Affine convex flow

M. Shkolnikov[1]

ICMS, IMI-BAS

based on joint projects with N. Kalinin, E. Lupercio and G. Mikhalkin

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Chapter 1

Tropical geometry

Origins of tropical geometry

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(works of I. Simon, term "tropical semiring" appears in 1988)
- High energy physics
(first pictures of tropical curves emerge in 1997)

arXiv:hep-th/9611230v3 11 Dec 1996

Type IIB Superstrings, BPS Monopoles, And Three-Dimensional Gauge Dynamics

Amihay Hanany and Edward Witten *

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School of Natural Sciences
Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA

Abstract

We propose an explanation via string theory of the correspondence between the Coulomb branch of certain three-dimensional supersymmetric gauge theories and certain moduli spaces of magnetic monopoles. The same construction also gives an explanation, via $SL(2, \mathbf{Z})$ duality of Type IIB superstrings, of the recently discovered "mirror symmetry" in three dimensions. New phase transitions in three dimensions as well as new infrared fixed points and even new coupling constants not present in the known Lagrangians are predicted from the string theory construction. An important role in the construction is played by a novel aspect of brane dynamics in which a third brane is created when two branes cross.

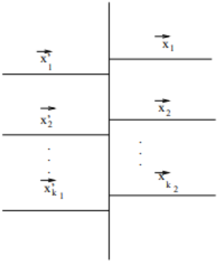

Figure 1: Here and in subsequent figures, vertical solid lines represent NS fivebranes in the 012345 directions, and horizontal lines represent 0126 threebranes. In the example depicted here, the threebranes come from left or right, and a massless hypermultiplet appears whenever a "left" and "right" threebrane meet.

Branes, Superpotentials and Superconformal Fixed Points

arXiv:hep-th/9704170v1 24 Apr 1997

Ofer Aharony[1] and Amihay Hanany[2]

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We analyze various brane configurations corresponding to field theories in three, four and five dimensions. We find brane configurations which correspond to three dimensional $N = 2$ and four dimensional $N = 1$ supersymmetric QCD theories with quartic superpotentials, in which what appear to be "hidden parameters" play an important role. We discuss the construction of five dimensional $N = 1$ supersymmetric gauge theories and superconformal fixed points using branes, which leads to new five dimensional $N = 1$ superconformal field theories. The same five dimensional theories are also used, in a surprising way, to describe new superconformal fixed points of three dimensional $N = 2$ supersymmetric theories, which have both "electric" and "magnetic" Coulomb branches.

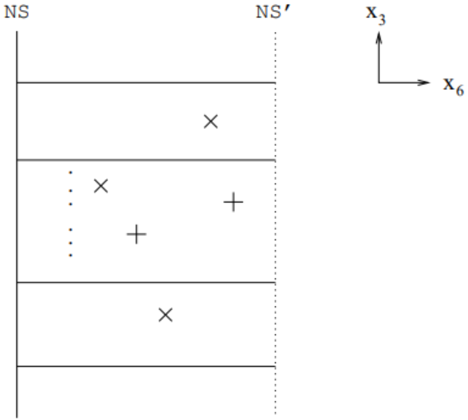

Figure 1: Three dimensional $N = 2$ supersymmetric gauge theories with gauge group $U(N_c)$ and N_f quarks. There are N_c D3-branes (horizontal lines), which are stretched in between two NS 5-branes (vertical lines). The figure is depicted in the 36 plane as indicated by the arrows in the upper right of the figure. The left 5-brane stretches along the 012345 directions and is denoted NS and the right 5-brane stretches along the 012389 directions and is denoted NS′. The "X"s denote D5-branes, and the "+"s denote D′ 5-branes, both of which give rise to quarks.

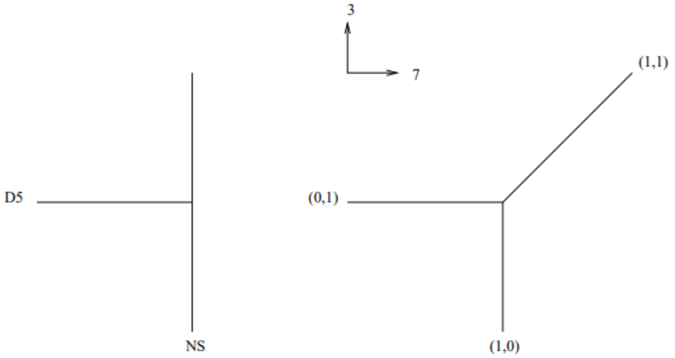

Figure 4: A D5-brane which ends on a NS 5-brane. The left side describes the naive configuration, and the right side the correct configuration, which implements conservation of charge at the vertex.

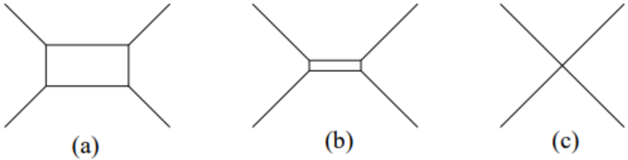

Figure 5: Pure $SU(2)$ gauge theory in five dimensions. Horizontal lines represent D5-branes, vertical lines represent NS 5-branes, and diagonal lines at an angle θ such that $\tan(\theta) = p/q$ represent (p, q) 5-branes. Figure (a) shows a generic point on the Coulomb branch, figure (b) shows a point near the origin of moduli space, and figure (c) corresponds to the strong coupling fixed point.

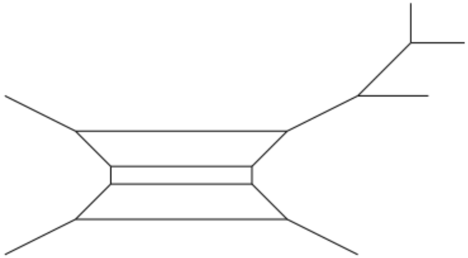

Figure 8: $SU(4)$ gauge theory with two (massive) flavors in five dimensions.

Webs of (p,q) 5-branes, Five Dimensional Field Theories and Grid Diagrams

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arXiv:hep-th/9710116v3 10 Nov 1997

Abstract

We continue to study 5d $N = 1$ supersymmetric field theories and their compactifications on a circle through brane configurations. We develop a model, which we call (p, q) Webs, which enables simple geometrical computations to reproduce the known results, and facilitates further study. The physical concepts of field theory are transparent in this picture, offering an interpretation for global symmetries, local symmetries, the effective (running) coupling, the Coulomb and Higgs branches, the monopole tensions, and the mass of BPS particles. A rule for the dimension of the Coulomb branch is found by introducing Grid Diagrams. Some known classifications of field theories are reproduced. In addition to the study of the vacuum manifold we develop methods to determine the BPS spectrum. Some states, such as quarks, correspond to instantons inside the 5-brane which we call strips. In general, these may not be identified with (p, q) strings. We describe how a strip can bend out of a 5-brane, becoming a string. A general BPS state corresponds to a Web of strings and strips. For special values of the string coupling a few strips can combine and leave the 5-brane as a string.

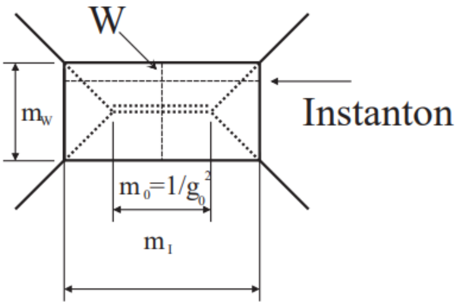


FIG. 2. The basic BPS states in the pure $SU(2)$ gauge theory – the W boson and the instanton.

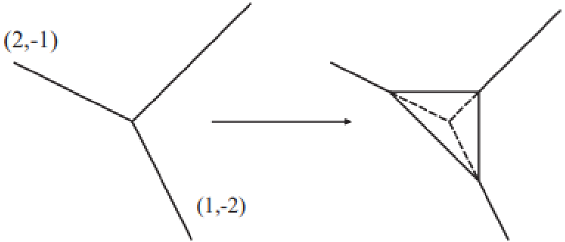


FIG. 4. A “hidden” face in the E_0 Web, realizing the 1d Coulomb branch.

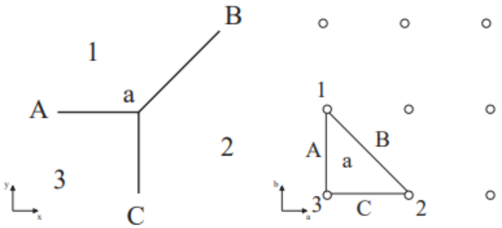


FIG. 5. The Grid diagram for the simple vertex of figure 1a. Vertices and corresponding polygons are marked a,b,c,..., edges and corresponding lines are marked A,B,C,..., and faces and corresponding points are marked $1, 2, 3, \dots$

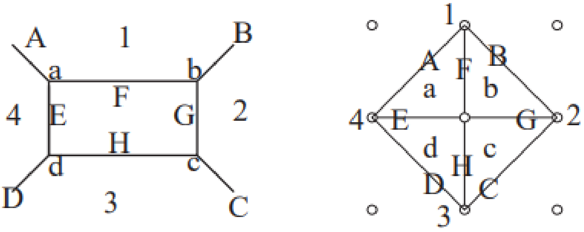


FIG. 6. The Grid diagram for the pure $SU(2)$ gauge theory of figure 1b.

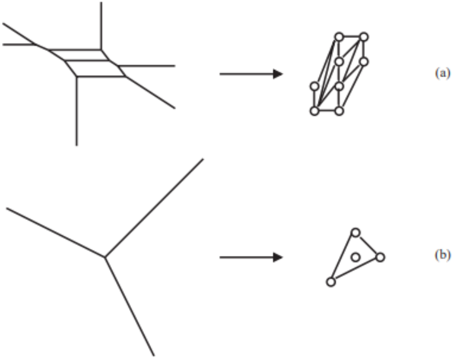


FIG. 7. Grid diagrams for (a) the $SU(3)$, $N_f = 2$ SQCD theory (figure 1c), and (b) the E_0

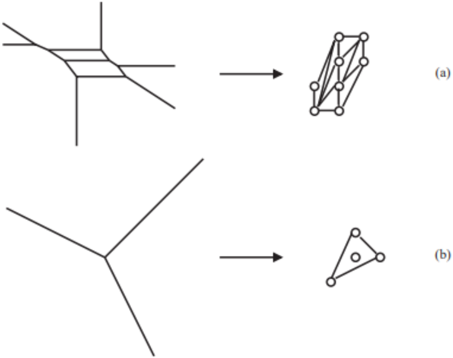


FIG. 7. Grid diagrams for (a) the $SU(3)$, $N_f = 2$ SQCD theory (figure 1c), and (b) the E_0

All these, mathematically, are examples of “dual subdivisions” of “Newton polygons” (appearing already in Viro’s work)

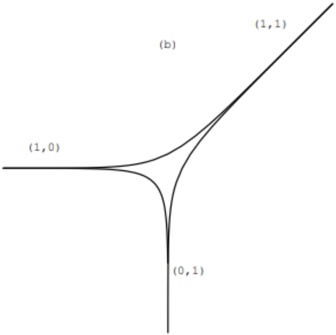
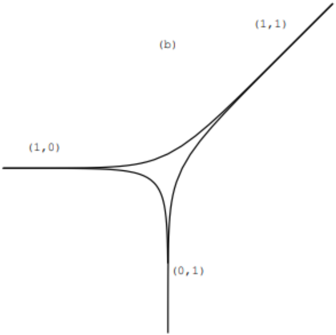


FIG. 8. A projection of the smooth curve for the simple vertex (figure 1a).

As we take the 5d limit $L_4 \to \infty$, the curve approaches the Web up to small corrections.



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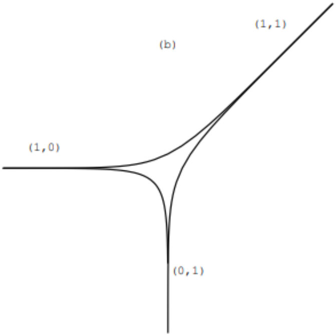

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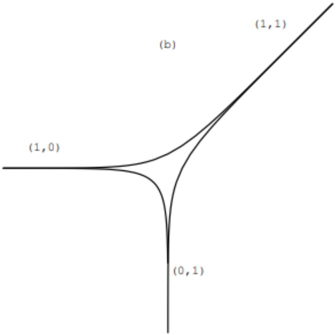

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When did tropical *geometry* emerge in mathematics?

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- B. Sturmfels, Solving systems of polynomial equations. No. 97. American Mathematical Soc., 2002.
(A very influential book on computational methods, the appearance of term "tropical algebraic geometry")

[math.AG] 31 Aug 2001

AMOEBAS OF ALGEBRAIC VARIETIES

GRIGORY MIKHALKIN

The notion of amoebas for algebraic varieties was introduced in 1994 by Gelfand, Kapranov and Zelevinski [7]. Some traces of amoebas were appearing from time to time, even before the formal introduction, as auxiliary tools in several problems (see e.g. [3]). After 1994 amoebas have been seen and studied in several areas of mathematics, from algebraic geometry and topology to complex analysis and combinatorics.

In particular, amoebas provided a very powerful tool for studying topology of algebraic varieties. The purpose of this survey is to summarize our current state of knowledge about amoebas and to outline their applications to real algebraic geometry and adjacent areas. Most proofs are omitted here. An expanded version of this survey is currently under preparation jointly with Oleg Viro [19].

Remark 4 (Non-Archimedean amoebas and Enumerative Geometry)**.** In a seminar talk in Paris, November 2000, Kontsevich noted a possibility of using non-Archimedean amoebas in enumerative geometry. As an example consider the problem of counting the number n_d of rational curves of degree d in $\mathbb{CP}^2$ which pass through $3d-1$ fixed generic points. A generic complex polynomial defines a curve of genus $(d-1)(d-2)/2$. The polynomials defining rational curves form a subset of codimension $(d-1)(d-2)/2$ and thus the rational curves form a $(3d-1)$-dimensional space (the space of curves has dimension one less than the dimension of the space of corresponding polynomials).

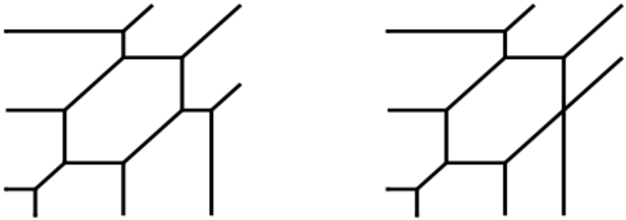


FIGURE 5. A smooth "non-Archimedian cubic amoeba" and a rational "non-Archimedian cubic amoeba".

My favorite introduction

For Proceedings of 21^{st} Gökova
Geometry-Topology Conference

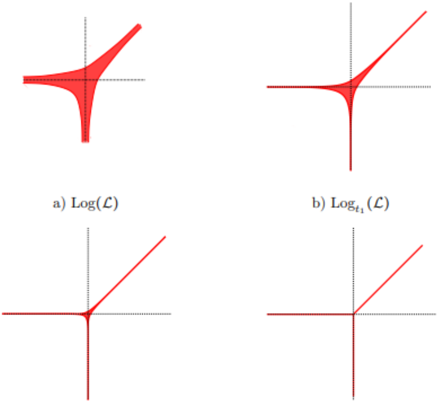
Brief introduction to tropical geometry

Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kristin Shaw

Abstract. The paper consists of lecture notes for a mini-course given by the authors at the Gökova Geometry & Topology conference in May 2014. We start the exposition with tropical curves in the plane and their applications to problems in classical enumerative geometry, and continue with a look at more general tropical varieties and their homology theories.

Tropicalization=Dequantization

Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Let $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ be given by $\mathrm{Log}(x, y) = (\log|x|, \log|y|)$. Let $\mathcal{L} = \{(x, y) \in (\mathbb{C}^*)^2 | x + y = 1\}$, then $\mathrm{Log}(\mathcal{L}) \subset \mathbb{R}^2$ is its **amoeba**.



a) $\mathrm{Log}(\mathcal{L})$ b) $\mathrm{Log}_{t_1}(\mathcal{L})$

c) $\mathrm{Log}_{t_2}(\mathcal{L})$ d) $\lim_{t\to\infty} \mathrm{Log}_t(\mathcal{L})$

FIGURE 6. Dequantization of a line ($e < t_1 < t_2$)

Tropicalization of planar curves

More generally, consider a family of complex curves $\mathcal{C}_t \subset (\mathbb{C}^*)^2$ given by $f_t = 0$, where f_t is a Laurent polynomial in two variables with coefficients analytically depending on t. Its tropicalization is the Hausdorff limit

$$\mathrm{Trop}(\mathcal{C}_t) = \lim_{t \to \infty} \mathrm{Log}_t(\mathcal{C}_t) \subset \mathbb{R}^2.$$

Such images are "planar tropical curves", i.e. graphs with straight edges of rational slope enhanced with **weights** satisfying the balancing condition at every vertex.

Balancing condition

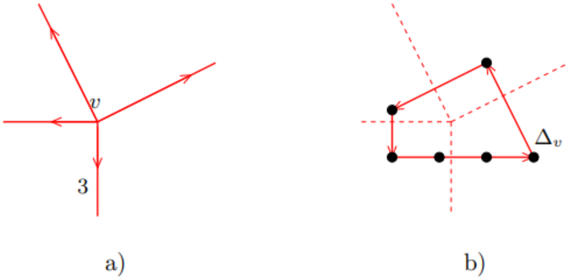


FIGURE 4. Balancing condition.

For every vertex, the weighted sum of outward primitive (with integer coprime coordinates) directions of adjacent edges is zero.

How to recover weights?

Let the defining Laurent polynomial f_t be written as a finite sum $\sum a_{m,n}(t)x^m y^n$, where $a_{m,n}(t) = c_{m,n}t^{\beta_{m,n}} + o(t^{\beta_{m,n}})$ as $t \to +\infty$. Denote by $\mathrm{Trop}(f_t) : \mathbb{R}^2 \to \mathbb{R}$ a function given by

$$(X, Y) \mapsto \max(\beta_{m,n} + mX + nY).$$

Then, the set of points on $\mathbb{R}^2$ where $\mathrm{Trop}(f_t)$ breaks (i.e. is not linear) is precisely $\mathrm{Trop}(\mathcal{C}_t)$. Every edge E of $\mathrm{Trop}(\mathcal{C}_t)$ separates to regions of linearity of $\mathrm{Trop}(f_t)$ on which the gradients are integer vectors (m_1, n_1) and (m_2, n_2). Then, the weight of E is the greatest common divisor of $m_1 - m_2$ and $n_1 - n_2$.

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$\mathrm{Trop}(f_t)$ **is called the tropical polynomial defining** $\mathrm{Trop}(\mathcal{C}_t)$.

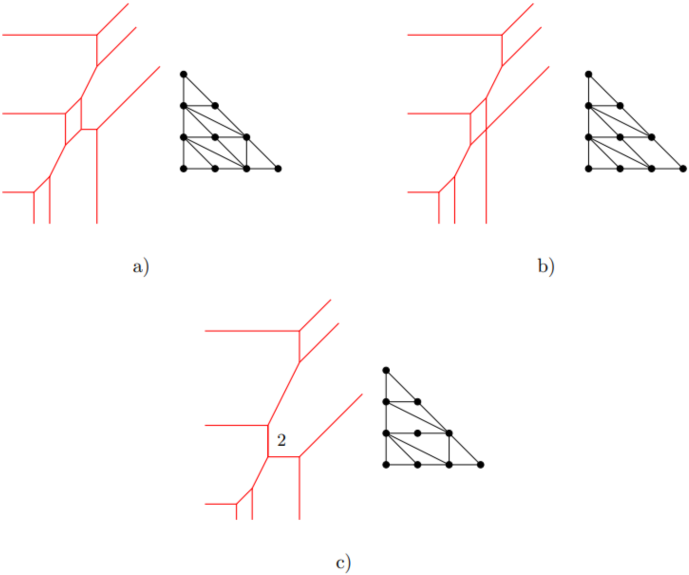

FIGURE 5. Some tropical cubics and their dual subdivisions

Min or Max?

The expression $\max(\beta_{m,n} + mX + nY) =$ "$(\sum \beta_{m,n} X^m Y^n)$" is indeed a two-variable tropical polynomial in X and Y: in tropical arithmetic one replaces the addition with taking the maximum and multiplication with the usual addition. If one takes the limit $t \to 0$, instead of $t \to \infty$, then, instead of the maximum, the minimum serves as an operation of tropical addition.

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The Min-convention will be adapted later on since it is slightly more convenient while working with self-organized criticality and tropical optics.

Chapter 2

Self-organized criticality

SOC started with...

Self-organized criticality: An explanation of the 1/*f* noise

P Bak, C Tang, K Wiesenfeld

Physical review letters, 1987 • APS

Abstract

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or 1/f noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

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Bak-Tang-Wiesenfeld model: topplings and relaxation

Consider a relatively large region of the square lattice $\Gamma \subset \mathbb{Z}^2$ (for example, take $\Gamma = \{1, 2, \ldots, 50\}^2$). A state of the system is a function ϕ from Γ to non-negative integers. The state ϕ is called stable if all of its values are in $\{0, 1, 2, 3\}$.

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Bak-Tang-Wiesenfeld model: the process

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Observation 1.

Performing measurements at each step, one observes power laws, similar to the behavior near a phase-transition but without the fine-tuning. This suggests a possibility for scaling limits.

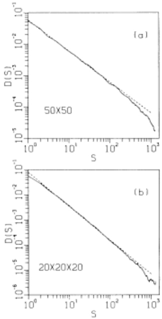

FIG. 2. Distribution of cluster sizes at criticality in two and three dimensions, computed dynamically as described in the text. (a) 50×50 array, averaged over 200 samples; (b) $20\times 20\times 20$ array, averaged over 200 samples. The data have been coarse grained.

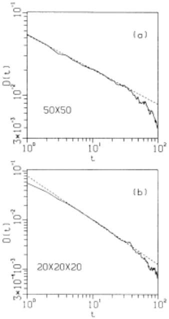

FIG. 3. Distribution of lifetimes corresponding to Fig. 2. (a) For the 50×50 array, the slope $\alpha\approx 0.42$, yielding a "$1/f$" noise spectrum $f^{-1.58}$; (b) $20\times 20\times 20$ array, $\alpha\approx 0.90$, yielding an $f^{-1.1}$ spectrum

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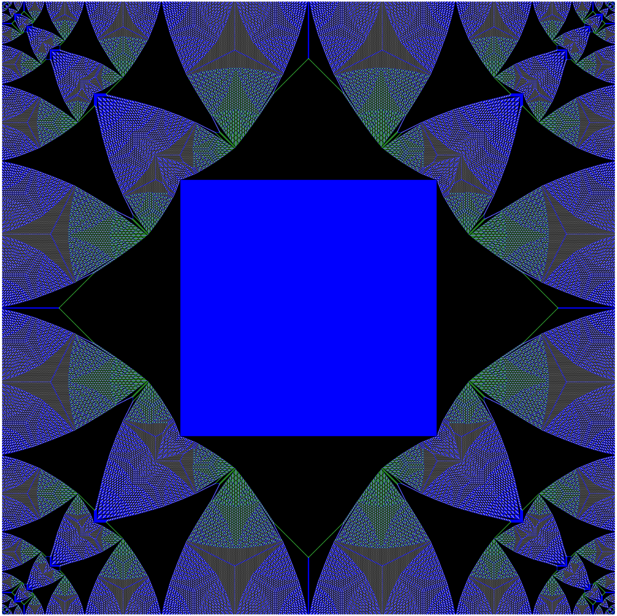
The neutral element of $G(\Gamma)$ is very beatiful and mysterious...

Identity element on 1000x1000 square



Strings in sandpiles – where our journey begins!

Conservation laws for strings in the Abelian Sandpile Model

Sergio Caracciolo,[1] Guglielmo Paoletti,[2] and Andrea Sportiello
[1] *Università degli Studi di Milano – Dipartimento di Fisica and INFN, via G. Celoria 16, 20133 Milano, Italy*
[2] *Università di Pisa – Dipartimento di Fisica and INFN, largo B. Pontecorvo 3, 56127 Pisa, Italy*
(Dated: November 7, 2018)

The Abelian Sandpile generates complex and beautiful patterns and seems to display allometry. On the plane, beyond *patches*, patterns periodic in both dimensions, we remark the presence of structures periodic in one dimension, that we call *strings*. We classify completely their constituents in terms of their principal periodic vector $\boldsymbol{k}$, that we call *momentum*. We derive a simple relation between the momentum of a string and its density of particles, E, which is reminiscent of a dispersion relation, $E = |\boldsymbol{k}|^2$. Strings interact: they can merge and split and within these processes momentum is conserved, $\sum_a \boldsymbol{k}_a = \boldsymbol{0}$. We reveal the role of the modular group $\mathrm{SL}(2, \mathbb{Z})$ behind these laws.

PACS numbers: 05.65.+b, 45.70.Qj, 89.75.Fb
Keywords: Sandpile Models, Lattice Automata, Pattern formation, Modular Invariance

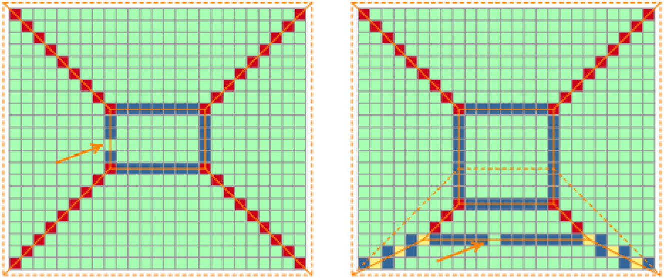


Figure 2. On the left, the configuration obtained after relaxation from $z_{\max}$ plus an extra grain of sand exactly at the vertex where a defect appears. On the right, the result after removing the defect and the addition of one more grain.

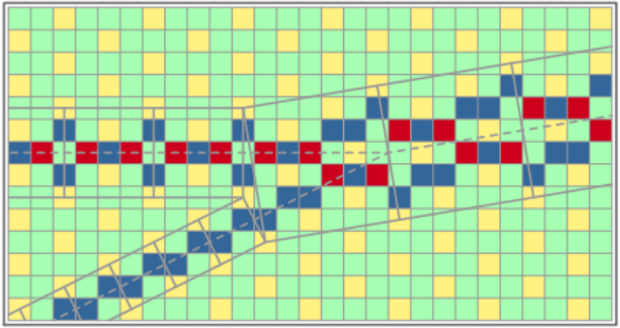


Figure 3. A scattering involving pseudo-propagators with momenta $(4, 0)$, $(2, 1)$ and $(6, 1)$, on the background pattern of Fig. 1 (also symbol code is as in Fig. 1).

Strings/solitons exist and are unique!

Sandpile solitons via smoothing of superharmonic functions

N Kalinin, M Shkolnikov

Communications in Mathematical Physics, 2020 • Springer

Abstract

Let $F:\mathbb{Z}^2 \to \mathbb{Z}$ be the pointwise minimum of several linear functions. The theory of *smoothing* allows us to prove that under certain conditions there exists the pointwise minimal function among all integer-valued superharmonic functions coinciding with F "at infinity". We develop such a theory to prove existence of so-called *solitons* (or strings) in a sandpile model, studied by S. Caracciolo, G. Paoletti, and A. Sportiello. Thus we made a step towards understanding the phenomena of the identity in the sandpile group for planar domains where solitons appear according to experiments. We prove that sandpile states, defined using our smoothing procedure, move changeless when we apply the wave operator (that is why we call them solitons), and can interact, forming *triads* and *nodes*.

Tropical scaling limit for sandpiles

[math.CO] 12 Dec 2023

TROPICAL CURVES IN SANDPILE MODELS

NIKITA KALININ, MIKHAIL SHKOLNIKOV

ABSTRACT. A sandpile is a cellular automaton on a graph that evolves by the following toppling rule: if the number of grains at a vertex is at least its valency, then this vertex sends one grain to each of its neighbors.

In the study of pattern formation in sandpiles on large subgraphs of the standard square lattice, S. Caracciolo, G. Paoletti, and A. Sportiello experimentally observed that the result of the relaxation of a small perturbation of the maximal stable state contains a clear visible thin balanced graph formed by its deviation (less than maximum) set. Such graphs are known as tropical curves.

During the early stage of our research, we have noticed that these tropical curves are approximately scale-invariant, that is the deviation set mimics an extremal tropical curve depending on the domain on the plane and the positions of the perturbation points, but not on the mesh of the lattice.

In this paper, we rigorously formulate these two facts in the form of a scaling limit theorem and prove it. We rely on the theory of tropical analytic series, which is used to describe the global features of the sandpile dynamic, and on the theory of smoothings of discrete superharmonic functions, which handles local questions.

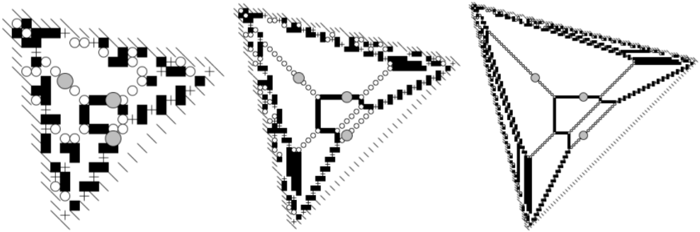


FIGURE 1. A thin balanced graph appears as a deviation set of a sandpile. See Example 2.11 for details. White corresponds to three grains, black to one, circles to two, crosses to zero, and skew lines are the boundary vertices (sinks). Grey rounds represent the positions of added grains.

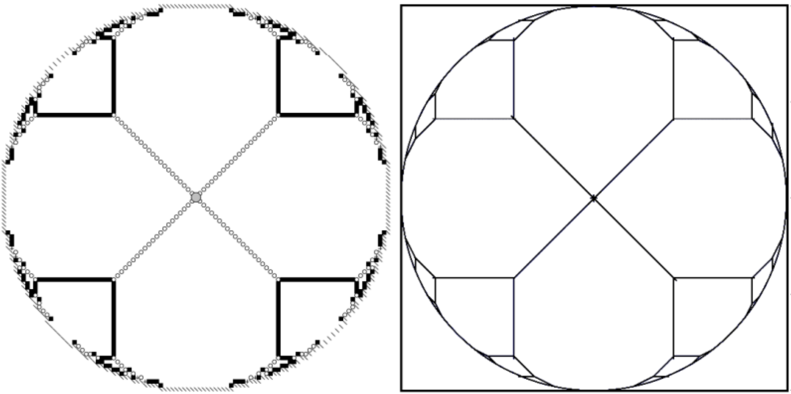
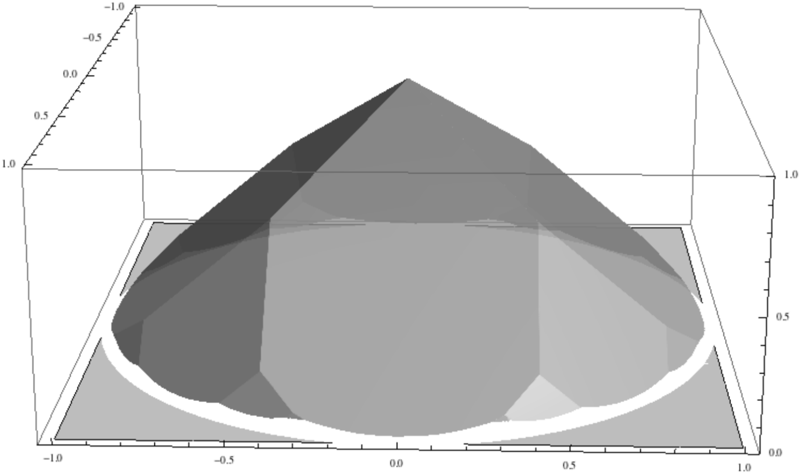


FIGURE 9. Left: the result of adding a single grain at the center of a disk to the maximal stable state and relaxing. Right: tropical caustic of the disk.

Tropical distance series of a disk

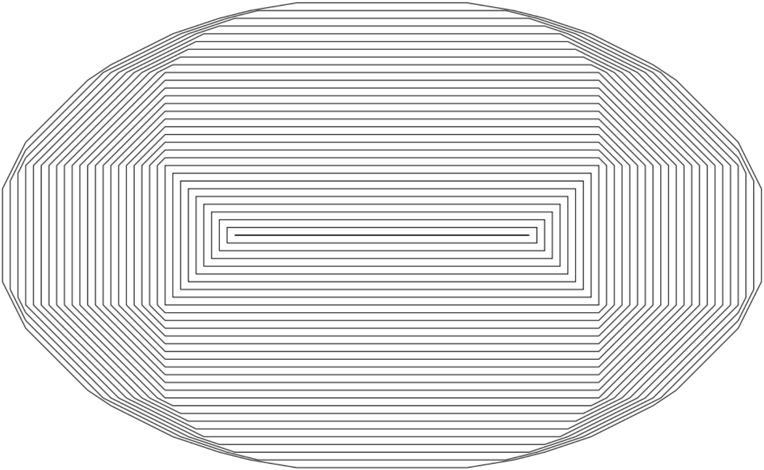


$$\mathcal{F}_{Disk}(x, y) = \inf_{(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}}(\sqrt{p^2+q^2} + px + qy)$$

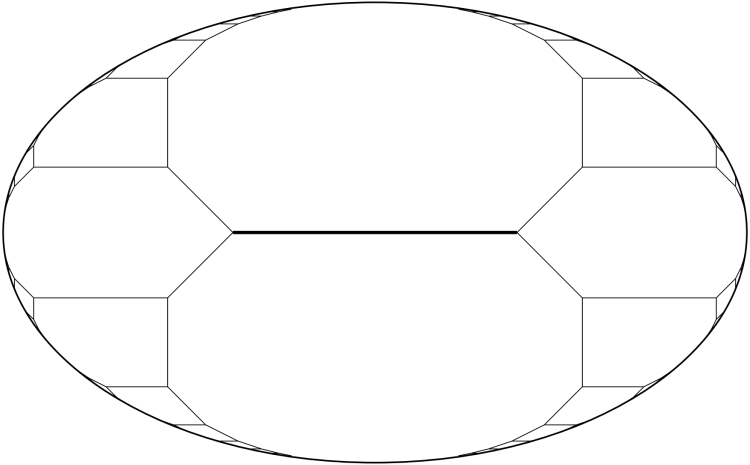
Chapter 3

Tropical optics

Tropical wave front of an ellipse



Tropical caustic of an ellipse



The weight of the horizontal edge in the middle is 2, all other weights are 1. All vertices are trivalent – i.e. the caustic is an infinite binary tree inscribed in the ellipse.

Formal definitions

Let $\Omega \subset \mathbb{R}^n$ be a convex domain. For $\lambda \in \mathbb{R}^n$ define $\sigma_\lambda(\Omega)$ as $-\inf_{p\in\Omega}(\lambda, p)$. The tropical distance series of Ω is given by

$$\mathcal{F}_\Omega(p) = \inf_{\lambda\in\mathbb{Z}^n\setminus\{0\}} (\sigma_\lambda(\Omega) + (\lambda, p)).$$

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Tropical Huygens' principle: $(\Omega(t))(s) = \Omega(t+s)$.

A place to read about tropical wave fronts and caustics

Proceedings of 28^{th} Gökova
Geometry-Topology Conference
pp. 11 – 48

Wave fronts and caustics in the tropical plane

Grigory Mikhalkin, Mikhail Shkolnikov

Abstract. The paper studies intrinsic geometry in the tropical plane. Tropical structure in the real affine n-space is determined by the integer tangent vectors. Tropical isomorphisms are affine transformations preserving the integer lattice of the tangent space, they may be identified with the group $\mathrm{GL_n}(\mathbb{Z})$ extended by arbitrary real translations. This geometric structure allows one to define *wave front propagation* for boundaries of convex domains. Interestingly enough, an arbitrary compact convex domain in the tropical plane evolves to a finite polygon after an arbitrarily small time. The caustic of a wave front evolution is a tropical analytic curve. The paper studies geometry of the tropical wave fronts and caustics. In particular, we relate the caustic of a tropical angle to the continued fraction expression of its slope, and treat it as a tropical trigonometry notion.

Affine distance function

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A technical realization: for a matrix $A \in Sl_n(\mathbb{R})$ and a convex domain $\Omega \subset \mathbb{R}^n$ define an A-transformed tropical series $\mathcal{F}_\Omega^A$ as $\mathcal{F}_\Omega^A(p) = \mathcal{F}_{A(\Omega)}(A(p))$.

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Define the affine distance function of Ω as

$$\mathcal{F}_\Omega^{aff}(p) = (Vol(Sl_n(\mathbb{Z})\backslash Sl_n(\mathbb{R})))^{-1} \int_{[A]\in Sl_n(\mathbb{Z})\backslash Sl_n(\mathbb{R})} \mathcal{F}_\Omega^A(p)d[A].$$

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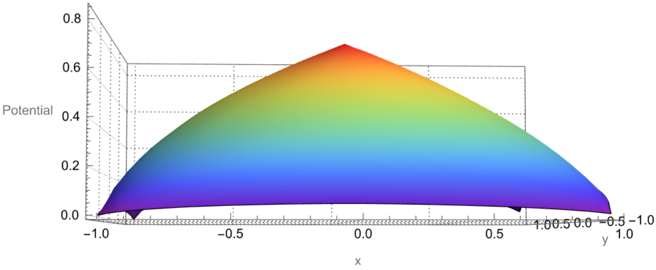
This new function is automatically affine invariant:
$\mathcal{F}_\Omega^{aff}(p) = \mathcal{F}_{A(\Omega)}^{aff}(A(p))$ for any $A \in Sl_n(\mathbb{R})$.

3d plot of affine distance for a square (E. Lupercio)



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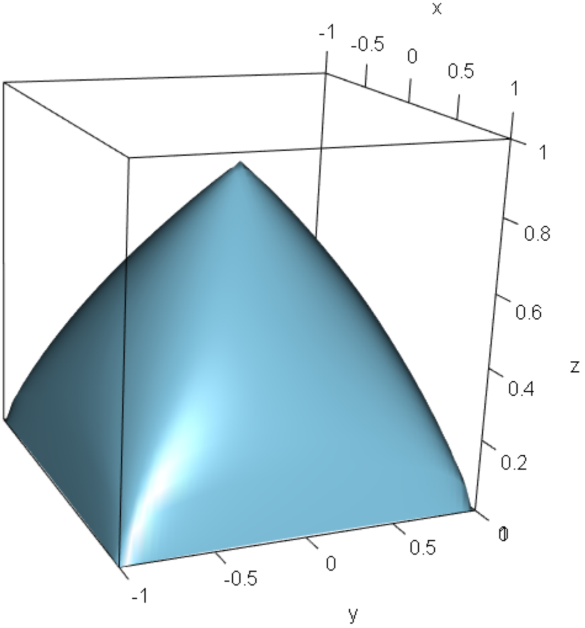
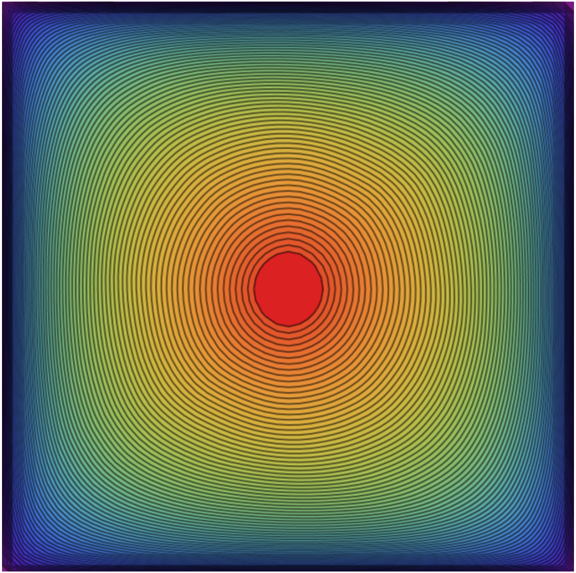


Contour plot of affine distance for a square (E. Lupercio)



Some conjectures for convex domains

Conjecture 1a:
The maximal value of $\mathcal{F}_{\Omega}^{aff}$ is attained at a single point.

Conjecture 1b:
The maximal value of $\mathcal{F}_{\Omega}^{aff}$ is attained at the center of mass of Ω.

Some conjectures for convex domains

Conjecture 2:

Assume that the maximal value M_Ω of $\mathcal{F}_\Omega^{aff}$ is attained at the origin. Then the following limit exists and is equal to an ellipsoid:

$$\lim_{t \to_- M_\Omega} (Vol((\mathcal{F}_\Omega^{aff})^{-1}[t, M_\Omega]))^{-(dim\Omega)^{-1}}((\mathcal{F}_\Omega^{aff})^{-1}(t)).$$

Some conjectures for convex domains

Recall that a polar set of a domain $\Omega \subset R^n$ is
$\Omega^\bullet = \{y \in \mathbb{R}^n : (y, x) \leq 1, \forall x \in \Omega\}$.
The Mahler volume of Ω is $Vol(\Omega)Vol(\Omega^\bullet)$.

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Conjecture 3:
The Mahler volume of $(\mathcal{F}_\Omega^{aff})^{-1}[t, M_\Omega]$ is increasing with t.

Thank you for your attention!