Set-indexed random fields and Euclidean quantum field theory

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Plan

1. Basic notions of axiomatic QFT
2. Algebraic Euclidean QFT
3. Background from probability theory
4. Non-Gaussian examples of Algebraic Euclidean QFT models
5. Current research and perspectives

Basic notions of axiomatic QFT

- In Wightman's axioms quantum fields are represented by point-like localized objects, namely, operator-valued distributions on Minkowski space.
- In the axioms of Glimm and Jaffe of Euclidean QFT quantum fields are represented by probability measures on distributions on Euclidean space.
- In the Haag-Kastler axioms of Algebraic QFT quantum fields are represented by open set indexed objects, namely nets (flabby pre-cosheaves) of operator algebras.
- Is there Euclidean analogue of Algebraic QFT? Yes, it is developed by D. Schlingemann, in 1999.

Reflection positive representations

Consider the data:

- A Hilbert space $\mathcal{H}_E$,
- A strongly continuous unitary representation on $\mathcal{H}_E$ of the d-dimensional Euclidean group $E(d)$,
- A fixed reflection θ in $\mathbb{R}^d$ along a "time axis" t,
- A family of $E(d)$-covariant closed subspaces $\{\mathcal{H}_V\}$ of $\mathcal{H}_E$ indexed by open sets $V \subset \mathbb{R}^d$.

We say that the above defines a reflection positive representation of $E(d)$ if $R := \Pi_+ \theta \Pi_+ \geq 0$, where Π_+ is the orthogonal projection on $\mathcal{H}_{t>0}$.

Theorem

(Klein-Landau, 1983) Every reflection positive representation of $E(d)$ on $\mathcal{H}_E$ gives rise to a strongly continuous unitary representation of the d-dimensional orthochronous Poincaré group $P(d)$ on a Hilbert space $\mathcal{H}_M$ satisfying the spectrum condition.

Euclidean Algebraic QFT

Consider the data:

- A Hilbert space $\mathcal{H}_E$,
- A strongly continuous unitary representation on $\mathcal{H}_E$ of the d-dimensional Euclidean group $E(d)$,
- A fixed reflection θ in $\mathbb{R}^d$
- A fixed $E(d)$-invariant vector $h_0 \in \mathcal{H}_E$,
- A net of $E(d)$-covariant unital commutative $*$-algebras $\{\mathcal{A}_V\}$ of bounded operators on $\mathcal{H}_E$ indexed by open sets $V \subset \mathbb{R}^d$.

We say that the above defines a a d-dimensional Euclidean Algebraic QFT if the subspaces $\overline{\mathcal{A}_V h_0}$ define a reflection positive representation of $E(d)$.

Euclidean Algebraic QFT

Theorem

(Schlingemann, 1999) (1) Every d-dimensional Euclidean Algebraic QFT gives rise to:
(1) strongly continuous unitary representation of $P(d)$ on a Hilbert space $\mathcal{H}_M$ satisfying the spectrum condition;
(2) A net of Banach algebras $\mathcal{B}_V$ of bounded operators on $\mathcal{H}_M$ indexed by doubles cones V in d-dimensional Minkowski space satisfying:
(a) $P(d)$-covariance
(b) locality/causality: $[\mathcal{B}_{V_1}, \mathcal{B}_{V_2}] = 0$ for space-like separated V_1 and V_2.

Notes:
1. The net $\mathcal{B}_V$ is obtained from $\mathcal{A}_{t>0}$ via the action of $P(d)$.
2. The locality property is proved by a Bargmann-Hall-Wightman type of theorem, using the commutativity of $\mathcal{A}_{t>0}$.
3. The original theorem of Schlingemann assumes the existence of sharp time localized operators and produces a net of C^*-algebras, i.e. the original Haag-Kastler axioms.

Constructing Gaussian measures and random fields

Some terminology:

- Fix a probability space $(\Omega, \mathcal{F}, \mu)$.
- Given a set T, by a random field indexed by T we mean a map $X : T \to L^2(\Omega, \mu)$.
- We write $X_t(\omega) = X(t, \omega)$ for the function on $T \times \Omega$ given by X. The functions $\{X(\cdot, \omega)\}_{\omega \in \Omega}$ are called the sample paths of X.
- The covariance of a random field X defined via

$$\mathrm{Cov}(X)(t_1, t_2) = \langle X_{t_1}, X_{t_2} \rangle_{L^2}, \quad t_1, t_2 \in T,$$

- A random field X is called Gaussian if all the random variables X_t are centered Gaussian.
- $\mathbb{R}^T$ stands for the set of all maps from T to $\mathbb{R}$.

Constructing Gaussian measures and random fields

Let $\mathcal{T}$ be the smallest σ-algebra on $\mathbb{R}^T$ with respect to which the canonical projections from $\mathbb{R}^T$ to $\mathbb{R}$ are measurable.

Theorem

Let $C : T \times T \to \mathbb{R}$ be positive definite and symmetric. Then there exists a centered Gaussian probability measure μ_C on $(\mathbb{R}^T, \mathcal{T})$ and a Gaussian random field X^C on $(\mathbb{R}^T, \mathcal{T}, \mu_C)$ indexed by T whose covariance is C.

The existence of μ_C is proved by the reducing to the finite dimensional case, using the Kolmogorov extension theorem; X^C is the canonical random field given by evaluation at points of $\mathbb{R}^T$.

Continuity of sample paths

- Let T be a compact space and let X be a Gaussian random field which is continuous as a map from T to $L^2(\Omega, \mu)$.
- Define a pseudo-metric on T:

$$d_X(t_1, t_2) = \|X_{t_1} - X_{t_2}\|_{L^2}.$$

- Let $N_X(\varepsilon)$ be the minimum number of balls of radius ε whose union covers T.

Theorem

(R. Dudley) If the integral

$$\int_0^\infty (\log N_X(\varepsilon))^{1/2} d\varepsilon$$

is finite, the Gaussian random field X has a modification with continuous sample paths on (T, d_X).

Constructing non-Gaussian Euclidean Algebraic QFTs

Consider the following input data:

- The covariance operator $C_m = (-\Delta + m^2)^{-1}$ of the free real scalar Euclidean quantum field with mass $m > 0$, where Δ is the Laplacian on $\mathbb{R}^d$.
- A continuous function $\phi : \mathbb{R} \to \mathbb{R}$

Theorem

(S.Z.) The pair (C_m, ψ) gives rise to (in general non-Gaussian) Euclidean Algebraic QFT.

Constructing non-Gaussian Euclidean Algebraic QFTs

Sketch of proof:

- Denote the set of all open balls in $\mathbb{R}^d$ by B_d; identify B_d with $\mathbb{R}^d \times (0, \infty)$ and equip it with the Euclidean metric on the latter.
- Set
$$C_m(x, r, y, s) := \langle v_r 1_{b(x,r)}, C_m v_s 1_{b(y,s)} \rangle_{L^2(\mathbb{R}^d)},$$
where v_r is the reciprocal of the volume of a d-dimensional ball with radius r, and $1_{b(x,r)}$ is the indicator function of $b(x, r)$, the ball in $\mathbb{R}^d$ with center x and radius r.
- $C_m(x, r, y, s)$ is a symmetric positive definite function on B_d, hence it defines a Gaussian probability measure μ_{C_m} on $\mathbb{R}^{\mathrm{B}_d}$.

- Let $C(\mathrm{B}_d)$ be the space of the continuous real-valued functions on B_d equipped with the topology of uniform convergence on compact sets.
- By Dudley's theorem, the canonical random field X^{C_m} associated to μ_{C_m} has a modification with continuous paths, thus one obtains a Borel Gaussian probability measure $\widetilde{\mu}_{C_m}$ on $C(\mathrm{B}_d)$.
- Denote by $X^{C_m,\psi}$ the composition of the canonical random field associated to $\widetilde{\mu}_{C_m}$ with the self-map of $C(\mathrm{B}_d)$ induced by ψ.

Constructing non-Gaussian Euclidean Algebraic QFTs

Sketch of proof, cont.

- For every open $V \subset \mathbb{R}^d$ define $\mathcal{A}_V$ to be the $*$-algebra generated by the bounded operators on $L^2_{\mathbb{C}}(C(\mathrm{B}_d), \widetilde{\mu}_{C_m})$ acting as multiplication by $e^{iX_b^{C_m,\Psi}}$, where the ball b is contained in V.
- Set $\mathcal{H}_E := \overline{\mathcal{A}_{\mathbb{R}^d} 1}$ and define the $E(d)$-covariant family of subspaces $\mathcal{H}_V := \overline{\mathcal{A}_V 1}$ of $\mathcal{H}_E$.
- The natural representation of $E(d)$ on $\mathcal{H}_E$ is:
  - strongly continuous since $E(d)$ acts continuously on $C(\mathrm{B}_d)$,
  - unitary since $\widetilde{\mu}_{C_m}$ is $E(d)$-invariant,
  - reflection positive (with respect to any coordinate reflection θ) since the operator C_m is reflection positive.
- The net of algebras $V \mapsto \mathcal{A}_V$ is clearly commutative and $E(d)$-covariant.

Current and future research

- Modify the construction so that one can obtain the full set of Haag-Kastler axioms.
- Construct the ϕ^4-model in this setting, utilizing an appropriate stochastic partial "differential" equation.
- Develop (a version of) the Haag-Kastler scattering theory for specific examples and show non-triviality of the S-matrix.

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