Covariant Quantization of the Partially Massless Graviton Field in de Sitter Spacetime

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Based on: (1) my recent work with the same title in collaboration with J.P. Gazeau, **ArXiv: 2306.10086 (2023)**;
(2) my book in collaboration with M. Enayati, J.P. Gazeau, A. Wang, *The de Sitter (dS) Group and its Representations: An Introduction to Elementary Systems and Modeling the Dark Energy Universe*, **Springer Nature Book Series (2022)**.

The main goal

This project is part of a series of books/papers serving a wider research plan that attempts to develop a consistent formulation of elementary systems in **the global structure** of dS and AdS spacetimes, *in the Wigner sense* [Ann, Math., 40, 149 (1939); Rev. Mod. Phys., 21, 400 (1949).], as associated with UIRs of the dS and AdS relativity groups, respectively.

It rests upon three basic observations:

- the (A)dS group representation theory (in the sense given by Wigner);
- the Wightman-Gärding axioms;
- analyticity prerequisites in the complexified pseudo-Riemanian manifold.

✓ (A)dS (Anti-)de Sitter
✓ UIR Unitary irreducible representation

Motivations

The motivation for this attempt is rooted:

in part in the key role that is played by the dS geometry in the inflationary cosmological scenarii;

in part in the desire to establish possible mechanisms for late-time cosmology;

in part in the need for a dS analogue of the so-called AdS/CFT correspondence (the dS/CFT correspondence).

✓ Yet, the underlying motivation behind this attempt stems from a more fundamental consideration/concern that we now elaborate on.

One notices that:

(i) Both field theoretical formulation and phenomenological treatment of an elementary system, on the level of interpretation in particular, rest on the concepts of energy, momentum, mass, and spin, whose existence is due to the principle of invariance under the Poincaré group (the relativity group of flat Minkowski spacetime).

- ✓ The rest mass m and the spin s of an (Einsteinian) elementary system living in flat Minkowski spacetime are the two invariants that specify the respective UIR of the Poincaré group.

(ii) In a curved spacetime, however, any interpretation with reference to the relativity group of *flat* Minkowski spacetime is physically irrelevant.

(iii) In curved spacetimes generally (with the exception of dS and AdS spacetimes), *no* non-trivial groups of motion, and consequently, *no* literal or unique extension of the aforementioned physical concepts exists.

dS and AdS spacetimes are the maximally symmetric solution to the vacuum Einstein's equations with respectively a positive and negative cosmological constant.

As Minkowski spacetime is the null-curvature limit of the ordinary dS and AdS spacetimes, the Poincaré group can be obtained as a contraction of either dS or AdS relativity groups.

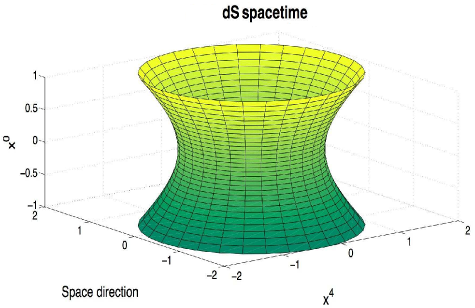
✓ UIRs of the dS and AdS groups, analogous to their shared Poincaré contraction limit, are characterized by two invariant parameters of the spin and energy scales (note that, in the AdS case, the latter should be read as the rest energy). These remarkable features, as already pointed out, allow the Wigner definition of elementary systems to be extended to dS and AdS relativities.

dS geometry and its relativity group

dS manifold can be conveniently visualized as a one-sheeted hyperboloid embedded in a $1+4$-dimensional Minkowski spacetime $\mathbb{R}^{1+4}$ (by abuse of notation, let us say $\mathbb{R}^5$):

$$M_R \equiv \left\{ x = (x^0,\ \ldots, x^4) \in \mathbb{R}^5 \; ; \; (x)^2 \equiv x \cdot x = \eta_{\alpha\beta} x^\alpha x^\beta = -R^2 \right\}, \tag{1}$$

where x^α (with $\alpha = 0, 1, 2, 3, 4$) stands for the corresponding Cartesian coordinates and $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$ for the ambient Minkowski metric. From a cosmological viewpoint: $R = c\, H^{-1}$, where c is the speed of light and H is the Hubble constant.



dS geometry and its relativity group

(i) The relativity group of dS spacetime, $\mathrm{SO}_0(1,4)$, is the ten-parameter group of all linear transformations in $\mathbb{R}^5$ which leave invariant the quadratic form $(x)^2 = \eta_{\alpha\beta}x^\alpha x^\beta$, have determinant 1, and do not reverse the direction of the "time" variable x^0.

(ii) The *universal covering* of the dS relativity group is the symplectic $\mathrm{Sp}(2,2)$ group. It is the group of all 2×2-matrices g, with *quaternionic components*:

$$\mathrm{Sp}(2,2) = \left\{ g = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \; ; \; \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{H}, \; \det(g) = 1, \; g^\dagger \gamma^0 g = \gamma^0 \right\}, \tag{2}$$

where $g^\dagger$ is the transpose, *quaternionic conjugate* of g and $\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$, and $\mathbf{1}$ and $\mathbf{0}$ being, respectively, the unit and zero 2×2 matrices.

(iii) A familiar realization of the corresponding Lie algebra is achieved by the linear span of the ten Killing vectors:

$$K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha \,, \qquad K_{\alpha\beta} = -K_{\beta\alpha} \,. \tag{3}$$

A discrete symmetry!

(i) Considering the following one-to-one mapping:

$$\mathbb{R}^5 \ni x \underset{1-1}{\Longleftrightarrow} \not{x} = x^A \gamma_A = \begin{pmatrix} \mathbf{1} x^0 & -\mathbf{x} \\ \mathbf{x}^\star & -\mathbf{1} x^0 \end{pmatrix} , \quad \mathbf{x} = (x^4, \vec{x}) \in \mathbb{H} , \; \mathbf{x}^\star \equiv (x^4, -\vec{x}) , \tag{4}$$

the action of the dS group on $\mathbb{R}^5$, for each element $g \in \mathrm{Sp}(2,2)$, is given by:

$$\not{x}' = g \not{x} g^{-1} = \begin{pmatrix} \mathbf{1} x'^0 & -\mathbf{x}' \\ \mathbf{x}'^\star & -\mathbf{1} x'^0 \end{pmatrix} . \tag{5}$$

(ii) The group action of γ^0 (as a particular member of the dS group) on $x \in \mathbb{R}^5$:

$$\not{x}' = \gamma^0 \not{x} \left(\gamma^0\right)^{-1} = \begin{pmatrix} \mathbf{1} x^0 & \mathbf{x} \\ -\mathbf{x}^\star & -\mathbf{1} x^0 \end{pmatrix} , \tag{6}$$

corresponds to the discrete symmetry:

$$x = (x^0, \mathbf{x}) \;\mapsto\; x' = (x^0, -\mathbf{x}) . \tag{7}$$

✓ The existence of this discrete symmetry in the dS group sending any point $(x^0, \mathbf{x}) \in M_R$ to its mirror image with respect to the x^0-axis, that is, $(x^0, -\mathbf{x}) \in M_R$, is indeed the origin of the well-known energy ambiguity in dS relativity.

A discrete symmetry!
Considering this discrete symmetry, the dS infinitesimal generators:

$$L_{\mathtt{A}0} = -\mathrm{i}\left(x_{\mathtt{A}} \frac{\partial}{\partial x^0} - x_0 \frac{\partial}{\partial x^{\mathtt{A}}} \right), \quad \mathtt{A} = 1, 2, 3, 4\,, \tag{8}$$

transform into their respective opposites with possibly different signs of the corresponding conserved charges, depending on the sign of **x**.

✓ This, for instance, implies that whether the generator L_{40}, which contracts to the Poincaré energy operator, moves us forwards or backward in time (towards increasing or decreasing x^0) depends on the sign of **x**, and hence, cannot be precisely determined. In this sense, this is the best we can do:

There is no positive conserved energy in dS spacetime (!?).

UIRs of the dS group and quantum version of dS motions

First ambient space notations:
According to ambient space notations, dS fields are identified with symmetric (spinor-)tensor fields $\Psi^{(r)}_{\alpha_1\ldots\alpha_n}(x)$ on the dS manifold ($x \in M_R$), such that the indices $\alpha_1, \ldots, \alpha_n$ take the values $0, 1, 2, 3, 4$ and r the values $n + 1/2$, n being the tensorial rank.

(i) These fields as functions of $\mathbb{R}^5$ are assumed to be homogeneous with some arbitrarily given degree of homogeneity ℓ:

$$x \cdot \partial \Psi^{(r)}_{\alpha_1\ldots\alpha_n}(x) \left(\equiv x^\alpha \frac{\partial}{\partial x^\alpha} \right) = \ell \Psi^{(r)}_{\alpha_1\ldots\alpha_n}(x)\,, \tag{9}$$

where, for the sake of simplicity, the degree of homogeneity ℓ is usually set to 0.
(ii) The fields are also assumed to be transitive with respect to all indices $\alpha_1, \ldots, \alpha_n$:

$$x^{\alpha_i} \Psi^{(r)}_{\alpha_1\ldots\alpha_i\ldots\alpha_n}(x) = (x \cdot \Psi)^{(r-1)}_{\alpha_1\ldots\breve{\alpha}_i\ldots\alpha_n}(x) = 0\,, \tag{10}$$

where by $\breve{\alpha}_i$ we mean this index is omitted. Clearly, the transversality requirement assures that the field $\Psi^{(r)}_{\alpha_1\ldots\alpha_n}(x)$ lies in the dS tangent spacetime.

Here, in view of the importance of the transversality requirement, the following symmetric, "transverse projector" is put forward:

$$\theta_{\alpha\beta} = \eta_{\alpha\beta} + R^{-2} x_\alpha x_\beta \,, \quad \theta_{\alpha\beta} x^\alpha = 0 = \theta_{\alpha\beta} x^\beta \,. \tag{11}$$

This operator is in fact the transverse form of the dS metric in ambient space formalism. Technically, $\theta_{\alpha\beta}$ is used to construct transverse entities, like the transverse derivative:

$$\bar{\partial}_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + R^{-2} x_\alpha x \cdot \partial \,, \tag{12}$$

for which:

$$\bar{\partial}_\alpha x_\beta = \theta_{\alpha\beta} \,, \quad \bar{\partial}_\alpha (x)^2 = 0 \,. \tag{13}$$

The latter reveals that $\bar{\partial}$ commutates with $(x)^2$, and hence, is intrinsically defined on the dS manifold $(x)^2 = -R^2$.

On the representation (quantum) level, in the Hilbert space of the symmetric, *square integrable* (with respect to some invariant inner product of Klein-Gordon type or else), (spinor-)tensors $\Psi^{(r)}_{\alpha_1\,\dots\,\alpha_n}(x)$ on M_R, the Killing vectors $K_{\alpha\beta}$ are represented by (essentially) self-adjoint operators $L^{(r)}_{\alpha\beta} = M_{\alpha\beta} + S^{(n)}_{\alpha\beta} + S^{(\frac{1}{2})}_{\alpha\beta}$, where the orbital part reads as:

$$M_{\alpha\beta} = -\mathrm{i}(x_\alpha\partial_\beta - x_\beta\partial_\alpha) = -\mathrm{i}(x_\alpha\bar{\partial}_\beta - x_\beta\bar{\partial}_\alpha)\,, \tag{14}$$

the action of the spinorial part $S^{(n)}_{\alpha\beta}$ on the tensorial indices is:

$$S^{(n)}_{\alpha\beta}\Psi^{(r)}_{\alpha_1\,\dots\,\alpha_n} = -\mathrm{i}\sum_{i=1}^{n}\Big(\eta_{\alpha\alpha_i}\Psi^{(r)}_{\alpha_1\,\dots\,(\alpha_i\mapsto\beta)\,\dots\,\alpha_n} - (\alpha\rightleftharpoons\beta)\Big)\,, \tag{15}$$

and finally the spinorial part $S^{(\frac{1}{2})}_{\alpha\beta}$ acts on the spinorial indices by $S^{(\frac{1}{2})}_{\alpha\beta} = -\frac{\mathrm{i}}{4}[\gamma_\alpha,\gamma_\beta]$, where γ_αs stand for the five 4×4-matrices generating the Clifford algebra. The self-adjoint operators $L^{(r)}_{\alpha\beta}$ obey the standard commutation relations of the dS Lie algebra:

$$\Big[L^{(r)}_{\alpha\beta}, L^{(r)}_{\gamma\delta}\Big] = -\mathrm{i}\left(\eta_{\alpha\gamma}L^{(r)}_{\beta\delta} + \eta_{\beta\delta}L^{(r)}_{\alpha\gamma} - \eta_{\alpha\delta}L^{(r)}_{\beta\gamma} - \eta_{\beta\gamma}L^{(r)}_{\alpha\delta}\right)\,. \tag{16}$$

Note that the operators $L^{(r)}_{\alpha\beta}$ are intrinsically defined on the dS$_4$ hyperboloid $(x)^2 = -R^2$, since we have $[L^{(r)}_{\alpha\beta}, (x)^2] = 0$.

In this group theoretical construction, there are two Casimir operators:

$$\begin{aligned} \text{quadratic} \quad &; \quad Q_r^{(1)} = -\frac{1}{2} L_{\alpha\beta}^{(r)} L^{(r)\alpha\beta} \,, \\ \text{quartic} \quad &; \quad Q_r^{(2)} = -W_\alpha^{(r)} W^{(r)\alpha} \,, \end{aligned} \tag{17}$$

where the dS counterpart of the Pauli-Lubanski operator $W_\alpha^{(r)} = -\frac{1}{8}\mathcal{E}_{\alpha\beta\delta\rho\sigma} L^{(r)\beta\delta} L^{(r)\rho\sigma}$, in which $\mathcal{E}_{\alpha\beta\delta\rho\sigma}$ refers to the five-dimensional totally antisymmetric Levi-Civita symbol. These two Casimir operators are also intrinsically defined on the dS_4 hyperboloid $(x)^2 = -R^2$ as $L_{\alpha\beta}^{(r)}$s do, since $[Q_r^{(1,2)}, (x)^2] = 0$.

Moreover, they commute with all generator representatives $L_{\alpha\beta}^{(r)}$, and hence, act like constants on all states in a certain dS UIR:

$$Q_r^{(1,2)} \Psi^{(r)} = \langle Q_r^{(1,2)} \rangle \Psi^{(r)}, \tag{18}$$

where the respective eigenvalues $\langle Q_r^{(1,2)} \rangle$, in the Dixmier notations, are determined in terms of a pair of parameters $\Delta(p,q)$, with $p \in \mathbb{N}/2$ and $q \in \mathbb{C}$, as:

$$\langle Q_r^{(1)} \rangle = \big(-p(p+1) - (q+1)(q-2) \big) \,, \tag{19}$$

$$\langle Q_r^{(2)} \rangle = \big(-p(p+1)q(q-1) \big) \,. \tag{20}$$

Therefore, the spectral values assumed by the Casimir operators (say, the allowed values of p and q) can be utilized to classify UIRs of the dS group. These UIRs fall basically into three distinguished series as we explain below.

Principal series representations $U^{\text{ps}}_{s,\nu}$ are characterized by $\Delta(p = s, q = \frac{1}{2} + i\nu)$. Two different cases must be distinguished:

The integer spin representations, with $\nu \in \mathbb{R}$ and $s = 0, 1, 2, \ldots$.

The half-integer spin representations, with $\nu \in \mathbb{R} - \{0\}$ and $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

Let us incorporate the parameters c (the speed of light) and $\hbar$ (the Planck constant) without setting them to unity. Let $\nu = mcR/\hbar$ define the relation between the representation parameter ν and the Poincaré-Minkowski mass m. Then, the Poincaré contraction limit of the representations yields:

$$U^{\text{ps}}_{s,\nu} \quad \underset{\substack{R\to\infty,\ \nu\to\infty \\ \hbar\nu/cR=m}}{\longrightarrow} \quad \mathcal{P}^{>}_{s,m} \ \oplus \ \underset{?}{\underline{\underline{\mathcal{P}^{<}_{s,m}}}} \,, \tag{21}$$

where $\mathcal{P}^{\gtrless}_{s,m}$ respectively refer to the positive/negative energy Wigner UIRs of the Poincaré group, with spin s and mass m.

✓ The dS principal UIRs are recognized as dS massive representations, in the above sense.

✓ Note that the spin-s fields associated with the dS (strictly) massive UIRs, admitting no gauge invariance, possess $2s + 1$ degrees of freedom.

Complementary series representations $U^{\textsf{cs}}_{s,\nu}$ are characterized by $\Delta(p = s, q = \frac{1}{2} + \nu)$. Two distinguished cases come to the fore:

The scalar representation $U^{\textsf{cs}}_{0,\nu}$, with $\nu \in \mathbb{R}$ and $0 < |\nu| < \frac{3}{2}$.

The spinorial representation $U^{\textsf{cs}}_{s,\nu}$, with $\nu \in \mathbb{R}$ and $0 < |\nu| < \frac{1}{2}$, while $s = 1, 2, 3, \dots$.

The only *meaningful* representation of the complementary series UIRs *from the point of view of a Minkowskian observer* is the scalar representation $U^{\textsf{cs}}_{s=0,\nu=\frac{1}{2}}$ $(\Delta(p = 0, q = 1))$:[1]

$$U^{\textsf{cs}}_{0,\frac{1}{2}} \hookrightarrow \begin{array}{c} \mathcal{C}^{>}_{1,0,0} \\ \oplus \\ \mathcal{C}^{<}_{-1,0,0} \end{array} \underset{R\to\infty}{\longrightarrow} \begin{array}{c} \mathcal{C}^{>}_{1,0,0} \\ \oplus \\ \mathcal{C}^{<}_{-1,0,0} \end{array} \begin{array}{c} \hookleftarrow \\ \\ \hookleftarrow \end{array} \begin{array}{c} \mathcal{P}^{>}_{0,0} \\ \oplus \\ \mathcal{P}^{<}_{0,0} \,. \end{array} \tag{22}$$

✓ In the above sense, the scalar representation $U^{\textsf{cs}}_{0,\frac{1}{2}}$ is called massless.

[1] Note that conformal invariance technically entails the discrete series representations (and their lower end) of the (universal covering of the) conformal group or its double covering group $\mathrm{SO}_0(2,4)$ or its fourth covering group $\mathrm{SU}(2,2)$. Here, the respective conformal UIRs are characterized by $\mathcal{C}^{\gtrless}_{E_0,j_l,j_r}$, with the parameter E_0 denoting the positive/negative conformal energy and $(j_l, j_r) \in \mathbb{N}/2 \times \mathbb{N}/2$ labeling the UIRs of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Discrete series representations $\Pi^{\pm}_{p,q}$ are characterized by $\Delta(p,q)$. For the symmetric cases $\Pi^{\pm}_{p=s,q=s}$, the parameter $p=s$ (with $s>0$) has a spin (helicity) meaning; the superscript '$\pm$' stands for the helicities $\pm s$. We have to distinguish between:

The nonsquare-integrable scalar case $\Pi_{p,0}$, with $p=1,2,...$.

The spinorial cases $\Pi^{\pm}_{p,q}$, with $p=\frac{1}{2},1,\frac{3}{2},...$ and $q=p,p-1,...,1$ or $\frac{1}{2}$ $(q>0)$; the representations characterized by $q=\frac{1}{2}$, namely, $\Pi^{\pm}_{p,\frac{1}{2}}$, are not square integrable.

The Minkowskian meaningful representations are $\Pi^{\pm}_{p=s,q=s}$ (with $s>0$):

$$\Pi^{+}_{s,s} \quad \hookrightarrow \quad \begin{array}{c} \mathcal{C}^{>}_{s+1,0,s} \\ \oplus \\ \mathcal{C}^{<}_{-s-1,0,s} \end{array} \quad \underset{R\to\infty}{\longrightarrow} \quad \begin{array}{c} \mathcal{C}^{>}_{s+1,0,s} \\ \oplus \\ \mathcal{C}^{<}_{-s-1,0,s} \end{array} \quad \begin{array}{c} \hookleftarrow \\ \\ \hookleftarrow \end{array} \quad \begin{array}{c} \mathcal{P}^{>}_{-s,0} \\ \oplus \\ \mathcal{P}^{<}_{-s,0} \end{array}, \tag{23}$$

$$\Pi^{-}_{s,s} \quad \hookrightarrow \quad \begin{array}{c} \mathcal{C}^{>}_{s+1,s,0} \\ \oplus \\ \mathcal{C}^{<}_{-s-1,s,0} \end{array} \quad \underset{R\to\infty}{\longrightarrow} \quad \begin{array}{c} \mathcal{C}^{>}_{s+1,s,0} \\ \oplus \\ \mathcal{C}^{<}_{-s-1,s,0} \end{array} \quad \begin{array}{c} \hookleftarrow \\ \\ \hookleftarrow \end{array} \quad \begin{array}{c} \mathcal{P}^{>}_{s,0} \\ \oplus \\ \mathcal{P}^{<}_{s,0} \end{array}. \tag{24}$$

✓ In the above sense, the discrete series UIRs $\Pi^{\pm}_{s,s}$ (with $s>0$) are called massless representations.
✓ For the spin-$s\,(>0)$ fields associated with the dS (strictly) massless UIRs, due to the gauge-invariant properties, the degrees of freedom reduce to 2, namely, the 2 modes of helicities $\pm s$, while the propagation of these modes is confined to the light cone.

dS field equations

(Projective) Hilbert spaces carrying the dS UIRs (in some restricted sense) identify the quantum ("one-particle") state spaces of the respective elementary systems living in dS spacetime. Such (projective) Hilbert spaces are densely generated by the *square integrable*[2], (spinor-)tensors $\Psi^{(r)}_{\alpha_1\,\dots\,\alpha_n}(x)$ on M_R. In practice, for a given dS UIR, the common dense subspace (of the respective Hilbert space) carrying the UIR is generated by the eigenfunctions of the dS Casimir operators $Q_r^{(1,2)}$ for the assumed eigenvalues $\langle Q_r^{(1,2)}\rangle$, namely:

$$\left(Q_r^{(1,2)} - \langle Q_r^{(1,2)}\rangle\right) \Psi^{(r)}_{\alpha_1\,\dots\,\alpha_n}(x) = 0\,. \tag{25}$$

The corresponding "wave (field) equation" is then identified with that of the quadratic Casimir operator $Q_r^{(1)}$:

$$\left(Q_r^{(1)} - \langle Q_r^{(1)}\rangle\right) \Psi^{(r)}_{\alpha_1\,\dots\,\alpha_n}(x) = 0\,. \tag{26}$$

✓ The equation involving the quartic Casimir operator $Q_r^{(2)}$, possessing higher derivatives, naturally entails "ghost" solutions.

[2] Again, with respect to some invariant inner product of Klein-Gordon type or else.

Let us now give the explicit form of the field equation associated with a rank-2 tensor field, say, $\Psi^{(2)}_{\alpha\beta} \equiv \mathcal{K}_{\alpha\beta}$, in terms of the ambient space notations. The explicit form of the quadratic Casimir operator $Q^{(1)}_2$, acting on the space generated by the rank-2 tensors $\mathcal{K}_{\alpha\beta}$, reads as:

$$Q^{(1)}_2\, \mathcal{K}_{\alpha\beta} = \Big(Q^{(1)}_0 - 6\Big)\mathcal{K}_{\alpha\beta} - 2\mathcal{S}\partial x \cdot \mathcal{K}_{\alpha\beta} + 2\mathcal{S}x\partial \cdot \mathcal{K}_{\alpha\beta} + 2\eta_{\alpha\beta}\mathcal{K}'\,, \tag{27}$$

where $Q^{(1)}_0 = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} = -R^2\overline{\partial}^2 = -R^2\Box_R$, the symmetrizer operator $\mathcal{S}$ acts as $\mathcal{S}(\zeta_\alpha\omega_\beta) = \zeta_\alpha\omega_\beta + \zeta_\beta\omega_\alpha$, and finally $\mathcal{K}' \equiv \eta^{\alpha\beta}\mathcal{K}_{\alpha\beta}$ denotes the trace of the field.

✓ Trivially, if one desires to be left with the space that merely carries the respective dS UIR, the requirements of homogeneity (reading here as $x \cdot \partial\mathcal{K} = 0$) and transversality ($x \cdot \mathcal{K} = 0$) intrinsic to a field in the ambient space notations must be supplemented by the divergenceless requirement ($\partial \cdot \mathcal{K} = 0$); note that the transversality and divergenceless requirements together entail $\mathcal{K}' = 0$.

For scalar fields $\Psi^{(0)} \equiv \phi$, the field equation is given by:

$$\Big(Q_0^{(1)} + \tau(\tau+3)\Big)\phi(x) = 0\,, \qquad Q_0 = -R^2\Box_R\,, \tag{28}$$

(i) $\tau = -q-1 = -3/2 - \mathrm{i}\nu$ with $\nu \in \mathbb{R}$, for the scalar principal series;

(ii) $\tau = -q-1 = -3/2 - \nu$ with $0 < |\nu| < 3/2$, for the scalar complementary series;

(iii) $\tau = -p-2$, with $p = 1, 2, \ldots$, for the scalar discrete series.

✓ From now on, we simplify our notations by dropping the superscript '(1)' from $Q_r^{(1)}$ ($Q_r \equiv Q_r^{(1)}$).

Partially massless graviton field equation

The partially massless spin-2 (say partially massless graviton) field is associated with the discrete series UIR $\Pi^{\pm}_{2,1}$, characterized by $\Delta(p=2, q=1)$, for which we have $\langle Q_2 \rangle = -4$. Therefore, the corresponding field equation should be as follows:

$$(Q_2 + 4)\,\mathcal{K} = 0\,, \tag{29}$$

while $x \cdot \partial \mathcal{K} = 0$, $x \cdot \mathcal{K} = 0$, and $\partial \cdot \mathcal{K} = 0$ (consequently, $\mathcal{K}' = 0$).

Generally, a symmetric transverse rank-2 tensor field $\mathcal{K}$ can be expressed in a dS-invariant way in terms of vector fields $\widetilde{K}$ and K ($x \cdot \widetilde{K} = 0 = x \cdot K$), and the scalar field ϕ as:

$$\mathcal{K} = \mathcal{S}\overline{Z}\widetilde{K} + D_2 K + \theta\phi\,, \tag{30}$$

where the three independent elements $\mathcal{S}\overline{Z}\widetilde{K}$, $D_2 K$, and $\theta\phi$ form a closed family under the action of Q_2, $D_2 = \mathcal{S}(D_1 - x)$, $D_1 = R^2\overline{\partial}$, and:

$$\mathcal{K}' = 2\big(Z \cdot \widetilde{K} + D_1 \cdot K + 2\phi\big) = 0\,. \tag{31}$$

✓ Note that the constant polarization five-vector $Z = Z_\alpha$ ($\overline{Z}_\alpha \equiv \theta_{\alpha\beta} Z^\beta$) carries the five-dimensional (nonunitary!) representation $(n_1 = 0, n_2 = 1)$ of the dS group.

On the other hand, the tensor field (30) has to verify the field equation (29), which implies that its ingredients $\widetilde{K}$, K, and ϕ have to respectively verify:

$$
\begin{aligned}
Q_1 \widetilde{K} &= 0\,, && (32)\\
(Q_1 + 4)\, K &= 2R^{-2}(Z \cdot x)\widetilde{K}\,, && (33)\\
(Q_0 + 4)\, \phi &= -4(Z \cdot \widetilde{K})\,. && (34)
\end{aligned}
$$

(i) The crucial observation to make in this context is that the recurrence formula (30) must have a group-theoretical interpretation.

(ii) Technically, it displays the reduction, through the leading term $\mathcal{S}\overline{Z}\widetilde{K}$, of the tensor product $(n_1 = 0, n_2 = 1) \otimes \Delta(p = 2 - 1, q = 1)$ which contains the UIR $\Pi^{\pm}_{2,1}$.

(iii) Accordingly, $\widetilde{K}$, verfying $Q_1 \widetilde{K} = 0$, has to obey the divergenceless condition $\partial \cdot \widetilde{K} = 0$ to be a carrier state for the UIR $\Pi^{\pm}_{1,1}$, characterized by $\Delta(p = 1, q = 1)$.

(i) But, once one imposes such a condition on $\widetilde{K}$, the solution to $Q_1\widetilde{K} = 0$ leads to a singularity of the type $1/\langle Q_1\rangle$, where $\langle Q_1\rangle$, being the quadratic Casimir eigenvalue associated with the UIR $\Pi^{\pm}_{1,1}$, is equal to zero.

(ii) To get rid of this singularity, the divergenceless condition on $\widetilde{K}$ ($\partial\cdot\widetilde{K} = 0$), must be relaxed. In other words, one must solve the field equation $Q_1\widetilde{K} = 0$ in a larger space which includes one more degree of freedom due to the $\partial\cdot\widetilde{K} \neq 0$ types of solutions.

(iii) Consequently, the equation $Q_1\widetilde{K} = 0$ turns into a gauge-invariant one $Q_1\widetilde{K} + D_1\partial\cdot\widetilde{K} = 0$ in such a way that $\widetilde{K} \mapsto \widetilde{K} + D_1\widetilde{\phi}_g$ is a solution to the field equation for any scalar field $\widetilde{\phi}_g$ as far as $\widetilde{K}$ is.

(iv) Then, there are three main types of solutions for $\widetilde{K}$, namely, **gauge solutions, physical solutions which are divergenceless, and solutions that are not divergenceless**.

After the gauge-fixing procedure, the explicit form of the solution reads as:

$$\widetilde{K} = \overline{\widetilde{Z}}\widetilde{\phi} - \frac{\widetilde{\lambda}}{2(1-\widetilde{\lambda})} D_1 \left(R^{-2}(\widetilde{Z}\cdot x)\widetilde{\phi} + \widetilde{Z}\cdot\bar{\partial}\widetilde{\phi} \right) + \underline{\underline{\frac{2-3\widetilde{\lambda}}{1-\widetilde{\lambda}} R^{-2} D_1 Q_0^{-1}(\widetilde{Z}\cdot x)\widetilde{\phi}}} + D_1 \widetilde{\phi}_g \,, \tag{35}$$

where $\widetilde{Z}$ denotes another constant polarization five-vector, $\widetilde{\lambda}(\neq 1)$ the gauge-fixing parameter, $\widetilde{\phi}$ the dS massless conformally coupled scalar field obeying the equation:

$$(Q_0 - 2)\widetilde{\phi} = 0 \,, \tag{36}$$

and corresponding to the scalar complementary (massless) UIR $U^{\text{cs}}_{0,\frac{1}{2}}$.

✓ A possible (*scalar plane wave*) solution to the latter equation is:

$$\widetilde{\phi}(x) = \left(\frac{x\cdot\xi}{R}\right)^{\tau}, \quad \tau = -1, -2 \,, \tag{37}$$

where x and ξ respectively live in M_R and in the null-cone C in $\mathbb{R}^5$:

$$C = \left\{ \xi \in \mathbb{R}^5 \,;\; (\xi)^2 \equiv \xi\cdot\xi = \eta_{\alpha\beta}\xi^{\alpha}\xi^{\beta} = 0 \right\}. \tag{38}$$

Note that:

(i) After the gauge fixing procedure (for $\widetilde{\lambda} \neq 1$), the gauge field $\widetilde{\phi}_g$ is determined by the equation $Q_0\widetilde{\phi}_g = 0$, which means that $\widetilde{\phi}_g$ is the minimally coupled scalar field corresponding to the scalar discrete UIR $\Pi_{1,0}$.

(ii) The term distinguished above by drawing a double line below is responsible for the appearance of logarithmic divergences in the field solutions which implies reverberation inside the light cone. Accordingly, contrary to the Minkowskian flat case, the "minimal" (or optimal) choice of $\widetilde{\lambda}$ is not zero. It is clearly $\widetilde{\lambda} = \frac{2}{3}$, which eliminates the logarithmic divergent term.

(iii) The solution (35), instead of the UIR $\Pi^{\pm}_{1,1}$, carries an indecomposable representation of the dS group containing $\Pi^{\pm}_{1,1}$ as its central (physical) part.

(iv) The physical part of the solution (35), carrying the representation $\Pi^{\pm}_{1,1}$, is obtained by imposing the divergenceless condition ($\partial \cdot \widetilde{K} = 0$) on (35) and eliminating the gauge solution $D_1 \widetilde{\phi}_g$:

$$\begin{aligned} \partial \cdot \widetilde{K} &= \frac{1}{1-\widetilde{\lambda}} \left((2+\tau) R^{-2} (\widetilde{Z} \cdot x) + \tau \frac{\widetilde{Z} \cdot \xi}{x \cdot \xi} \right) \widetilde{\phi} \\ &= 0 \quad \text{if and only if } \tau = -2 \text{ and } \widetilde{Z} \cdot \xi = 0 \,. \end{aligned} \tag{39}$$

✓ It is evident that the solutions satisfying the divergencelessness condition are not determined by a particular choice of $\widetilde{\lambda}$ (for $\widetilde{\lambda} \neq 1$). Hence, one has the freedom to select the specific value $\widetilde{\lambda} = \frac{2}{3}$.

Therefore, the plane-wave reading of the physical part of the solution ($\partial \cdot \widetilde{K} = 0$) is:

$$\widetilde{K} \equiv \widetilde{K}(x, \xi; \widetilde{Z}) = 2 \left(\overline{\widetilde{Z}} - \frac{\widetilde{Z} \cdot x}{x \cdot \xi} \, \bar{\xi} \right) \left(\frac{x \cdot \xi}{R} \right)^{-2} \equiv 2 \, \varepsilon(x, \xi; \widetilde{Z}) \left(\frac{x \cdot \xi}{R} \right)^{-2} . \tag{40}$$

One notices that $\xi \cdot \varepsilon(x, \xi; \widetilde{Z}) = \bar{\xi} \cdot \varepsilon(x, \xi; \widetilde{Z}) = 0$, as $\xi \cdot \widetilde{Z} = 0$.

Systematically, having the above scenario for the vector field $\widetilde{K}$ entails the traceless ($\mathcal{K}' \neq 0$) and the divergenceless ($\partial \cdot \mathcal{K} \neq 0$) conditions being relaxed from the very beginning in the case of the tensor field $\mathcal{K}$. Of course, to restrict the relaxed degrees of freedom to 1, we also impose:

$$\partial_2 \cdot \mathcal{K} = \frac{1}{2}\overline{\partial}\mathcal{K}' \,, \tag{41}$$

where '$\partial_2\cdot$' is called the generalized divergence on the dS hyperboloid and technically defined as $\partial_2 \cdot \mathcal{K} = \partial \cdot \mathcal{K} - R^{-2}x\mathcal{K}' - \frac{1}{2}\overline{\partial}\mathcal{K}'$.

The field equation (29) then turns into the following gauge-invariant one:

$$(Q_2 + 4)\,\mathcal{K} + D_2\partial_2 \cdot \mathcal{K} - \theta\mathcal{K}' = 0\,, \tag{42}$$

such that:

$$\mathcal{K} \mapsto \mathcal{K} + D_2 D_1 \phi_g - 2R^2\theta\phi_g\,, \tag{43}$$

represents a solution to the field equation as far as $\mathcal{K}$ does (ϕ_g being an arbitrary dS scalar field). Now, we introduce a gauge fixing parameter λ. The field equation then reads:

$$(Q_2 + 4)\,\mathcal{K} + \lambda D_2\partial_2 \cdot \mathcal{K} - \lambda\theta\mathcal{K}' = 0\,, \tag{44}$$

while we have in mind the constraint (41).

Precision on the field equation:
Substituting the generic solution (30) into (44), we get three counterparts of Eqs. (32)-(34) respectively:

$$Q_1\widetilde{K}+\lambda D_1\partial\cdot\widetilde{K} = 0\,, \tag{45}$$

$$(Q_1+4)K = 2R^{-2}(Z\cdot x)\widetilde{K}+\lambda\overline{Z}(\partial\cdot\widetilde{K})-\frac{1}{2}\lambda\bar{\partial}\mathcal{K}'\,, \tag{46}$$

$$(Q_0+4)\phi = -4(Z\cdot\widetilde{K})-2\lambda(Z\cdot x)\partial\cdot\widetilde{K}+\lambda\mathcal{K}'\,, \tag{47}$$

where $\mathfrak{T}Z\cdot\bar{\partial}\widetilde{K}\equiv Z\cdot\bar{\partial}\widetilde{K}-R^{-2}x(Z\cdot\widetilde{K})$.

✓ Here, we would like to point out that the above equations, as already expected, are invariant under the transformations $K\mapsto K+D_1\phi_g$ and $\phi\mapsto\phi-2R^2\phi_g$, issued from the gauge transformation (43), such that: if $\lambda=1$, the scalar field ϕ_g remains arbitrary (except for the restrictions imposed by ordinary differentiability prerequisites); if $\lambda\neq 1$, ϕ_g is restricted by $(Q_0+4)\phi_g=0$.

The Gupta-Bleuler triplet

Considering the field equation (44), the corresponding dS-invariant bilinear form (inner product) on the space of solutions is:

$$\langle \mathcal{K}_1, \mathcal{K}_2 \rangle = \mathrm{i}R^2 \int_{\mathbb{S}^3, \rho=0} \Big[(\mathcal{K}_1)^* \cdot\cdot \partial_\rho \mathcal{K}_2 - \underline{\underline{2\lambda\big((\partial_\rho x)\cdot(\mathcal{K}_1)^*\big)\cdot(\partial\cdot\mathcal{K}_2)}} - (1^* \leftrightharpoons 2) \Big] \mathrm{d}\Omega\,,$$

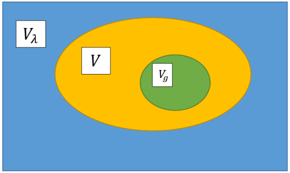
where $\mathcal{K}_1$ and $\mathcal{K}_2$ stand for two modes in the space of solutions, and $\mathrm{d}\Omega$ for the invariant measure on $\mathbb{S}^3$.

✓ Note that, above, we have employed a system of bounded global intrinsic coordinates (X^μ) well suited for describing a bounded copy of dS spacetime, that is, $\mathbb{S}^3 \times \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. This coordinate system, which is known in the literature under the name of conformal coordinates, is characterized by:

$$x = \big(x^0 = R\tan\rho, R(\cos\rho)^{-1} u \big)\,, \tag{48}$$

where $-\frac{\pi}{2} < \rho < \frac{\pi}{2}$ and $u \in \mathbb{S}^3$.

We now elaborate the Gupta-Bleuler triplet $V_g \subset V \subset V_\lambda$ which carries the dS indecomposable group representation structure corresponding to our problem:



(i) V_λ is the space of all square-integrable (according to (48)) solutions to the field equation (44). In V_λ, the inner product is indefinite. This means that V_λ includes negative norm solutions as well.
(ii) V is the space of the divergenceless and/or traceless solutions (recall that the constraint (41)). It forms a closed subspace of V_λ and, according to the field equation (44), is clearly λ independent. Of course, one must notice that the invariant subspace V is not invariantly complemented in V_λ. The inner product in V is semidefinite.
(iii) V_g is the space of the gauge solutions of the form $\mathcal{K}_g = D_2 D_1 \phi_g - 2R^2 \theta \phi_g$. It forms a closed, *null-norm* subspace of V. Of course, again, the invariant subspace V_g is not invariantly complemented in V.

Gauge states space; V_g

Considering the gauge solutions $\mathcal{K} = \mathcal{K}_g \equiv D_2 D_1 \phi_g - 2R^2 \theta \phi_g$, the field equation (44) reduces to:

$$(1-\lambda)(D_2 D_1 - 2R^2\theta)(Q_0 + 4)\phi_g = 0\,. \tag{49}$$

✓ If $\lambda = 1$, the scalar field ϕ_g remains arbitrary.
✓ If $\lambda \neq 1$, the scalar field ϕ_g is restricted by the equation $(Q_0 + 4)\phi_g = 0$ (possibly up to the addition of a specific solution to the inhomogeneous equation $(Q_0 + 4)\phi_g = \psi_g$, such that $(D_2 D_1 - 2R^2\theta)\psi_g = 0$).

(i) Since:

$$L^{(2)}_{\alpha\beta}\mathcal{K}_g = \Big(D_2 D_1 - 2R^2\theta\Big) M_{\alpha\beta}\phi_g\,, \tag{50}$$

the gauge solutions $\mathcal{K}_g$ do not carry any spin; $\mathcal{K}_g$s are entirely characterized by their scalar content ϕ_g corresponding to the scalar discrete UIR $\Pi_{p=2,q=0}$ (since, we have $(Q_0 + 4)\phi_g = 0$).

(ii) It is also useful to point out that:

$$\mathcal{K}'_g = -2R^2(Q_0+4)\phi_g\,, \quad \partial_2 \cdot \mathcal{K}_g = -D_1(Q_0+4)\phi_g\,. \tag{51}$$

Therefore, as far as $\lambda \neq 1$, and consequently, ϕ_g is restricted by $(Q_0 + 4)\phi_g = 0$, the gauge solutions are tracelessness/divergencelessness.

Physical states space (central part); V/V_g

The physical state space is λ independent. It consists of the solutions $\mathcal{K} = \mathcal{K}_{phys}$, which are defined up to the gauge solutions $\mathcal{K}_g$ and obtained by imposing the divergenceless and/or traceless condition on the field equation (44):

$$(Q_2 + 4)\,\mathcal{K}_{phys} = 0\,. \tag{52}$$

The dS group acts on physical states space V/V_g by the UIR $\Pi^{\pm}_{2,1}$. In the sequel, we will show that these modes propagate on the light cone.

Scalar states space (pure-trace part); V_λ/V

Then, the scalar states (or the pure-trace part)[3] obey:

$$(1-\lambda)(Q_0 + 4)\mathcal{K}' = 0\,. \tag{53}$$

✓ If $\lambda = 1$, $\mathcal{K}'$ remains unrestricted. Then, one naturally loses the ability to restrain the space V_λ.

✓ If $\lambda \neq 1$, $\mathcal{K}'$ is restricted by $(Q_0 + 4)\mathcal{K}' = 0$, which implies that $\mathcal{K}'$ also corresponds to the scalar discrete UIR $\Pi_{p=2,q=0}$.

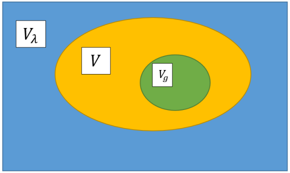
[3] Recall that, due to the constraint (41), the relaxed degrees of freedom reduce to 1. This is the fact that assures us that the exterior part of the Gupta-Bleuler triplet (i.e., V_λ/V) is merely characterized by the pure-trace solutions.

Indecomposable group representation structure

The indecomposable group representation structure, carried by the above Gupta-Bleuler triplet, associated with the dS partially massless graviton field, then reads:

$$\underbrace{\Pi_{2,0}}_{V_\lambda/V} \dashrightarrow \underbrace{\Pi^{\pm}_{2,1}}_{V/V_g} \dashrightarrow \underbrace{\Pi_{2,0}}_{V_g}, \tag{54}$$

where the arrows stand for the leaks under the dS group action. Note that the UIRs $\Pi^{\pm}_{2,1}$ and $\Pi_{2,0}$ share the same Casimir eigenvalue, and hence, they are Weyl equivalent.



Field equation solutions

For the sake of simplicity, we merely restrict our attention to the physical sector of the vector field $\widetilde{K}$ ($\partial \cdot \widetilde{K} = 0$), which is free of singularities. Then, we get the simplest forms of Eqs. (45)-(47):

$$Q_1 \widetilde{K} = 0\,, \tag{55}$$

$$(Q_1 + 4)K = 2R^{-2}(Z \cdot x)\widetilde{K} - \frac{1}{2}\lambda\bar{\partial}\mathcal{K}'\,, \tag{56}$$

$$(Q_0 + 4)\phi = -4(Z \cdot \widetilde{K}) + \lambda\mathcal{K}'\,. \tag{57}$$

(i) The imposition of $\partial \cdot \widetilde{K} = 0$ merely confines the nonphysical (dependent on λ) components of the equations, consequently merely affecting the nonphysical (dependent on λ) part of the corresponding solution.

(ii) This simplified set of equations, similar to its previous version, remains unchanged under the transformations $K \mapsto K + D_1\phi_g$ and $\phi \mapsto \phi - 2R^2\phi_g$, issued from the gauge invariance of the theory.

Considering the above set of equations, we obtain:

$$\phi = -\frac{2}{3}(Z\cdot\widetilde{K}) + \underline{\underline{\lambda(Q_0+4)^{-1}\mathcal{K}'}}\,. \tag{58}$$

$$K = \frac{3}{2}R^{-2}(Z\cdot x)\widetilde{K} + \frac{1}{2}\mathfrak{T}(Z\cdot\bar{\partial})\widetilde{K} - \frac{1}{6}\bar{\partial}(Z\cdot\widetilde{K}) \underline{\underline{-\frac{1}{2}\lambda(Q_1+4)^{-1}\bar{\partial}\mathcal{K}'}}\,. \tag{59}$$

✓ The last term in both cases bears a logarithmic singularity since $(Q_0+4)\,\mathcal{K}'=0$.

Accordingly, a general solution to the field equation (44) can be expressed as follows:

$$\begin{aligned}\mathcal{K} = & -\frac{2}{3}\theta(Z\cdot\widetilde{K}) + \mathcal{S}\overline{Z}\widetilde{K} + \frac{3}{2}D_2R^{-2}(Z\cdot x)\widetilde{K} \\ & + \frac{1}{2}D_2\mathfrak{T}(Z\cdot\bar{\partial})\widetilde{K} - \frac{1}{6}D_2\bar{\partial}(Z\cdot\widetilde{K}) \\ & - \frac{\lambda}{2}R^{-2}\Big(D_2(Q_1+4)^{-1}D_1\mathcal{K}' - 2R^2\theta(Q_0+4)^{-1}\mathcal{K}'\Big)\,. \end{aligned} \tag{60}$$

✓ The gauge solution $\mathcal{K}_g = D_2D_1\phi_g - 2R^2\theta\phi_g$ is coupled to the λ-dependent part of the solution (given in the last line).

✓ It is evident that the optimal choice for the gauge-fixing parameter is $\lambda=0$, which effectively removes all singular terms.

Here, it is worth recalling:

$$\widetilde{K}(x) = 2\,\varepsilon(x,\xi;\widetilde{Z}^{\kappa})\left(\frac{x\cdot\xi}{R}\right)^{-2} = 2\left(\overline{\widetilde{Z}}^{\kappa} - \frac{\widetilde{Z}^{\kappa}\cdot x}{x\cdot\xi}\,\bar{\xi}\right)\left(\frac{x\cdot\xi}{R}\right)^{-2}. \tag{61}$$

Note that when choosing constant five-vectors, such as $\widetilde{Z}_\alpha$, they are commonly labeled with $\kappa = 0,1,2,3$. These vectors then can be denoted as $\widetilde{Z}_\alpha^{(\kappa)}$ or, for the sake of convenience in expressions, as $\widetilde{Z}_\alpha^{\kappa}$. Now, let us impose the following conditions on $\widetilde{Z}_\alpha^{\kappa}$:

$$\widetilde{Z}^{\kappa}\cdot\widetilde{Z}^{\kappa'} = \eta^{\kappa\kappa'}\,, \quad \sum_{\kappa=1}^{3}\widetilde{Z}_\alpha^{\kappa}\widetilde{Z}_\beta^{\kappa} = -\eta_{\alpha\beta}\,, \quad \sum_{\kappa=1}^{3}\widetilde{Z}_4^{\kappa}\widetilde{Z}_\mu^{\kappa} = 0\,. \tag{62}$$

It then follows that the characteristics of the dS polarization vector $\varepsilon(x,\xi;\widetilde{Z}^{\kappa}) \equiv \varepsilon^{\kappa}(x,\xi)$, bear a striking resemblance to the Minkowskian scenario:

$$\sum_{\kappa=1}^{3}\varepsilon_\alpha^{\kappa}(x,\xi)\,\varepsilon_\beta^{\kappa}(x,\xi) \quad = \quad -\left(\theta_{\alpha\beta} - \frac{\bar{\xi}_\alpha\bar{\xi}_\beta}{(x\cdot\xi/R)^2}\right), \tag{63}$$

$$\varepsilon^{\kappa}(x,\xi)\cdot\varepsilon^{\kappa'}(x,\xi) \quad = \quad \widetilde{Z}^{\kappa}\cdot\varepsilon^{\kappa'}(x,\xi) = \varepsilon^{\kappa}(x,\xi)\cdot\varepsilon^{\kappa'}(x',\xi) = \widetilde{Z}^{\kappa}\cdot\widetilde{Z}^{\kappa'} = \eta^{\kappa\kappa'}\,. \tag{64}$$

It follows from the above equations that the plane-wave reading of the solution to the field equation (44) is:[4]

$$\mathcal{K}_{\alpha\beta}(x) = a_0\, \mathcal{E}_{\alpha\beta}(x,\xi;Z,\widetilde{Z}) \left(\frac{x\cdot\xi}{R}\right)^{-2}, \tag{65}$$

with $a_0 = 6\,c_0$, where c_0 is a normalization constant, and $\mathcal{E}_{\alpha\beta}(x,\xi;Z,\widetilde{Z})$ represents the corresponding polarization tensor. Here, we consider an explicit realization of the latter, if we select Z to be equal to $\widetilde{Z}$ and refer to both as Z hereafter. Then, the polarization tensor $\mathcal{E}_{\alpha\beta}(x,\xi;Z,\widetilde{Z}) \equiv \mathcal{E}_{\alpha\beta}(x,\xi;Z)$ adopts the following straightforward expression:

$$\begin{aligned}\mathcal{E}_{\alpha\beta}(x,\xi;Z) &\equiv \mathcal{E}^{\kappa\kappa'}_{\alpha\beta}(x,\xi)\\ &= \frac{1}{2}\left(\mathcal{S}\,\varepsilon^{\kappa}_{\alpha}(x,\xi)\,\varepsilon^{\kappa'}_{\beta}(x,\xi) - \frac{2}{3}\left(\theta_{\alpha\beta} - \frac{\bar{\xi}_\alpha\bar{\xi}_\beta}{(x\cdot\xi/R)^2}\right)\varepsilon^{\kappa}(x,\xi)\cdot\varepsilon^{\kappa'}(x,\xi)\right), \end{aligned}\tag{66}$$

where, again, $\varepsilon(x,\xi;\widetilde{Z}^{\kappa}) \equiv \varepsilon^{\kappa}(x,\xi)$.

[4]Here, for simplicity, we again exclude the superscript 'κ'.

(i) Evidently, we have $\eta^{\alpha\beta}\mathcal{E}_{\alpha\beta}(x,\xi;Z) \equiv \mathcal{E}'(x,\xi;Z) = 0.$ This means that when the gauge-fixing parameter λ is set to zero, we are effectively constrained to the traceless and/or divergenceless part of the solution living in the subspace V.

(ii) Moreover, $\xi \cdot \mathcal{E} = \overline{\xi} \cdot \mathcal{E} = 0$. These transversality conditions are valid only for the physical part of the solution residing in the quotient space V/V_g, rather than the entire solution belonging to V, which includes both the physical and the gauge parts.

(iii) The dS tensor wave $\mathcal{K}_{\alpha\beta}(x)$ exhibits a homogeneity property with the degree of $\tau = -2$ when considered on both the null-cone C and the dS submanifold M_R $\left(\subset \mathbb{R}^5\right)$, which is defined by the condition $x \cdot x = -R^2$, where R remains constant.

(iv) Within this framework, it becomes feasible to develop the covariant quantization of the field by virtue of the closure of $\mathcal{K}_{\alpha\beta}(x)$ under the dS group action:

$$\begin{aligned}\Big(U(g)\mathcal{K}\Big)_{\alpha\beta}(x) = g_\alpha^\gamma g_\beta^\delta \mathcal{K}_{\gamma\delta}(g^{-1}x) =& g_\alpha^\gamma g_\beta^\delta\, a_0\, \mathcal{E}_{\gamma\delta}(g^{-1}x,\xi;Z)\left(\frac{g^{-1}x\cdot\xi}{R}\right)^{-2} \\ =& a_0\, \mathcal{E}_{\alpha\beta}(x,g\xi;gZ)\left(\frac{x\cdot g\xi}{R}\right)^{-2}. \end{aligned} \tag{67}$$

(v) The final point to be discussed here is that the wave $\mathcal{K}_{\alpha\beta}(x)$, as a *function* on M_R, is only locally defined on connected open subsets of M_R. To ensure a global definition of this solution an analytical continuation becomes crucial. Consequently, we define the dS tensor wave $\mathcal{K}_{\alpha\beta}(x,\xi)$ as the boundary value of the analytic continuation of the solution (65) to the forward-tube $\mathcal{T}^+$; for $z \in \mathcal{T}^+ = \left\{\mathbb{R}^5 + \mathrm{i}\mathring{V}^+\right\} \cap M_R^{(\mathbb{C})}$ and $\xi \in C^+$ (where $\mathring{V}^+ \equiv \left\{y \in \mathbb{R}^5\ ;\ (y)^2 > 0, y^0 > 0\right\}$ originate from the causal structure of M_R.), the following holds:

$$K_{\alpha\beta}(z,\xi) = a_0\, \mathcal{E}^{\kappa\kappa'}_{\alpha\beta}(z,\xi) \left(\frac{z\cdot\xi}{R}\right)^{-2}. \tag{68}$$

Then, the corresponding boundary value *(in the distribution sense)* of the above-complexified waves yields a single-valued global plane-wave reading of the solution (65) as:

$$\text{bv}\ K_{\alpha\beta}(z,\xi) \equiv \mathcal{K}_{\alpha\beta}(x,\xi) = a_0\, \mathcal{E}^{\kappa\kappa'}_{\alpha\beta}(x,\xi) \left(\frac{(x+\mathrm{i}y)\cdot\xi}{R}\right)^{-2}\Bigg|_{\xi\in C^+,\ y\in\mathring{V}^+,\ y\to 0}. \tag{69}$$

Two-point function and quantum field

In this study, our focus lies on the free field part of the theory, where all truncated correlation functions vanish. Consequently, the complete characterization of the respective QFT is entirely captured by the corresponding Wightman two-point function $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$, where $\alpha,\beta,\alpha',\beta'=0,1,\ldots,4$. The two-point function must meet the following criteria:

Indefinite sesquilinear form:

$$\int_{M_R\times M_R} f^{*\alpha\beta}(x)\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')f^{\alpha'\beta'}(x')\,\mathrm{d}\mu(x)\mathrm{d}\mu(x')\,, \tag{70}$$

where $f_{\alpha\beta}$ stands for a test function in the space of functions C^∞ with compact support in M_R, and $\mathrm{d}\mu(x)$ for the invariant measure on M_R.

Covariance:

$$(g^{-1})^{\gamma}_{\alpha}(g^{-1})^{\delta}_{\beta}\mathcal{W}_{\gamma\delta\gamma'\delta'}(gx,gx')g^{\gamma'}_{\alpha'}g^{\delta'}_{\beta'}=\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')\,, \tag{71}$$

for all $g\in SO_0(1,4)$.

Locality:

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')=\mathcal{W}_{\alpha'\beta'\alpha\beta}(x',x)\,, \tag{72}$$

if x and x' are spacelike separated ($x\cdot x'>-R^2$).

Index symmetrizer:

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \mathcal{W}_{\alpha\beta\beta'\alpha'}(x,x') = \mathcal{W}_{\beta\alpha\alpha'\beta'}(x,x')\,. \tag{73}$$

Transversality:

$$x^{\alpha}\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = 0 = x'^{\alpha'}\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')\,. \tag{74}$$

Normal analyticity: The Wightman two-point function $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$ is the boundary value (in the sense of distribution) of a function $W_{\alpha\beta\alpha'\beta'}(z,z')$ that is analytic in the tuboid domain:

$$\mathcal{T}^{+(2)} = \Big\{(z,z')\,;\; z\in\mathcal{T}^{-},\; z'\in\mathcal{T}^{+}\Big\}, \tag{75}$$

where, again, $\mathcal{T}^{\pm}$ are respectively the forward and backward tubes of $M_R^{(\mathbb{C})}$, respectively, defined by:

$$\mathcal{T}^{\pm} = \Big\{\mathbb{R}^5 + \mathrm{i}\mathring{V}^{\pm}\Big\}\cap M_R^{(\mathbb{C})}\,, \tag{76}$$

and the domains $\mathring{V}^{\pm} \equiv \big\{y\in\mathbb{R}^5\,;\;(y)^2>0, y^0 \gtrless 0\big\}$ originate from the causal structure of M_R.

Based on the previously mentioned condition of normal analyticity, we can derive the following conclusions:

(i) The function $W_{\alpha\beta\alpha'\beta'}(z,z')$ exhibits maximal analyticity, allowing for its analytic continuation to the "cut domain":

$$\Delta = \Big\{(z,z') \in M_R^{(\mathbb{C})} \times M_R^{(\mathbb{C})}\ ;\ (z-z')^2 < 0\Big\}. \tag{77}$$

(ii) The "permuted Wightman two-point function" $\mathcal{W}_{\alpha'\beta'\alpha\beta}(x',x)$ corresponds to the boundary value of $W_{\alpha\beta\alpha'\beta'}(z,z')$ from the domain:

$$\mathcal{T}^{-(2)} = \{(z,z')\ ;\ z \in \mathcal{T}^+,\ z' \in \mathcal{T}^-\}, \tag{78}$$

defined on $M_R^{(\mathbb{C})} \times M_R^{(\mathbb{C})}$. Notably, the permuted two-point function fulfills all the aforementioned requirements, as well.

The explicit definition of the analytic two-point function $W_{\alpha\beta\alpha'\beta'}(z,z')$ can be derived from the provided solution (65), using the following class of integral representations:

$$W_{\alpha\beta\alpha'\beta'}(z,z') = a_0^2 \int_\gamma \left(\frac{z\cdot\xi}{R}\right)^{-2} \left(\frac{z'\cdot\xi}{R}\right)^{-2} \sum_{\kappa,\kappa'=1}^{3} \mathcal{E}^{\kappa\kappa'}_{\alpha\beta}(z,\xi)\, \mathcal{E}^{*\kappa\kappa'}_{\alpha'\beta'}(z'^{*},\xi)\, \mathrm{d}\mu_\gamma(\xi)\,,$$

where the integration takes place over any orbital basis γ of the future null-cone $C^+ \equiv \left\{\xi \in \mathbb{R}^5\,;\; (\xi)^2 = 0,\; \xi^0 > 0\right\}$, $\mathrm{d}\mu_\gamma$ denotes the intrinsic C^+ invariant measure on γ and is derived from the Lebesgue measure of $\mathbb{R}^5$, and the normalization factor a_0 is determined subsequently by applying the local Hadamard condition. Then:

$$W(z,z') = \Delta(z,z')\widetilde{W}_1(z,z') + \Theta(z,z')\widetilde{W}_0(z,z')\,,$$

$$\Delta(z,z') = -\,\frac{9}{4}\left(\mathcal{S}\mathcal{S}'\,\theta\cdot\theta' + \frac{\mathcal{S}\,R^{-2}(\theta\cdot z')D_2'}{3} + \frac{\mathcal{S}'\,R^{-2}(\theta'\cdot z)D_2}{3} - \frac{R^{-2}\mathcal{Z}D_2D_2'}{9}\right), \tag{79}$$

$$\Theta(z,z') = -\,\frac{16}{3}\left(\theta - \frac{D_2D_1}{8R^2}\right)\left(\theta' - \frac{D_2'D_1'}{8R^2}\right),$$

in which $\mathcal{Z} \equiv -z\cdot z'/R^2$ is a dS-invariant length, **AND ...**

$$\widetilde{W}_1(z,z') = 4c_0 \int_\gamma \sum_{\kappa=1}^{3} \varepsilon^{\kappa}(z,\xi)\, \varepsilon^{*\kappa}(z'^*,\xi) \left(\frac{z\cdot\xi}{R}\right)^{-2} \left(\frac{z'\cdot\xi}{R}\right)^{-2} \mathrm{d}\mu_\gamma(\xi)$$
$$= -4\left(\theta\cdot\theta' + \frac{R^{-2}(\theta\cdot z')D'_1}{2} + \frac{R^{-2}(\theta'\cdot z)D_1}{2} - \frac{R^{-2}\mathcal{Z}D_1D'_1}{4}\right)\widetilde{W}_0(z,z'), \quad (80)$$

$$\widetilde{W}_0(z,z') = c_0 \int_\gamma \left(\frac{z\cdot\xi}{R}\right)^{-2} \left(\frac{z'\cdot\xi}{R}\right)^{-2} \mathrm{d}\mu_\gamma(\xi). \quad (81)$$

Consequently, the analytic tensor two-point function can be expressed in terms of the scalar analytic two-point function as:

$$W_{\alpha\beta\alpha'\beta'}(z,z') = \mathfrak{D}_{\alpha\beta\alpha'\beta'}(z,z')\,\widetilde{W}_0(z,z'). \quad (82)$$

✓ Note that the integral representation (81) of the scalar analytic two-point function, for well-chosen points $z, z' \in \mathcal{T}^{+(2)}$ in the domain determined by $(z-z')^2 < 0$, yields:

$$\widetilde{W}_0(z,z') = \frac{-1}{8\pi^2R^2}\,\frac{1}{1-\mathcal{Z}(z,z')}, \quad (83)$$

where the Hadamard condition gives a fixed value for the normalization factor c_0.

Eventually, taking the boundary value of the analytic two-point function (82) yields the corresponding Wightman two-point function:

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \text{bv}\ W_{\alpha\beta\alpha'\beta'}(z,z') = \mathfrak{D}_{\alpha\beta\alpha'\beta'}(x,x')\left(\text{bv}\ \widetilde{W}_0(z,z')\right), \qquad (84)$$

with:

$$\widetilde{\mathcal{W}}_0(x,x') = \text{bv}\ \widetilde{W}_0(z,z') = \frac{-1}{8\pi^2 R^2}\left(\mathbf{P}\frac{1}{1-\mathcal{Z}(x,x')} - i\pi\,\epsilon(x^0 - x'^0)\,\delta\big(1-\mathcal{Z}(x,x')\big)\right), \qquad (85)$$

where **P** stands for the principal part and $\epsilon(x^0 - x'^0) = 1, 0, -1$ for $(x^0 - x'^0) >, =,$ or < 0, respectively.

✓ Here, it must be underlined that the derived kernel satisfies, through its construction, the previously mentioned conditions of indefinite sesquilinearity, covariance, locality, index symmetrization, transversality, and normal analyticity. These conditions are essential for obtaining a Wightman two-point function. It is also crucial to emphasize that the existence of a Wightman two-point function is a fundamental requirement in dS axiomatic QFT.

Light-cone propagation

Now without going into the technical details, let $\hat{\mathcal{K}}_{\alpha\beta}(x)$ denote the corresponding quantum field operator in the introduced context. The field commutation relations are:

$$\Big[\hat{\mathcal{K}}_{\alpha\beta}(x),\,\hat{\mathcal{K}}_{\alpha'\beta'}(x')\Big] = 2\mathrm{i}\,\mathrm{Im}\,\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = 2\mathrm{i}\,\mathfrak{D}_{\alpha\beta\alpha'\beta'}\,\mathrm{Im}\,\widetilde{\mathcal{W}}_0(x,x')\,, \tag{86}$$

where we have used Eq. (82) while, following the relation (85), we have:

$$\mathrm{Im}\,\widetilde{\mathcal{W}}_0(x,x') = \frac{1}{8\pi R^2}\,\epsilon(x^0 - x'^0)\,\delta\big(1-\mathcal{Z}(x,x')\big)\,. \tag{87}$$

Finally, we have the commutator:

$$\mathrm{i}\,G_{\alpha\beta\alpha'\beta'}(x,x') = \Big[\hat{\mathcal{K}}_{\alpha\beta}(x),\,\hat{\mathcal{K}}_{\alpha'\beta'}(x')\Big] = \frac{\mathrm{i}}{4\pi R^2}\,\mathfrak{D}_{\alpha\beta\alpha'\beta'}\,\epsilon(x^0 - x'^0)\,\delta\big(1-\mathcal{Z}(x,x')\big)\,. \tag{88}$$

✓ The right-hand side of the above equation clearly demonstrates the light-cone propagation of the dS partially massless graviton field.

Thank you!