma_1\right)}-M^2},\\
C_\chi &= -\frac{2\sin\gamma_1}{M^2\beta\left(1-\sin^2\theta_1\cos^2\gamma_1\right)}.
\end{aligned}$$

Comparing (A.3) with (3.61), one sees that the solution found in [63] corresponds actually to $M^2 = 1$ which leads to $A_{SS} = 0$. Hence, the "dual" CSG solution is of the type (3.63).

C Explicit exact solutions in $AdS_3 \times S^3 \times T^4$ with NS-NS B-field

Let start with the solutions for the string coordinates in AdS_3 subspace. By using (3.15), (3.144) and (3.145), one can find that the scalar potential U_r in (3.147) is given by

$$U_r(r) = \frac{1}{2(\alpha^2-\beta^2)}\left[\left(\alpha\Lambda^{\phi}\right)^2 r^2 - \left(\alpha\Lambda^{t}\right)^2(1+r^2) + \frac{\left(C_{\phi}+q\alpha\Lambda^{t}r^2\right)^2}{r^2} - \frac{\left(C_t - q\alpha\Lambda^{\phi}r^2\right)^2}{1+r^2}\right]. \tag{A.4}$$

After introducing the variable

$$y = r^2, \tag{A.5}$$

and replacing (A.4) into (3.149) one can rewrite it in the following form

$$d\xi = \frac{\alpha^2-\beta^2}{2\alpha\sqrt{(1-q^2)\left[(\Lambda^{\phi})^2-(\Lambda^{t})^2\right]}}\frac{dy}{\sqrt{(y_p-y)(y-y_m)(y-y_n)}}, \tag{A.6}$$

where

$$0 \le y_m < y < y_p, \quad y_n < 0,$$

and y_p, y_m, y_n satisfy the relations

$$\begin{aligned}
y_p+y_m+y_n &= \frac{1}{\alpha^2(1-q^2)\left[(\Lambda^{\phi})^2-(\Lambda^{t})^2\right]} \\
&\left[C_r(\alpha^2-\beta^2) - \alpha\left(\alpha\left(\Lambda^{\phi}\right)^2 - 2\alpha\left(\Lambda^{t}\right)^2\right) + 2q\left(C_{\phi}\Lambda^{t}+C_t\Lambda^{\phi}\right) + q^2\alpha\left(\Lambda^{t}\right)^2\right], \\
y_py_m+y_py_n+y_my_n &= -\frac{1}{\alpha^2(1-q^2)\left[(\Lambda^{\phi})^2-(\Lambda^{t})^2\right]} \\
&\left[C_r(\alpha^2-\beta^2) + C_t^2 - C_{\phi}^2 + \alpha^2\left(\Lambda^{t}\right)^2 - 2q\alpha C_{\phi}\Lambda^{t}\right], \\
y_py_my_n &= -\frac{C_{\phi}^2}{\alpha^2(1-q^2)\left[(\Lambda^{\phi})^2-(\Lambda^{t})^2\right]}.
\end{aligned} \tag{A.7}$$

Integrating (A.6) and inverting

$$\xi(y) = \frac{\alpha^2-\beta^2}{\alpha\sqrt{(1-q^2)\left[(\Lambda^{\phi})^2-(\Lambda^{t})^2\right](y_p-y_n)}}\, F\left(\arcsin\sqrt{\frac{y_p-y}{y_p-y_m}}, \frac{y_p-y_m}{y_p-y_n}\right)$$

to $y(\xi)$, one finds the following solution

$$y(\xi) = (y_p - y_n)\ \mathbf{dn}^2 \left[\frac{\alpha\sqrt{(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right](y_p - y_n)}}{\alpha^2 - \beta^2}\ \xi, \frac{y_p - y_m}{y_p - y_n} \right] + y_n. \tag{A.8}$$

Next, we will compute $\tilde{X}^t(\xi)$ and $\tilde{X}^\phi(\xi)$ entering (3.146). Integrating

$$\frac{d\tilde{X}^t}{d\xi} = \frac{1}{\alpha^2 - \beta^2}\left[\beta\Lambda^t + q\alpha\Lambda^\phi - \left(C_t + q\alpha\Lambda^\phi\right)\frac{1}{1+y}\right],$$
$$\frac{d\tilde{X}^\phi}{d\xi} = \frac{1}{\alpha^2 - \beta^2}\left(\beta\Lambda^\phi + q\alpha\Lambda^t + \frac{C_\phi}{y}\right),$$

and using (A.8), we obtain the following solutions for the string coordinates t, ϕ, in accordance with our ansatz

$$t(\tau,\sigma) = \Lambda^t\tau + \frac{1}{\alpha\sqrt{(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right](y_p - y_n)}} \tag{A.9}$$
$$\left[\left(\beta\Lambda^t + q\alpha\Lambda^\phi\right)\ F\left(\arcsin\sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p - y_n}\right)\right.$$
$$\left. - \frac{C_t + q\alpha\Lambda^\phi}{1 + y_p}\ \Pi\left(\arcsin\sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{1 + y_p}, \frac{y_p - y_m}{y_p - y_n}\right)\right]$$
$$\phi(\tau,\sigma) = \Lambda^\phi\tau + \frac{1}{\alpha\sqrt{(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right](y_p - y_n)}} \tag{A.10}$$
$$\left[\left(\beta\Lambda^\phi + q\alpha\Lambda^t\right)\ F\left(\arcsin\sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p - y_n}\right)\right.$$
$$\left. + \frac{C_\phi}{y_p}\ \Pi\left(\arcsin\sqrt{\frac{y_p - y}{y_p - y_m}}, \frac{y_p - y_m}{y_p}, \frac{y_p - y_m}{y_p - y_n}\right)\right].$$

Let us compute now the string energy and spin on the solutions found. Starting from (3.151), (3.152), we obtain

$$E_s = \frac{2T}{\sqrt{(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right](y_p - y_n)}} \tag{A.11}$$
$$\left[\left(\Lambda^t - \frac{\beta}{\alpha^2}C_t - q\frac{C_\phi}{\alpha}\right)\mathbf{K}\left(1 - \frac{y_m - y_n}{y_p - y_n}\right) + \right.$$
$$\left.(1-q^2)\Lambda^t\left(y_n\,\mathbf{K}\left(1 - \frac{y_m - y_n}{y_p - y_n}\right) + (y_p - y_n)\,\mathbf{E}\left(1 - \frac{y_m - y_n}{y_p - y_n}\right)\right)\right],$$
$$S = \frac{2T}{\sqrt{(1-q^2)\left[(\Lambda^\phi)^2 - (\Lambda^t)^2\right](y_p - y_n)}} \tag{A.12}$$
$$\left[\left(\frac{\beta}{\alpha^2}C_\phi + q\frac{C_t}{\alpha} + \Lambda^\phi q^2\right)\mathbf{K}\left(1 - \frac{y_m - y_n}{y_p - y_n}\right) + \right.$$
$$(1-q^2)\Lambda^\phi\left(y_n\,\mathbf{K}\left(1 - \frac{y_m - y_n}{y_p - y_n}\right) + (y_p - y_n)\,\mathbf{E}\left(1 - \frac{y_m - y_n}{y_p - y_n}\right)\right)$$
$$\left.-\frac{q\frac{C_t}{\alpha} + q^2\Lambda^\phi}{1 + y_p}\,\mathbf{\Pi}\left(\frac{y_p - y_m}{1 + y_p}, 1 - \frac{y_m - y_n}{y_p - y_n}\right)\right].$$

Now we turn to the S^3 subspace. By using (3.15), (3.144) and (3.145), one can show that the scalar potential U_θ in (3.149) can be written as

$$U_\theta(\theta) = \frac{1}{2(\alpha^2 - \beta^2)}\left[\frac{\left(C_{\phi_2} - q\alpha\Lambda^{\phi_1}\,\chi\right)^2}{\chi} + \frac{\left(C_{\phi_1} + q\alpha\Lambda^{\phi_2}\,\chi\right)^2}{1-\chi}\right. \tag{A.13}$$
$$\left.+\alpha^2\left(\Lambda^{\phi_2}\right)^2\chi + \alpha^2\left(\Lambda^{\phi_1}\right)^2(1-\chi)\right],$$

where we introduced the notation

$$\chi \equiv \cos^2\theta. \tag{A.14}$$

Replacing (A.13) in (3.149), one can see that it can be written in the form

$$d\xi = \frac{\alpha^2 - \beta^2}{2\alpha\sqrt{(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)}}\frac{d\chi}{\sqrt{(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}, \tag{A.15}$$

where

$$0 \le \chi_m < \chi < \chi_p \le 1, \qquad \chi_n \le 0,$$

and

$$\chi_p + \chi_m + \chi_n = \frac{1}{\alpha^2(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)}$$
$$\left[-C_\theta(\alpha^2 - \beta^2) - \left(\alpha\Lambda^{\phi_2}\right)^2 + (2-q^2)\left(\alpha\Lambda^{\phi_1}\right)^2\right.$$
$$\left.-2q\alpha\left(C_{\phi_2}\Lambda^{\phi_1} + C_{\phi_1}\Lambda^{\phi_2}\right)\right],$$

$$\chi_p\chi_m + \chi_p\chi_n + \chi_m\chi_n = \frac{1}{\alpha^2(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)}$$
$$\left[\left(\alpha\Lambda^{\phi_1}\right)^2 + C_{\phi_1}^2 - C_{\phi_2}^2 - C_\theta(\alpha^2-\beta^2) - 2q\alpha C_{\phi_2}\Lambda^{\phi_1}\right],$$

$$\chi_p\chi_m\chi_n = -\frac{(C_{\phi_2})^2}{\alpha^2(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)}.$$

Integrating (A.15), one finds the following solution for χ

$$\chi(\xi) = (\chi_p - \chi_n)\,\mathbf{dn}^2\left[\frac{\alpha\sqrt{(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)(\chi_p-\chi_n)}}{\alpha^2-\beta^2}\,\xi, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right] + \chi_n. \quad \text{(A.16)}$$

Now we are ready to find the "orbit" $r = r(x)$. Written in terms of y and χ, it is given by

$$y = (y_p - y_n)\,\mathbf{dn}^2\left[\frac{\sqrt{\left((\Lambda^{\phi})^2 - (\Lambda^t)^2\right)(y_p-y_n)}}{\sqrt{\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)(\chi_p-\chi_n)}}\right. \quad \text{(A.17)}$$
$$\left.\times F\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right), \frac{y_p-y_m}{y_p-y_n}\right] + y_n.$$

Next, we compute $\tilde{X}^{\phi_1}(\xi)$ and $\tilde{X}^{\phi_2}(\xi)$. Replacing the results in our ansatz, we derive the following solutions for the isometric coordinates on S^3

$$\phi_1 = \Lambda^{\phi_1}\tau + \frac{1}{\alpha\sqrt{(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)(\chi_p-\chi_n)}} \quad \text{(A.18)}$$
$$\left[\left(\beta\Lambda^{\phi_1} - q\alpha\Lambda^{\phi_2}\right) F\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right.$$
$$\left.+\frac{C_{\phi_1} + q\alpha\Lambda^{\phi_2}}{1-\chi_p}\,\Pi\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, -\frac{\chi_p-\chi_m}{1-\chi_p}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right].$$

$$\phi_2 = \Lambda^{\phi_2}\tau + \frac{1}{\alpha\sqrt{(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)(\chi_p - \chi_n)}}$$
$$\left[\left(\beta\Lambda^{\phi_2} - q\alpha\Lambda^{\phi_1}\right)\ F\left(\arcsin\sqrt{\frac{\chi_p - \chi}{\chi_p - \chi_m}}, \frac{\chi_p - \chi_m}{\chi_p - \chi_n}\right)\right.$$
$$\left. + \frac{C_{\phi_1}}{\chi_p}\ \Pi\left(\arcsin\sqrt{\frac{\chi_p - \chi}{\chi_p - \chi_m}}, 1 - \frac{\chi_m}{\chi_p}, \frac{\chi_p - \chi_m}{\chi_p - \chi_n}\right)\right], \tag{A.19}$$

Based on (3.153) and the solutions for the string coordinates on S^3 we found, we can write down the explicit expressions for the conserved angular momenta J_1 and J_2 computed on the solutions. The result is

$$J_1 = \frac{2T}{\sqrt{(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)(\chi_p - \chi_n)}}$$
$$\left[\left(\frac{\beta}{\alpha^2}C_{\phi_1} + \Lambda^{\phi_1} - q\frac{C_{\phi_2}}{\alpha}\right)\mathbf{K}\left(1 - \frac{\chi_m - \chi_n}{\chi_p - \chi_n}\right)\right.$$
$$\left. -(1-q^2)\Lambda^{\phi_1}\left(\chi_n\ \mathbf{K}\left(1 - \frac{\chi_m - \chi_n}{\chi_p - \chi_n}\right) + (\chi_p - \chi_n)\ \mathbf{E}\left(1 - \frac{\chi_m - \chi_n}{\chi_p - \chi_n}\right)\right)\right]. \tag{A.20}$$

$$J_2 = \frac{2T}{\sqrt{(1-q^2)\left((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2\right)(\chi_p - \chi_n)}}$$
$$\left[\left(\frac{\beta}{\alpha^2}C_{\phi_2} - q\left(\frac{C_{\phi_1}}{\alpha} + q\Lambda^{\phi_2}\right)\right)\mathbf{K}\left(1 - \frac{\chi_m - \chi_n}{\chi_p - \chi_n}\right)\right.$$
$$+(1-q^2)\Lambda^{\phi_2}\left(\chi_n\ \mathbf{K}\left(1 - \frac{\chi_m - \chi_n}{\chi_p - \chi_n}\right) + (\chi_p - \chi_n)\ \mathbf{E}\left(1 - \frac{\chi_m - \chi_n}{\chi_p - \chi_n}\right)\right)$$
$$\left. + \frac{q\left(\frac{C_{\phi_1}}{\alpha} + q\Lambda^{\phi_2}\right)}{1 - \chi_p}\ \mathbf{\Pi}\left(-\frac{\chi_p - \chi_m}{1 - \chi_p}, 1 - \frac{\chi_m - \chi_n}{\chi_p - \chi_n}\right)\right], \tag{A.21}$$

Now, let us go to the T^4 subspace. Since in terms of φ^i coordinates the metric is flat and there is no B-field, the solutions for the string coordinates are simple and given by

$$\varphi^i(\tau, \sigma) = \Lambda^i\tau + \frac{1}{\alpha^2 - \beta^2}\left(C_i + \beta\Lambda^i\right)\xi. \tag{A.22}$$

The conserved charges (3.154) can be computed to be

$$J_i^T = \frac{2\pi\alpha T}{\alpha^2 - \beta^2}\left(\frac{\beta}{\alpha}C_i + \alpha\Lambda^i\right). \tag{A.23}$$

If we impose the periodicity conditions

$$\varphi^i(\tau,\sigma) = \varphi^i(\tau,\sigma+2L) + 2\pi n_i, \qquad n_i \in \mathbb{Z}_i,$$

the integration constants C_i are fixed in terms of the embedding parameters. Namely,

$$C_i = \frac{\pi n_i}{L\alpha}(\alpha^2-\beta^2) - \beta\Lambda^i. \tag{A.24}$$

Replacing (A.24) into (A.22) and (A.23), one finally finds

$$\begin{aligned} \varphi^i &= \left(\Lambda^i + \frac{\beta}{\alpha}\frac{\pi n_i}{L}\right)\tau + \frac{\pi n_i}{L}\sigma, \\ J_i^T &= 2\pi T\left(\Lambda^i + \frac{\beta}{\alpha}\frac{\pi n_i}{L}\right). \end{aligned} \tag{A.25}$$

Let us finally point out that the Virasoro constraints impose the following two conditions on the embedding parameters and integrations constants in the solutions found

$$\begin{aligned} &C_r + C_\theta = 0, \\ &\Lambda^t C_t + \Lambda^\phi C_\phi + \Lambda^{\phi_1}C_{\phi_1} + \Lambda^{\phi_2}C_{\phi_2} - \Lambda^i\left(\beta\Lambda^i - (\alpha^2-\beta^2)\frac{\pi n_i}{\alpha L}\right) = 0. \end{aligned} \tag{A.26}$$

D M2-brane GM and SS

For the GM-like case by using that $\tilde{C}_2 = 0$, $\tilde{\kappa}^2 = \omega_1^2$ in (3.336), (3.337), one finds

$$\begin{aligned} &\cos\theta(\xi) = \frac{\cos\tilde{\theta}_0}{\cosh(D_0\xi)}, \quad \sin^2\tilde{\theta}_0 = \frac{\beta^2\omega_1^2}{\tilde{A}^2(\omega_1^2-\omega_2^2)}, \quad D_0 = \frac{\tilde{A}\sqrt{\omega_1^2-\omega_2^2}}{\tilde{A}^2-\beta^2}\cos\tilde{\theta}_0, \\ &\varphi_1(\tau,\xi) = \omega_1\tau + \arctan\left[\cot\tilde{\theta}_0\tanh(D_0\xi)\right], \quad \varphi_2(\tau,\xi) = \omega_2\left(\tau + \frac{\beta}{\tilde{A}^2-\beta^2}\xi\right). \end{aligned}$$

For the SS-like solutions when $\tilde{C}_2 = 0$, $\tilde{\kappa}^2 = \omega_1^2\tilde{A}^2/\beta^2$, by solving the equations (3.336), (3.337), one arrives at

$$\begin{aligned} &\cos\theta(\xi) = \frac{\cos\tilde{\theta}_1}{\cosh(D_1\xi)}, \quad \sin^2\tilde{\theta}_1 = \frac{\tilde{A}^2\omega_1^2}{\beta^2(\omega_1^2-\omega_2^2)}, \quad D_1 = \frac{\tilde{A}\sqrt{\omega_1^2-\omega_2^2}}{\tilde{A}^2-\beta^2}\cos\tilde{\theta}_1, \\ &\varphi_1(\tau,\xi) = \omega_1\left(\tau - \frac{\xi}{\beta}\right) - \arctan\left[\cot\tilde{\theta}_1\tanh(D_1\xi)\right], \quad \varphi_2(\tau,\xi) = \omega_2\left(\tau + \frac{\beta}{\tilde{A}^2-\beta^2}\xi\right). \end{aligned}$$

The energy-charge relations computed on the above membrane solutions were found in [89], and in our notations read

$$\sqrt{1-r_0^2}E - \frac{J_1}{2} = \sqrt{\left(\frac{J_2}{2}\right)^2 + \frac{\tilde{\lambda}}{\pi^2}\sin^2\frac{p}{2}}, \quad \frac{p}{2} = \frac{\pi}{2} - \tilde{\theta}_0, \tag{A.27}$$

for the GM-like case, and

$$\sqrt{1-r_0^2}E - \frac{\sqrt{\tilde{\lambda}}}{2\pi}\Delta\varphi_1 = \frac{\sqrt{\tilde{\lambda}}}{\pi}\frac{p}{2}, \quad \frac{J_1}{2} = \sqrt{\left(\frac{J_2}{2}\right)^2 + \frac{\tilde{\lambda}}{\pi^2}\sin^2\frac{p}{2}}, \quad \frac{p}{2} = \frac{\pi}{2} - \tilde{\theta}_1, \tag{A.28}$$

for the SS-like solution, where

$$\tilde{\lambda} = \left[2\pi^2 T_2 R^3 r_0 (1-r_0^2)\gamma\right]^2. \tag{A.29}$$

D.1 Finite-Size Effects

For $\tilde{C}_2 = 0$, Eq.(3.336) can be written as

$$(\cos\theta)' = \mp\frac{\tilde{A}\sqrt{\omega_1^2-\omega_2^2}}{\tilde{A}^2-\beta^2}\sqrt{(z_+^2-\cos^2\theta)(\cos^2\theta - z_-^2)}, \tag{A.30}$$

where

$$z_\pm^2 = \frac{1}{2(1-\frac{\omega_2^2}{\omega_1^2})}\left\{q_1+q_2-\frac{\omega_2^2}{\omega_1^2} \pm \sqrt{(q_1-q_2)^2 - \left[2\left(q_1+q_2-2q_1q_2\right) - \frac{\omega_2^2}{\omega_1^2}\right]\frac{\omega_2^2}{\omega_1^2}}\right\},$$
$$q_1 = 1-\tilde{\kappa}^2/\omega_1^2, \quad q_2 = 1-\beta^2\tilde{\kappa}^2/\tilde{A}^2\omega_1^2.$$

The solution of (A.30) is

$$\cos\theta = z_+ \mathbf{dn}\left(C\xi|m\right), \quad C = \mp\frac{\tilde{A}\sqrt{\omega_1^2-\omega_2^2}}{\tilde{A}^2-\beta^2}z_+, \quad m \equiv 1 - z_-^2/z_+^2. \tag{A.31}$$

The solutions of Eqs.(3.337) now read

$$\mu_1 = \frac{2\beta/\tilde{A}}{z_+\sqrt{1-\omega_2^2/\omega_1^2}}\left[C\xi - \frac{\tilde{\kappa}^2/\omega_1^2}{1-z_+^2}\Pi\left(am(C\xi), -\frac{z_+^2-z_-^2}{1-z_+^2}\middle| m\right)\right],$$
$$\mu_2 = \frac{2\beta\omega_2/\tilde{A}\omega_1}{z_+\sqrt{1-\omega_2^2/\omega_1^2}}C\xi.$$

Our next task is to find out what kind of energy-charge relations can appear for the M2-brane solution in the limit when the energy $E \to \infty$. It turns out that the semiclassical behavior depends crucially on the sign of the difference $\tilde{A}^2 - \beta^2$.

D.1.1 The M2-brane GM

We begin with the M2-brane GM, i.e. $\tilde{A}^2 > \beta^2$. In this case, one obtains from (3.335) the following expressions for the conserved energy E and the angular momenta J_1, J_2

$$\begin{aligned}
\mathcal{E} &= \frac{2\tilde{\kappa}(1-\beta^2/\tilde{A}^2)}{\omega_1 z_+\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{K}\left(1-z_-^2/z_+^2\right), \\
\mathcal{J}_1 &= \frac{2z_+}{\sqrt{1-\omega_2^2/\omega_1^2}}\left[\frac{1-\beta^2\tilde{\kappa}^2/\tilde{A}^2\omega_1^2}{z_+^2}\mathbf{K}\left(1-z_-^2/z_+^2\right)-\mathbf{E}\left(1-z_-^2/z_+^2\right)\right], \\
\mathcal{J}_2 &= \frac{2z_+\omega_2/\omega_1}{\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{E}\left(1-z_-^2/z_+^2\right).
\end{aligned} \tag{A.32}$$

Here, we have used the notations

$$\mathcal{E} = \frac{2\pi}{\sqrt{\tilde{\lambda}}}\sqrt{1-r_0^2}E, \quad \mathcal{J}_1 = \frac{2\pi}{\sqrt{\tilde{\lambda}}}\frac{J_1}{2}, \quad \mathcal{J}_2 = \frac{2\pi}{\sqrt{\tilde{\lambda}}}\frac{J_2}{2}, \tag{A.33}$$

where $\tilde{\lambda}$ is defined in (A.29). The computation of $\Delta\varphi_1$ gives

$$\begin{aligned}
p \equiv \Delta\varphi_1 &= 2\int_{\theta_{min}}^{\theta_{max}}\frac{d\theta}{\theta'}\mu_1' = \\
&- \frac{2\beta/\tilde{A}}{z_+\sqrt{1-\omega_2^2/\omega_1^2}}\left[\frac{\tilde{\kappa}^2/\omega_1^2}{1-z_+^2}\mathbf{\Pi}\left(-\frac{z_+^2-z_-^2}{1-z_+^2}\middle|1-z_-^2/z_+^2\right)-\mathbf{K}\left(1-z_-^2/z_+^2\right)\right].
\end{aligned} \tag{A.34}$$

Expanding the elliptic integrals about $z_-^2 = 0$, one arrives at

$$\begin{aligned}
\mathcal{E}-\mathcal{J}_1 &= \sqrt{\mathcal{J}_2^2+4\sin^2(p/2)} - \frac{16\sin^4(p/2)}{\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}} \\
&\exp\left[-\frac{2\left(\mathcal{J}_1+\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2+4\sin^2(p/2)}\sin^2(p/2)}{\mathcal{J}_2^2+4\sin^4(p/2)}\right].
\end{aligned} \tag{A.35}$$

It is easy to check that the energy-charge relation (A.35) coincides with the one found in [65], describing the finite-size effects for dyonic GM. The difference is that in the string case the relations between $\mathcal{E}$, $\mathcal{J}_1$, $\mathcal{J}_2$ and E, J_1, J_2 are given by

$$\mathcal{E} = \frac{2\pi}{\sqrt{\lambda}}E, \quad \mathcal{J}_1 = \frac{2\pi}{\sqrt{\lambda}}J_1, \quad \mathcal{J}_2 = \frac{2\pi}{\sqrt{\lambda}}J_2,$$

while for the M2-brane they are written in (A.33).

D.1.2 The M2-brane SS

Let us turn our attention to the M2-brane SS, when $\tilde{A}^2 < \beta^2$. The computation of the conserved quantities (3.335) and $\Delta\varphi_1$ now gives

$$\mathcal{E} = \frac{2\tilde{\kappa}(\beta^2/\tilde{A}^2 - 1)}{\omega_1\sqrt{1-\omega_2^2/\omega_1^2}z_+}\mathbf{K}\left(1 - z_-^2/z_+^2\right),$$
$$\mathcal{J}_1 = \frac{2z_+}{\sqrt{1-\omega_2^2/\omega_1^2}}\left[\mathbf{E}\left(1 - z_-^2/z_+^2\right) - \frac{1-\beta^2\tilde{\kappa}^2/\tilde{A}^2\omega_1^2}{z_+^2}\mathbf{K}\left(1 - z_-^2/z_+^2\right)\right],$$
$$\mathcal{J}_2 = -\frac{2z_+\omega_2/\omega_1}{\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{E}\left(1 - z_-^2/z_+^2\right),$$
$$\Delta\varphi_1 = -\frac{2\beta/\tilde{A}}{\sqrt{1-\omega_2^2/\omega_1^2}z_+}\left[\frac{\tilde{\kappa}^2/\omega_1^2}{1-z_+^2}\mathbf{\Pi}\left(-\frac{z_+^2-z_-^2}{1-z_+^2}|1 - z_-^2/z_+^2\right) - \mathbf{K}\left(1 - z_-^2/z_+^2\right)\right].$$

$\mathcal{E} - \Delta\varphi_1$ can be derived as

$$\begin{aligned} \mathcal{E} - \Delta\varphi_1 &= \arcsin N(\mathcal{J}_1,\mathcal{J}_2) + 2\left(\mathcal{J}_1^2 - \mathcal{J}_2^2\right)\sqrt{\frac{4}{\left[4-\left(\mathcal{J}_1^2 - \mathcal{J}_2^2\right)\right]} - 1} \\ &\times \exp\left[-\frac{2\left(\mathcal{J}_1^2 - \mathcal{J}_2^2\right)N(\mathcal{J}_1,\mathcal{J}_2)}{\left(\mathcal{J}_1^2 - \mathcal{J}_2^2\right)^2 + 4\mathcal{J}_2^2}\left[\Delta\varphi + \arcsin N(\mathcal{J}_1,\mathcal{J}_2)\right]\right], \\ N(\mathcal{J}_1,\mathcal{J}_2) &\equiv \frac{1}{2}\left[4-\left(\mathcal{J}_1^2 - \mathcal{J}_2^2\right)\right]\sqrt{\frac{4}{\left[4-\left(\mathcal{J}_1^2 - \mathcal{J}_2^2\right)\right]} - 1}. \end{aligned} \tag{A.36}$$

Finally, by using the SS relation between the angular momenta

$$\mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)},$$

we obtain

$$\mathcal{E} - \Delta\varphi_1 = p + 8\sin^2\frac{p}{2}\tan\frac{p}{2}\exp\left(-\frac{\tan\frac{p}{2}(\Delta\varphi_1 + p)}{\tan^2\frac{p}{2} + \mathcal{J}_2^2\csc^2 p}\right). \tag{A.37}$$

This result coincides with the string result found in [14]. As in the GM case, the difference is in the identification (A.33).

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