Precision M5-brane holography

Vasil Dimitrov

May 31, 2023

1 Statement of the AdS/CFT conjecture

The AdS/CFT conjecture proposes a duality between a quantum gravity theory on asymptotically AdS_d spacetime times a compact space and a $(d-1)$-dimensional conformal field theory on. Here we focus on an $\mathrm{AdS}_5/\mathrm{CFT}_4$, example coming from M-theory. Practically, AdS/CFT is the statement that the M-theory and the CFT partition functions are the same

$$\mathcal{Z}_{\mathrm{M}}[\varphi = \phi(r \to \infty)] \text{ on } \mathrm{AdS}_5 \times \mathcal{M}_6 = \mathcal{Z}_{\mathrm{CFT}}[\varphi] \text{ on } \mathbb{R} \times S^3\,, \tag{1}$$

where $\mathcal{M}_6$ is a compact manifold and φ is a set of refinement fugacities for the CFT partition function, that can be seen as boundary conditions for the bulk fields ϕ as they approach the AdS boundary (at radial coordinate $r \to \infty$).

Summary 1

(1) A theory without gravity in 4 dimensions is equivalent to a theory with gravity in 11 dimensions: holographic principle

(2) The partition function carries all the dynamical information of the two theories (can compute all correlation functions)

(3) In M-theory there is one parameter N — the “number” of M-branes. On the gravity side the branes backreacts to form the $\mathrm{AdS}_5 \times \mathcal{M}_6$ geometry. The dual field theory lives on the worldvolume of the branes.

(4) At arbitrary N both $\mathcal{Z}_{\mathrm{CFT}}$ and $\mathcal{Z}_{\mathrm{M}}$ are very hard to compute (computing the path integral is generically hard)

(5) At $N \to \infty$ the field theory is strongly coupled but there are tricks to obtain the large-N scaling of $\mathcal{Z}_{\mathrm{CFT}}$. At $N \to \infty$, the gravity side is easy to compute: $\mathcal{Z}_{\mathrm{M}} \approx e^{-I_{\text{on-shell}}}$, where $I_{\text{on-shell}}$ is the Euclidean on-shell action of the dominant saddle

(6) Our aim: match $\mathcal{Z}_{\mathrm{M}}$ and $\mathcal{Z}_{\mathrm{CFT}}$ beyond leading order in large-N

2 M5-brane holography

The concrete setup is N M5-branes, wrapping a Riemann surface $\Sigma_{\mathfrak{g}}$, which resides in a Calabi-Yau three fold ($\mathcal{M}_6 = \mathrm{CY}_3$). Locally one can view CY_3 as a fibration of two line bundles over $\Sigma_{\mathfrak{g}}$

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \to \Sigma_{\mathfrak{g}}\,. \tag{2}$$

Aside from N in the UV there are two other dimensionless parameters: the first Chern numbers of the two line bundles. We package this data in terms of the genus $\mathfrak{g}$ and a rational number z

$$c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) = 2\mathfrak{g} - 2\,, \quad \frac{c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2)}{2\mathfrak{g} - 2} = z\,. \tag{3}$$

We have an infinite collection of AdS/CFT duals for every value of $(\mathfrak{g}, z)$ [1, 2]. Concretely

$$\text{M-theory on } \mathrm{AdS}_5 \times \Sigma_{\mathfrak{g}} \times S^4 \iff \mathcal{N} = 1 \text{ SCFT of class } \mathcal{S} \text{ on } \mathbb{R} \times S^3 \tag{4}$$

3 Partition function of a 4d $\mathcal{N}=1$ SCFT

Consider a generic 4d $\mathcal{N}=1$ SCFT with $\mathrm{U}(1)_R$ R-symmetry. We put the theory on the following $\mathbb{R}\times S^3$ background

$$\begin{aligned} \mathrm{d}s^2 &= -\,\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\phi_1 + \Omega_1\,\mathrm{d}t)^2 + \cos^2\theta(\mathrm{d}\phi_2 + \Omega_2\,\mathrm{d}t)^2\,, \\ A &= -\Phi\,\mathrm{d}t\ , \end{aligned} \tag{5}$$

where A is the background $\mathrm{U}(1)_R$ field, and (Ω_1,Ω_2,Φ) are a set of fixed background chemical potentials. Initially, consider the theory at temperature $T=1/\beta$. The partition function is

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H + \beta\Omega_1 J_1 + \beta\Omega_2 J_2 + \beta\Phi R}\,, \tag{6}$$

where H is the Hamiltonian, $J_{1,2}$ are a set of charges associated with the S^3 and R is the $\mathrm{U}(1)_R$ charge, considered as operators on the Hilbert space. There is a complex supercharge satisfying

$$\left\{\mathcal{Q},\mathcal{Q}^\dagger\right\} = H - J_1 - J_2 - \frac{3}{2}R\,. \tag{7}$$

It is well known that

$$\text{QFT at finite temperature } T \longleftrightarrow \text{QFT on a thermal circle } S^1_\beta,\ \beta = 1/T\,, \tag{8}$$

thus we Wick rotate the background

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\phi_1 - \mathrm{i}\Omega_1\,\mathrm{d}\tau)^2 + \cos^2\theta(\mathrm{d}\phi_2 - \mathrm{i}\Omega_2\,\mathrm{d}\tau)^2\,, \\ A &= \mathrm{i}\Phi\,\mathrm{d}\tau\ , \end{aligned} \tag{9}$$

where the background is now $S^1_\beta \times S^3$ with thermal boundary conditions

$$(\tau,\phi_1,\phi_2) \sim (\tau+\beta,\phi_1,\phi_2)\,. \tag{10}$$

Supersymmetry (or the existence of a Killing spinor on this background) dictates [7]

$$\beta(1+\Omega_1+\Omega_2-2\Phi) = 2\pi\mathrm{i}n\,, \quad n\in\mathbb{Z}\,. \tag{11}$$

Re-defining the chemical potentials as

$$\omega_1 = \beta(\Omega_1-1)\,, \quad \omega_2 = \beta(\Omega_2-1)\,, \quad \varphi = \beta\left(\Phi - \frac{3}{2}\right), \tag{12}$$

and using (7) and (11) we rewrite the partition function as

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta\left\{\mathcal{Q},\mathcal{Q}^\dagger\right\} + \omega_1\left(J_1+\frac{R}{2}\right) + \omega_2\left(J_2+\frac{R}{2}\right)} e^{-\pi\mathrm{i}nR}\,. \tag{13}$$

We use the charge assignment of the operators to see that for odd n: $e^{\pi\mathrm{i}nR} = (-1)^F$, where $F=\pm1$ is the fermion number of the state. Thus, for odd n, the partition function is

$$\mathcal{Z} = \mathrm{Tr}\,(-1)^F e^{-\beta\left(\mathcal{Q},\mathcal{Q}^\dagger\right) + \omega_1\left(J_1+\frac{R}{2}\right) + \omega_2\left(J_2+\frac{R}{2}\right)}\,. \tag{14}$$

The Hilbert space is naturally split into

$$\begin{aligned} \text{BPS states:}\quad & \mathcal{Q}\,|\psi\rangle = 0,\ \text{with energy } E_0\,, \\ \text{excited states:}\quad & \mathcal{Q}\,|\chi\rangle \neq 0,\ \text{boson-fermion paired states with the same energy } E > E_0\,. \end{aligned} \tag{15}$$

Due to the factor $(-1)^F$ the excited states cancel each other out and the above partition function only counts the degeneracy of the ground state, for which $\mathcal{Q}=0$. Thus, the partition function can be written as [4, 3, 7]

$$\mathcal{Z} = e^{-\mathcal{F}}\mathcal{I}\,, \quad \mathcal{I} = \mathrm{Tr}|_{\mathcal{Q}=0}\,(-1)^F e^{\omega_1\left(J_1+\frac{R}{2}\right) + \omega_2\left(J_2+\frac{R}{2}\right)}\,, \tag{16}$$

where $\mathcal{F}$ is related to the so called *supersymmetric Casimir energy* (essentially captures the energy of the ground state) and $\mathcal{I}$ is the so called *superconformal index*. The point of this rewriting is that in the Cardy limit ($\beta \ll l_3$, where l_3 is the size of the S^3) $\mathcal{F}$ and $\mathcal{I}$ are entirely controlled by anomalies (symmetries) and can be deduced for any 4d $\mathcal{N}=1$ SCFT [8]

$$-\log\mathcal{Z} = \mathcal{F} - \log\mathcal{I} = \frac{\varphi^3}{6\,\omega_1\omega_2}\mathrm{Tr}\,R^3 - \frac{\varphi(\omega_1^2+\omega_2^2-4\pi^2)}{24\,\omega_1\omega_2}\mathrm{Tr}\,R + O\!\left(e^{-l_3/\beta}\right), \tag{17}$$

where $\operatorname{Tr} R^3$ and $\operatorname{Tr} R$ are the cubic and linear 't Hooft anomalies which are related to the $\mathfrak{a}$ and $\mathfrak{c}$ conformal anomalies as

$$\operatorname{Tr} R^3 = \frac{16}{9}(5\mathfrak{a} - 3\mathfrak{c}) \,, \quad \operatorname{Tr} R = 16(\mathfrak{a} - \mathfrak{c}) \,. \tag{18}$$

For the concrete SU(N) class $\mathcal{S}$ 4d $\mathcal{N} = 1$ theory arising from N M5-banes wrapping a Riemann surface of genus $\mathfrak{g}$ inside a Calabi-Yau one has [2]

$$\begin{aligned} \operatorname{Tr} R^3 &= \frac{2(\mathfrak{g}-1)}{27z^2}\Big[9z^2 - 1 + \big(3z^2+1\big)^{3/2}\Big] N^3 \\ &\quad - \frac{\mathfrak{g}-1}{9z^2}\Big[\Big(\sqrt{3z^2+1} - 1\Big)(2+3z^2) - 3z^2\Big] N + \dots \,, \\ \operatorname{Tr} R &= \frac{\mathfrak{g}-1}{3}\Big[4 - \sqrt{3z^2+1}\Big] N + \dots \,. \end{aligned} \tag{19}$$

Summary 2

(1) Overall for the theory labelled by $(\mathfrak{g}, z)$ the logarithm of the partition function at large-N scales as

$$\mathcal{Z}_{\mathfrak{g},z} = f(\omega_1, \omega_2, \varphi; \mathfrak{g}, z) N^3 + g(\omega_1, \omega_2, \varphi; \mathfrak{g}, z) N + \dots \,, \tag{20}$$

where f and g are known function of the chemical potentials $(\omega_1, \omega_2, \varphi)$ given for chosen values of $(\mathfrak{g}, z)$

(2) The chemical potentials $(\omega_1, \omega_2, \varphi)$ are defined such that they remain finite in the simultaneous limit to zero temperature: $\beta \to \infty$, and supersymmetry: $\Omega_{1,2} \to 1$, $\Phi \to 3/2$

(3) The supersymmetry condition (11) written in terms of $(\omega_1, \omega_2, \varphi)$

$$\omega_1 + \omega_2 - 2\varphi = 2\pi \mathrm{i} n \tag{21}$$

dictates that for any $n \neq 0$ the chemical potentials are complex

(4) The real and imaginary parts of $\omega_{1,2}$ have the following interpretation

$$\omega_1 = \frac{\beta}{l_3} b + \mathrm{i} \operatorname{Im} \omega_1 \,, \quad \omega_2 = \frac{\beta}{l_3} b^{-1} + \mathrm{i} \operatorname{Im} \omega_2 \,, \tag{22}$$

where b is the squashing of the S^3 and the imaginary parts encode rotation parameters due to twisting S^3 over S^1_β

(5) In the supersymmetric limit the partition function is explicitly independent of β and is written entirely in terms of the suitable chemical potentials $(\omega_1, \omega_2, \varphi)$. This makes the $\beta \to \infty$ limit trivial to take

(6) The final scaling of the partition function with N is obtained in the Cardy limit where size$(S^1_\beta) \ll$ size(S^3) or $\beta/l_3 \to 0$. Apriory this is distinct from evaluating $\log \mathcal{Z}$ in the explicit large-N limit

(7) The benefit of the Cardy limit is that the final N^3 and N terms are entirely controlled by anomalies and hold for any 4d $\mathcal{N} = 1$ SCFT, even for non-Lagrangian theories such as class $\mathcal{S}$, where explicit localization calculations are not possible

4 Higher derivative corrections and the gravity partition function

Unfortunately, we lack fundamental understanding of the M-theory partition function. However, it is known that in the low-energy limit, M-theory reduces to the unique 11d supergravity. Compactified on $\mathcal{M}_6$ this gives an effective 5d supergravity theory with Euclidean Lagrangian schematically given by

$$e^{-1}\mathcal{L}^{(0)}(g_{\mu\nu}, \phi) = \frac{1}{16\pi G_5}\left(R + \frac{12}{L^2}\right) + e^{-1}\mathcal{L}^{(0)}_{\text{matter}}(\phi) \,, \tag{23}$$

where G_5 is the 5d Newton constant and L is the AdS$_5$ scale (and, since we are dealing with gauged supergravity, thus no scale separation) also the $\mathcal{M}_6$ scale. In particular ϕ collectively denotes the massless matter fields and the infinite tower of massive KK modes. The full partition function is given by

$$Z_{\mathrm{M}}[\varphi] = \int [Dg_{\mu\nu} D\phi] e^{-\int \mathrm{d}^5 x \mathcal{L}(g_{\mu\nu},\phi)} \,, \tag{24}$$

where φ collectively denotes a set of fixed boundary conditions for the bulk fields and $\mathcal{L}$ above includes an infinite series of higher derivative corrections to $\mathcal{L}^{(0)}$. Using the saddle point approximation we write the above path integral as

$$\mathcal{Z}_{\mathrm{M}}[\varphi] = \sum_{\alpha\,\in\,\text{saddles}} e^{-I_\alpha[\varphi]} \mathcal{Z}^{\alpha}_{\text{loops}}[\varphi] \,, \tag{25}$$

where $I_\alpha[\varphi]$ is the collection of Euclidean on-shell actions of all the gravitational saddles (solutions to the Euclidean equations of motion) that have fixed boundary conditions φ for the bulk fields and $\mathcal{Z}^{\alpha}_{\text{loops}}$ is the contribution from all loops of all fields around the classical saddles. Keeping track of the contributions from

(i) all higher derivative corrections

(ii) all loops

(iii) all KK modes

(iv) all saddles

is a tall order. However, at large-N it turns out that the leading behavior is controlled by the dominant saddle (the Wick rotation of a large AdS_5 black hole) at 2∂ level and the subleading behavior is controlled by 4∂ corrections on top of this saddle. To obtain the 2∂ one needs to find the supergravity on-shell action $I^{(0)}_{\text{dominant}}$. Generically

$$-\log \mathcal{Z}^{(0)}_{\mathrm{M}}[\varphi] = f(\varphi)\frac{L^3}{G_5} + \dots \,, \tag{26}$$

where L^3/G_5 is a dimensionless combination of the two scales in the problem and $f(\varphi)$ is a calculable function of the thermodynamic black hole potentials. Here the $\dots$ denote subleading contributions coming from loops, KK modes and subdominant saddles. We include 4∂ corrections to the theory (think Wilsonian effective theory) as

$$e^{-1}\mathcal{L}^{(1)}(g_{\mu\nu},\phi) = \frac{1}{16\pi G^{(1)}_5}\left(R + \frac{12}{L^2_{(1)}}\right) + c^{(1)}_1 R^2 + c^{(2)}_2 R^2_{\mu\nu} + c^{(1)}_3 R^2_{\mu\nu\rho\sigma} + e^{-1}\mathcal{L}_{\text{rest}}(g_{\mu\nu},\phi) \,, \tag{27}$$

where the coefficients $c^{(1)}_i$ are small in the sense

$$Lc^{(1)}_i \ll \frac{L^3}{G_5} \,, \tag{28}$$

the bare couplings G_5 and L get renormalized to $G^{(1)}_5$ and $L_{(1)}$ with corrections of order $Lc^{(1)}_i$ and $\mathcal{L}_{\text{rest}}$ includes all other non-pure-gravity 4∂ terms made up from the fields in the theory (possibly introducing more couplings to the list $c^{(1)}_i$). Evaluating the on-shell action of the dominant saddle in the 4∂ theory one obtains (schematically)

$$-\log \mathcal{Z}^{(1)}_{\mathrm{M}}[\varphi] = f(\varphi)\frac{L^3_{(1)}}{G^{(1)}_5} + h^i(\varphi)Lc^{(1)}_i + \dots = f(\varphi)\frac{L^3}{G_5} + \widetilde{h}^i(\varphi)Lc^{(1)}_i + \dots \,, \tag{29}$$

where we have expanded the combination $L^3_{(1)}/G^{(1)}_5$ to order $Lc^{(1)}_i$ and we have assumed that the $Lc^{(1)}_i$ effects are parametrically larger than the KK modes, the loops and the contributions form the other saddles. As the M5-brane setup discussed above contains a single large dimensionless parameter N (for fixed $\mathfrak{g}$ and z) we expect that

$$\frac{L^3}{G_5} = u(\mathfrak{g},z)N^3 \,, \quad Lc^{(1)}_i = v_i(\mathfrak{g},z)N \,. \tag{30}$$

Below we show that this is indeed the case and that the functions $u(\mathfrak{g},z)$ and $v_i(\mathfrak{g},z)$ together with the functions $f(\varphi)$ and $\widetilde{h}^i(\varphi)$ precisely agree with the field theory prediction (think of φ collectively denoting $(\omega_1,\omega_2,\varphi)$ from the previous section).

5 The CCLP saddle and matching the leading terms

5d $\mathcal{N}=2$ minimal supergravity with 2∂ action

$$I=\frac{1}{16\pi G_5}\int\left[\star\left(R+\frac{12}{L^2}\right)-\frac{1}{2}F\wedge\star F+\frac{1}{3\sqrt{3}}A\wedge F\wedge F\right] \tag{31}$$

contains a universal black hole solution that is $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathbb{R}$ isometric and is specified by four real parameters (a,b,m,q) [11]. Wick rotating the solution as $t=-i\tau$ and demanding regularity in the deep IR (when the $r\to r_+$, with r_+ being the outer horizon) we see that

$$(\tau,\xi_1,\xi_2)\sim(\tau+\beta,\xi_1-\mathrm{i}\beta\Omega_1,\xi_2-\mathrm{i}\beta\Omega_2)\,, \tag{32}$$

which determines the inverse temperature β and the two angular velocities (Ω_1,Ω_2) in terms of the metric parameters (a,b,m,q). The final thermodynamic potential is given by

$$\Phi=V^\mu A_\mu|_{r\to r_+}-V^\mu A_\mu|_{r\to\infty}\,, \tag{33}$$

again in terms of (a,b,m,q), where

$$V=\frac{\partial}{\partial t}+\Omega_1\frac{\partial}{\partial\xi_1}+\Omega_2\frac{\partial}{\partial x_2} \tag{34}$$

is a linear combination of the Killing vectors of the solution. The Euclidean solution conformally asymptotes to the $S^1_\beta\times S^3$ background discussed above. We calculate the asymptotic charges via Komar integrals

$$J_i=\frac{1}{16\pi G_5}\int_{S^3_\infty}\star\mathrm{d}\left[\left(\frac{\partial}{\partial\xi_i}\right)_\mu\mathrm{d}x^\mu\right]\,,\quad Q=-\frac{1}{16\pi G_5}\int_{S^3_\infty}\left(\star F-\frac{1}{\sqrt{3}}F\wedge A\right), \tag{35}$$

and the entropy as 1/4 the horizon area

$$S=\frac{1}{4G_5}\int_{S^3_{r_+}}\mathrm{vol}\left(S^3_{r_+}\right). \tag{36}$$

The asymptotic mass and the Euclidean on-shell action are trickier due to UV divergences. We follow the holographic renormalization procedure and supplement the action with boundary terms

$$I_{\mathrm{bdy}}=\frac{1}{8\pi G_5}\int_{\mathbb{R}\times S^3}\sqrt{-\gamma}\left(K-\frac{3}{L}-\frac{L}{4}\mathcal{R}+\zeta\mathcal{R}^2\right), \tag{37}$$

where the four terms in the bracket are: the Gibbons-Hawking boundary term, two gravitational counterterms and a finite counterterm allowed by the symmetries. Then the asymptotic mass is given by

$$E=\int_{S^3_\infty}\sqrt{-\gamma}\,T_{\mathrm{bdy}}{}^t{}_t\,, \tag{38}$$

and the Euclidean on-shell action is the following

$$I^E_{\mathrm{ren}}=I^E+I^E_{\mathrm{bdy}}\,, \tag{39}$$

evaluated on the smooth Euclidean CCLP solution with boundary $S^1_\beta\times S^3$. Having obtained everything explicitly in terms of the black hole parameters (a,b,m,q) we verify the quantum statistical relation and the first law of thermodynamics

$$\begin{aligned}I^E_{\mathrm{ren}}&=\beta E-S-\beta\Omega_1J_1-\beta\Omega_2J_2-\beta\Phi Q\,,\\ \mathrm{d}E&=\frac{1}{\beta}\,\mathrm{d}S+\Omega_1\,\mathrm{d}J_1+\Omega_2\,\mathrm{d}J_2+\Phi\,\mathrm{d}Q\,.\end{aligned} \tag{40}$$

Supersymmetry (existence of a Killing spinor with antiperiodic boundary conditions at the horizon) dictates that

$$\beta(1+\Omega_1+\Omega_2-2\Phi)=2\pi\mathrm{i}\,. \tag{41}$$

Practically, it is implemented by relating the parameters $q=q(a,b,m)$. Further, it constraints the charges as

$$E=J_1+J_2+\frac{3}{2}Q\,. \tag{42}$$

Summary 3

(1) The Lorentzian supersymmetric CCLP has CTCs (no such issue with the Euclidean supersymmetric CCLP),

(2) The thermodynamic potentials $(\beta, \Omega_1, \Omega_2, \Phi)$, the charges (E, J_1, J_2, Q), the entropy (S) and the on-shell action (I^E_{ren}) are all complex in the supersymmetric limit

(3) In particular, the supersymmetric solution has non-zero temperature (supersymmetry in AdS does NOT imply extremality)

(4) Defining new thermodynamic potentials $\omega_i = \beta(\Omega_i - 1)$ and $\varphi = \beta(\Phi - 3/2)$ the on-shell action takes a remarkably simple form

$$I^E_{\text{ren}} = \frac{16}{27}\frac{\pi L^3}{8G_5}\frac{\varphi^3}{\omega_1\omega_2} \tag{43}$$

(5) From AdS_5/CFT_4 holography it is well known that at leading order in large-N the $\mathfrak{a}$ and $\mathfrak{c}$ conformal anomalies are given by

$$\mathfrak{a} \approx \mathfrak{c} \approx \frac{\pi L^3}{8G_5}\,, \tag{44}$$

Then

$$I^E_{\text{ren}} = \frac{16\mathfrak{a}}{27}\frac{\varphi^3}{\omega_1\omega_2} = \frac{1}{6}\frac{32\mathfrak{a}}{9}\frac{\varphi^3}{\omega_1\omega_2} = \frac{\operatorname{Tr} R^3}{6}\frac{\varphi^3}{\omega_1\omega_2} \approx -\log \mathcal{Z}_{\text{SCFT}}\,. \tag{45}$$

Holography works as advertized at leading order.

(6) The supersymmetry condition for the SCFT fugacities

$$\omega_1 + \omega_2 - 2\varphi = 2\pi i n\,, \tag{46}$$

is recovered (independently) by supergravity for $n = 1$.

(7) The non-supersymmetric result for the on-shell action appears to be proportional to β, which naively indicates an IR divergence for I^E_{ren} when $\beta \to \infty$. However, first taking the supersymmetric limit, and only then the limit to extremality, one obtains a perfectly finite on-shell action in the BPS ("= supersymmetry + extremality") limit. In the BPS limit, the thermodynamic potentials $(\omega_1, \omega_2, \varphi)$ and the on-shell action (I^E_{ren}) remain complex, but the charges (J_1, J_2, Q) and the entropy

$$S_{\text{BPS}} = \pi\sqrt{3Q^2 - 8\mathfrak{a}(J_1 + J_2)} \tag{47}$$

become real.

(8) The supergravity BPS limit implements a mysterious (from the SCFT point of view) non-linear relation between the charges

$$(3Q + 4\mathfrak{a})\big[3Q^2 - 2\mathfrak{a}(J_1 + J_2)\big] = Q^3 + 16\mathfrak{a}J_1J_2\,. \tag{48}$$

6 Matching the subleading term

One can use the formalism of 5d superconformal minimal supergravity to deduce that there are only two supersymmetric 4∂ invariants: Weyl^2 and Ricci^2 with coefficients c_1 and c_2. Gauge fixing to 5d $\boxdot$ minimal supergravity one obtains the following mess of a Lagrangian [6]

$$
\begin{aligned}
e^{-1}\mathcal{L}_{4\partial} = & -\Big[\frac{1}{\kappa^2}+\frac{(5\,c_1+24\,c_2)g^2}{2\sqrt{3}}\Big]R+\Big[\frac{1}{4\kappa^2}+\frac{7(5\,c_1-12\,c_2)g^2}{24\sqrt{3}}\Big]F_{ab}^2 \\
& -\Big[\frac{12g^2}{\kappa^2}+\frac{1}{\sqrt{3}}\,(25\,c_1+156\,c_2)\,g^4\Big]-\frac{\mathrm{i}}{12\sqrt{3}}\Big[\frac{1}{\kappa^2}-\frac{3\sqrt{3}(c_1+6\,c_2)}{2}g^2\Big]e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau} \\
& -\frac{\mathrm{i}c_1}{16}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}A_\mu R_{\nu\rho}{}^{\lambda\epsilon}R_{\sigma\tau\lambda\epsilon}-\frac{(2\,c_1-3\,c_2)}{24\sqrt{3}}R\,F_{ab}^2+\frac{5c_1}{4\sqrt{3}}R^{ab}F_{ac}F_b{}^c \\
& -\frac{\sqrt{3}c_1}{16}R_{abcd}F^{ab}F^{cd}-\frac{(c_1+6\,c_2)}{8\sqrt{3}}R^2+\frac{c_1}{2\sqrt{3}}R_{ab}^2-\frac{\sqrt{3}c_1}{8}(R_{abcd})^2 \\
& -\frac{5\sqrt{3}}{64}\,c_1\,F^{ab}F_a{}^cF_b{}^dF_{cd}+\frac{(61\,c_1-6\,c_2)}{1152\sqrt{3}}F_{ab}^2\,F_{cd}^2+\frac{\sqrt{3}c_1}{2}(\nabla_aF_{bc})(\nabla^{[a}F^{b]c}) \\
& +\frac{\sqrt{3}c_1}{2}F_{ab}\nabla^b\nabla_cF^{ac}-\frac{\mathrm{i}c_1}{8}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}F_\mu{}^\lambda F_{\sigma\tau}\Big(\frac{3}{2}\,\nabla_\nu F_{\lambda\rho}-\nabla_\lambda F_{\nu\rho}\Big) \\
& -\frac{3\mathrm{i}c_1}{32}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}F_{\rho\sigma}\nabla^\lambda F_{\lambda\tau}+\mathcal{O}(c_i^2)\,,
\end{aligned}
\tag{49}
$$

where $\kappa^2 = 16\pi G_5$ and $g = 1/L$. Indeed we see that at $c_{1,2} \to 0$, we recover the 2∂ action and that the presence of the higher derivative terms renormalizes the effective Newton constant (the piece in front of R) and the effective AdS scale (the piece in front of g^2). In principle, we should find a new saddle (a corrected CCLP) that solves the equations of motion of the above 4∂ theory. However, it turns out that [9]

$$
I_{4\partial}[\mathrm{CCLP}_{4\partial}] = I_{4\partial}[\mathrm{CCLP}_{2\partial}] + O(c_i^2)\,. \tag{50}
$$

Thus, we proceed and evaluate the Euclidean on shell action and renormalize the result as before. After the dust settles one obtains (in the supersymmetric limit) [5, 9]

$$
I_{\mathrm{ren}}^E = \left[\frac{2\pi}{27}\frac{L^3}{G_5}-\frac{16\pi^2}{3\sqrt{3}}L(c_1+6c_2)\right]\frac{\varphi^3}{\omega_1\omega_2}+\frac{4\pi^2}{\sqrt{3}}Lc_1\frac{\varphi(\omega_1^2+\omega_2^2-4\pi^2)}{\omega_1\omega_2}\,. \tag{51}
$$

By considering an empty AdS_5 background one can deduce the relation between the $\mathfrak{a}$ and $\mathfrak{c}$ anomalies and the 4∂ coefficients c_i

$$
\frac{16}{9}(5\mathfrak{a}-3\mathfrak{c}) = \frac{4\pi L^3}{9G_5}-\frac{32\pi^2}{\sqrt{3}}L(c_1+6c_2)\,,\quad 16(\mathfrak{a}-\mathfrak{c}) = -32\sqrt{3}\pi^2Lc_1\,. \tag{52}
$$

Rewriting the final answer:

$$
I_{\mathrm{ren}}^E = \frac{8(5\mathfrak{a}-3\mathfrak{c})}{27}\frac{\varphi^3}{\omega_1\omega_2}-\frac{2(\mathfrak{a}-\mathfrak{c})}{3}\frac{\varphi(\omega_1^2+\omega_2^2-4\pi^2)}{\omega_1\omega_2}\,, \tag{53}
$$

we observe a precise match with the SCFT answer upon using the relation between the conformal anomalies and the 't Hooft anomalies

$$
\frac{8(5\mathfrak{a}-3\mathfrak{c})}{27} = \frac{1}{6}\mathrm{Tr}\,R^3\,,\quad \frac{2(\mathfrak{a}-\mathfrak{c})}{3} = \frac{1}{24}\mathrm{Tr}\,R\,. \tag{54}
$$

Summary 4

(1) The ensemble used in this calculation is the one that keeps the thermodynamic potentials fixed, meaning that

$$(\omega_1^{4\partial}, \omega_2^{4\partial}, \varphi^{4\partial}) = (\omega_1^{2\partial}, \omega_2^{2\partial}, \varphi^{2\partial}) \tag{55}$$

(2) The above answer for the 4∂ on-shell action is valid in the supersymmetric limit, and to obtain it one does NOT need to correct the CCLP solution. However, to obtain the 4∂ corrected charges and entropy, one DOES need to correct the solution (see [10])

(3) Additionally taking the limit to extremality ($\beta \to \infty$) works the same as before. The BPS entropy can be obtained from a (constrained) Legendre transform of I_{ren}^E and gets corrected as

$$S_{\text{BPS}} = \pi\sqrt{3Q^2 - 8\mathfrak{a}(J_1 + J_2) - \frac{16\mathfrak{a}(\mathfrak{a} - \mathfrak{c})(J_1 - J_2)^2}{Q^2 - 2\mathfrak{a}(J_1 + J_2)}}\,. \tag{56}$$

(4) Similarly to the 2∂ case, 4∂ supergravity dictates a non-linear constraint between the BPS charges

$$[3Q + 4(2\mathfrak{a} - \mathfrak{c})]\left[3Q^2 - 8\mathfrak{c}(J_1 + J_2)\right] = Q^3 + 16(3\mathfrak{c} - 2\mathfrak{a})J_1 J_2 + 64\mathfrak{a}(\mathfrak{a} - \mathfrak{c})\frac{(Q + \mathfrak{a})(J_1 - J_2)^2}{Q^2 - 2\mathfrak{a}(J_1 + J_2)}\,. \tag{57}$$

References

[1] Ibrahima Bah, Christopher Beem, Nikolay Bobev, and Brian Wecht. AdS/CFT Dual Pairs from M5-Branes on Riemann Surfaces. *Phys. Rev. D*, 85:121901, 2012.

[2] Ibrahima Bah, Christopher Beem, Nikolay Bobev, and Brian Wecht. Four-Dimensional SCFTs from M5-Branes. *JHEP*, 06:005, 2012.

[3] Pietro Benetti Genolini, Davide Cassani, Dario Martelli, and James Sparks. The holographic supersymmetric Casimir energy. *Phys. Rev. D*, 95(2):021902, 2017.

[4] Nikolay Bobev, Mathew Bullimore, and Hee-Cheol Kim. Supersymmetric Casimir Energy and the Anomaly Polynomial. *JHEP*, 09:142, 2015.

[5] Nikolay Bobev, Vasil Dimitrov, Valentin Reys, and Annelien Vekemans. Higher derivative corrections and AdS5 black holes. *Phys. Rev. D*, 106(12):L121903, 2022.

[6] Nikolay Bobev, Kiril Hristov, and Valentin Reys. AdS_5 holography and higher-derivative supergravity. *JHEP*, 04:088, 2022.

[7] Alejandro Cabo-Bizet, Davide Cassani, Dario Martelli, and Sameer Murthy. Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes. *JHEP*, 10:062, 2019.

[8] Davide Cassani and Zohar Komargodski. EFT and the SUSY Index on the 2nd Sheet. *SciPost Phys.*, 11:004, 2021.

[9] Davide Cassani, Alejandro Ruipérez, and Enrico Turetta. Corrections to AdS_5 black hole thermodynamics from higher-derivative supergravity. *JHEP*, 11:059, 2022.

[10] Davide Cassani, Alejandro Ruipérez, and Enrico Turetta. Boundary terms and conserved charges in higher-derivative gauged supergravity. 4 2023.

[11] Z. W. Chong, Mirjam Cvetic, H. Lu, and C. N. Pope. General non-extremal rotating black holes in minimal five-dimensional gauged supergravity. *Phys. Rev. Lett.*, 95:161301, 2005.