On (some of) the uses of random unitary matrices

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Outline

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General definitions

- For $N \in \mathbb{N}$, let $\mathcal{U}(N)$ be the space of unitary matrices

$$\mathcal{U}(N) := \{M \in \mathcal{M}(N) \,/\, M^* M = MM^* = I_N\}$$

- Let $U = (U_{k,l})_{k,l \leqslant N}$ be a $N \times N$ unitary matrix with randomly distributed elements. This is a random variable $U : \Omega \longrightarrow \mathcal{U}(N)$ (with $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space) and the randomness is measured with respect to a certain probability measure $\mathbb{P}$.

- The most natural probability is the (normalised) Haar measure $\mathbb{P}_N$ of the group $\mathcal{U}(N)$.

- Vocabulary : if we endow $\mathcal{U}(N)$ with its normalised (probability) Haar measure $\mathbb{P}_N$, the probability space $(\mathcal{U}(N), \mathbb{P}_N)$ is called the **Circular Unitary Ensemble**, in short *$CUE(N)$*.

Probability on $\mathcal{U}(N)$

- A left Haar measure μ on a group G is a measure that is left invariant by elements of G :

$$\mu(gE) = \mu(E) \quad \forall\, g \in G, \quad \forall\, E \in \mathcal{F}$$

  A Haar measure is a left Haar measure and a right Haar measure. On a compact group, we can turn it into a probability measure by rescaling it with the total mass.

- A class function f on $\mathcal{U}(N)$ is invariant by conjugacy by an element of $\mathcal{U}(N)$, hence, only depends on the eigenvalues/eigenangles of the matrix. By abuse of notations, we write :

$$f(U) = f(V^{-1}UV) \equiv f(\theta_1, \dots, \theta_N) \quad \text{with } \mathrm{Sp}(U) = \left\{ e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_N} \right\}$$

The Haar measure on $\mathcal{U}(N)$

- One of the main tool to compute integrals on $\mathcal{U}(N)$ is the following :

Theorem (Weyl, 30')

Let f be a bounded class function (i.e. $f(V^{-1}UV) = f(U)$). Then, with an abuse of notation :

$$\begin{aligned}
\mathbb{E}_N(f(U)) &:= \int_{\mathcal{U}(N)} f(U)\, d\mathbb{P}_N(U) = \int_{\mathcal{U}(N)} f(V^{-1}\Theta V)\, d\mathbb{P}'_N(V, \Theta) \\
&\equiv \mathbb{E}_N(f(\theta_1, \dots, \theta_N)) \\
&= \frac{1}{N!} \int_{(-\pi,\pi]^N} f(\theta_1, \dots, \theta_N) \prod_{1 \leqslant k < \ell \leqslant N} \left| e^{\mathrm{i}\theta_\ell} - e^{\mathrm{i}\theta_k} \right|^2 \frac{d\theta_1 \cdots d\theta_N}{(2\pi)^N} \\
&=: \frac{1}{N!} \int_{(-\pi,\pi]^N} f(\theta_1, \dots, \theta_N) \left| \Delta(e^{i\theta_1}, \dots, e^{i\theta_N}) \right|^2 \frac{d\theta_1 \cdots d\theta_N}{(2\pi)^N}
\end{aligned}$$

Statistical mechanics intuition of the eigenangles

- The weight in Weyl's integration reads

$$\prod_{1\leqslant k<\ell\leqslant N}\left|e^{\mathrm{i}\theta_\ell}-e^{\mathrm{i}\theta_k}\right|^{2}=\left|\Delta\left(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_N}\right)\right|^{2}$$

with Δ the Vandermonde determinant. It can be rewritten into

$$\exp\left(-2\sum_{1\leqslant k<\ell\leqslant N}-\log\left|1-e^{\mathrm{i}(\theta_k-\theta_\ell)}\right|\right)=\exp\left(-2\sum_{1\leqslant k<\ell\leqslant N}H(\theta_k-\theta_\ell)\right)$$

- This is the interaction potential of a Coulomb gaz of N repelling particules confined on a circle at “inverse temperature" 2. The electrostatic potential prevents the particles from clustering (they “repell” from each other like fermions).

Orthogonal polynomials for $L^2(\mathbb{P}_N)$

- The orthogonal polynomials for the Haar measure of $\mathcal{U}_N$ are the **Schur polynomials** defined by

$$s_\lambda(z_1, \dots, z_N) := \frac{\det\left(z_i^{\lambda_j + N - j}\right)_{1 \leqslant i,j \leqslant N}}{\Delta(z_1, \dots, z_N)}, \qquad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_L \geqslant 0$$

- These polynomials are *symmetric* (as $Z \mapsto |\Delta(Z)|^2$ is symmetric). Here, λ is a partition with length $L := \ell(\lambda) \leqslant N$. One completes the partition with 0's to have a length N.

- The orthogonality relation reads ($\boldsymbol{u} := (u_1, \dots, u_N)$, $d^*x := \frac{dx}{2\mathbf{i}\pi}$)

$$\int_{\mathcal{U}_N} s_\lambda(U) s_\mu(U^{-1}) dU = \frac{1}{N!} \oint_{\mathbb{U}^N} s_\lambda(\boldsymbol{u}) s_\mu(\boldsymbol{u}^{-1}) \left|\Delta(\boldsymbol{u})\right|^2 \frac{d^*\boldsymbol{u}}{\boldsymbol{u}} = \delta_{\lambda,\mu} \mathbb{1}_{\{\ell(\lambda) \leqslant N\}}$$

How to compute on $(\mathcal{U}(N), \mathbb{P}_N)$?

1. analysis of Toeplitz and Fredholm determinants (Onsager, Szegö, Johansson, Deift-Its-Krasovsky, Borodin-Okounkov),
2. Orthogonal Polynomials on the Unit Circle (Killip-Nenciu, Najnudel),
3. mathematical physics (Kostov) and supersymmetry (Conrey-Farmer-Zirnbauer, Fyodorov),
4. integrable systems (Adler-Van Moerbeke, Forrester),
5. algebraic combinatorics (Biane, Dehaye, Féray),
6. representation theory and symmetric functions (Bump-Gamburd, Diaconis-Shahshahani, Dehaye, B.-A.),
7. probability theory (Bourgade-Hughes-Nikeghbali-Yor, B.-A.),
8. Integrable/determinantal probability (Borodin-Strahov),
9. Weingarten calculus (Collins, Weingarten, Matsumoto-Novak),
10. Itô calculus (Bru, Katori-Tanemura), etc.

Outline

Andréieff-Heine-Szegö and/or Cauchy-Binet formula

Theorem

- ***Cauchy-Binet formula*** *(1815) : if* $A, B \in \mathcal{M}_{k\times N}$

$$\det(AB^T)_{k\times k} = \det\left(\sum_{\ell=1}^{N} A_{i,\ell} B_{j,\ell}\right)_{1\leqslant i,j\leqslant k}$$

$$= \sum_{1\leqslant \ell_1 < \cdots < \ell_k \leqslant N} \det\left(A_{i,\ell_j}\right)_{1\leqslant i,j\leqslant k} \det\left(B_{i,\ell_j}\right)_{1\leqslant i,j\leqslant k}$$

$$= \frac{1}{k!} \sum_{1\leqslant \ell_1, \dots, \ell_k \leqslant N} \det\left(A_{i,\ell_j}\right)_{1\leqslant i,j\leqslant k} \det\left(B_{i,\ell_j}\right)_{1\leqslant i,j\leqslant k}$$

- ***Andréieff/Heine-Szegö identity*** *(1886) : if* $f_i, g_i \in L^2(\mathfrak{X}, \mu)$

$$\det\left(\int_{\mathfrak{X}} f_i(x) g_j(x)\, d\mu(x)\right)_{1\leqslant i,j\leqslant k}$$

$$= \frac{1}{k!} \int_{\mathfrak{X}^k} \det(f_i(x_j))_{1\leqslant i,j\leqslant k} \det(g_i(x_j))_{1\leqslant i,j\leqslant k}\, d\mu^{\otimes k}(\boldsymbol{x})$$

- Specialising the AHS formula with the discrete measure $\mu = \sum_{\ell=1}^N \delta_\ell$ gives the Cauchy-Binet formula.
- Taking a Riemann sum approximation in the Cauchy-Binet formula gives the AHS formula. The two formuli are thus equivalent.
- The proof is very simple :

$$\begin{aligned}
&\frac{1}{k!}\int_{\mathfrak{X}} \det(f_i(x_j))_{1\leqslant i,j\leqslant k} \det(g_i(x_j))_{1\leqslant i,j\leqslant k}\, d\mu^{\otimes k}(\boldsymbol{x}) \\
&\qquad = \frac{1}{k!}\sum_{\sigma,\tau\in\mathfrak{S}_k} \varepsilon(\sigma)\varepsilon(\tau)\int_{\mathfrak{X}^k}\prod_{i=1}^k f_{\sigma(i)}(x_i) g_{\tau(i)}(x_i)\, d\mu(x_i) \\
&\qquad = \frac{1}{k!}\sum_{\sigma,\tau\in\mathfrak{S}_k} \varepsilon(\sigma\tau^{-1})\prod_{j=1}^k \int_{\mathfrak{X}} f_{\sigma\tau^{-1}(j)}(x) g_j(x)\, d\mu(x) \\
&\qquad = \det\left(\int_{\mathfrak{X}} f_i(x) g_j(x)\, d\mu(x)\right)_{1\leqslant i,j\leqslant k}
\end{aligned}$$

Applications (1/2)

- Recall that

$$\Delta(z_1, \dots, z_N) = \det\left(z_i^{j-1}\right)_{1\leqslant i,j\leqslant N} =: \det(f_j(z_i))_{i,j}$$

- Proof of the orthogonality of Schur functions, with $\ell(\lambda), \ell(\mu) \leqslant N$:

$$\begin{aligned}
\langle s_\lambda, s_\mu \rangle_{L^2(\mathcal{U}_N)} &= \frac{1}{N!} \oint_{\mathbb{U}^N} s_\lambda(\boldsymbol{z}) s_\mu(\boldsymbol{z}^{-1}) \, |\Delta(\boldsymbol{z})|^2 \, \frac{d^*\boldsymbol{z}}{\boldsymbol{z}} \\
&= \frac{1}{N!} \oint_{\mathbb{U}^N} \det\left(z_i^{\lambda_j+N-j}\right)_{1\leqslant i,j\leqslant N} \det\left(z_i^{-(\mu_j+N-j)}\right)_{1\leqslant i,j\leqslant N} \frac{d^*\boldsymbol{z}}{\boldsymbol{z}} \\
&\overset{(AHS)}{=} \det\left(\oint_{\mathbb{U}} z^{\lambda_i+N-i-(\mu_j+N-j)} \frac{d^*z}{z}\right)_{1\leqslant i,j\leqslant N} \\
&= \det\left(\delta_{\lambda_i+N-i,\mu_j+N-j}\right)_{1\leqslant i,j\leqslant N} = (\lambda,\mu)\text{-minor of } I_{N+|\lambda|} \\
&= \prod_{k=1}^{N} \delta_{\lambda_i-i,\mu_i-i} \\
&=: \delta_{\lambda,\mu}
\end{aligned}$$

Applications (2/2), the *Toeplitz connection*

- Setting $f = g^{\otimes N}$ in the Weyl formula gives

$$\begin{aligned}
\int_{\mathcal{U}_N} \det(g(U))dU &= \frac{1}{N!}\oint_{\mathbb{U}^N} |\Delta(Z)|^2 \prod_{k=1}^N g(z_k)\frac{dz_k}{2\mathbf{i}\pi z_k} \\
&= \frac{1}{N!}\oint_{\mathbb{U}^N} \Delta(Z)\Delta(Z^{-1}) \prod_{k=1}^N g(z_k)\frac{dz_k}{2\mathbf{i}\pi z_k} \\
&= \frac{1}{N!}\oint_{\mathbb{U}^N} \det\left(z_i^{j-1}\right)_{i,j\leqslant N} \det\left(z_i^{-(j-1)}\right)_{i,j\leqslant N} \prod_{k=1}^N g(z_k)\frac{d^* z_k}{z_k} \\
&\overset{(AHS)}{=} \det\left(\oint_{\mathbb{U}} z^{i-j} g(z)\frac{d^* z}{z}\right)_{1\leqslant i,j\leqslant N} \\
&=: \det(\boldsymbol{T}_N(g)) =: \boldsymbol{D}_N(g)
\end{aligned}$$

- $\boldsymbol{T}_N(g)$ is a *Toeplitz matrix* and $\boldsymbol{D}_N(g)$ is a *Toeplitz determinant* with *symbol* g.

Applications (2/2), the *Toeplitz connection*, example

- For $g(z) = e^{x(z+z^{-1})}$, one has $\det(g(U)) = e^{x \operatorname{tr}(U+U^{-1})}$ hence

$$\mathbb{E}\left(e^{x \operatorname{tr}(U_N + U_N^{-1})}\right) := \int_{\mathcal{U}_N} e^{x(\operatorname{tr}(U)+\operatorname{tr}(U^{-1}))} dU$$
$$= \det\left(\oint_{\mathbb{U}} z^{i-j} e^{x(z+z^{-1})} \frac{dz}{2\mathbf{i}\pi z}\right)_{1\leqslant i,j\leqslant N}$$
$$=: \det(J_{i-j}(2x))_{1\leqslant i,j\leqslant N}$$

  where $J_x(z)$ is the Bessel function of first kind.
- The **Strong Szegö Theorem** gives for $f(z) = \sum_{k\in\mathbb{Z}} f_k z^k$

$$\boldsymbol{D}_n(e^f) = e^{nf_0 + V(f) + o(1)}, \qquad V(f) = \sum_{k\geqslant 1} k f_k f_{-k} \quad \text{if} \quad \sum_k \left|\sqrt{k}\, f_k\right|^2 < \infty$$

- This translates into a probabilistic result, a Central Limit Theorem for the real part of the trace of $U_N \sim CUE(N)$:

$$\mathbb{E}\left(e^{\frac{x}{2} \operatorname{tr}(U_N + U_N^{-1})}\right) := \mathbb{E}\left(e^{x \,\mathfrak{Re}(\operatorname{tr}(U_N))}\right) = e^{\frac{x^2}{2} + o(1)}$$

Outline

OPUCs

- The AHS formula allows one more description of a Toeplitz determinant. Taking linear combinations of lines and/or columns in the Vandermonde determinant gives

$$
\begin{aligned}
\int_{\mathcal{U}_N} \det(g(U))dU &= \frac{1}{N!}\oint_{\mathbb{U}^N} |\Delta(Z)|^2 \prod_{k=1}^N g(z_k)\frac{dz_k}{2\mathbf{i}\pi z_k} \\
&= \frac{1}{N!}\oint_{\mathbb{U}^N} \det\left(z_i^{j-1}\right)_{i,j\leqslant N} \det\left(z_i^{-(j-1)}\right)_{i,j\leqslant N} \prod_{k=1}^N g(z_k)\frac{d^*z_k}{z_k} \\
&= \frac{1}{N!}\oint_{\mathbb{U}^N} \det(P_{j-1}(z_i))_{i,j\leqslant N} \det\left(\overline{Q_{j-1}(z_i)}\right)_{i,j\leqslant N} g^{\otimes N}(\boldsymbol{z})\frac{d^*\boldsymbol{z}}{\boldsymbol{z}} \\
&\overset{(AHS)}{=} \det\left(\oint_{\mathbb{U}} P_{i-1}(z)\overline{Q_{j-1}(z)}g(z)\frac{d^*z}{z}\right)_{1\leqslant i,j\leqslant N} \\
&= \det\left(\langle P_{i-1}, Q_{j-1}\rangle_{L^2(g\bullet\mathrm{Leb})}\right)_{1\leqslant i,j\leqslant N} \qquad \text{(Gram matrix)}
\end{aligned}
$$

- P_j and Q_j are monic and of degree j.

- One can choose $P_j = Q_j$ to be orthogonal in $L^2(g(e^{i\theta})d\theta)$ if g is an acceptable function (non negative integrable on $\mathbb{U}$). The Gram matrix then becomes diagonal :

$$\int_{\mathcal{U}_N} \det(g(U))dU = \det\Big(\langle P_{i-1}, P_{j-1}\rangle_{L^2(g\bullet \mathrm{Leb})}\Big)_{1\leqslant i,j\leqslant N} = \prod_{k=0}^{N-1} \|P_k\|^2_{L^2(g(e^{i\theta})d\theta)}$$

- The theory of Orthogonal Polynomials on the Unit Circle (OPUC) mimics the theory of Orthogonal Polynomials on the Real Line (OPRL) with a few differences.
- Such computations were used e.g. in the study of partition functions in 2D Yang-Mills theory :

D. J. Gross, A. Matytsin, *Instanton induced large N phase transitions in two and four dimensional QCD*, Nucl. Phys. B 429(1):50-74 (**1994**).

D. J. Gross, E. Witten, *Possible third-order phase transition in the large-N lattice gauge theory*, Phys. Rev. D 21(2):446-453 (**1980**).

Outline

Cauchy identities and specialisations

- The Cauchy identity (Cauchy-Binet formula for the Cauchy determinant) reads

$$H[XY] := \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(X) s_\lambda(Y)$$

- By taking logarithms and supposing $|x_i y_j| < 1$, one also has

$$H[XY] = \exp\left(\sum_{k \geqslant 1} \frac{1}{k} p_k(X) p_k(Y) \right), \quad p_k(X) = \sum_j x_j^k$$

- Taking this last expansion as a definition of $H[XY]$, one can *specialise* the $p_k(X)$ in any possible way to get an identity involving Schur functions (there is an exact formula involving the s_λ's and the p_k's).

Character expansion

- We now define

$$H\left[X\mathcal{A}\right] = \exp\left(\sum_{k\geqslant 1}\frac{1}{k}p_k(X)p_k\left[\mathcal{A}\right]\right) = \sum_{\lambda} s_\lambda(X)s_\lambda\left[\mathcal{A}\right]$$

and consider

$$\int_{\mathcal{U}_N} H\left[\mathcal{A}U\right] H\left[\mathcal{B}U^{-1}\right] dU = \frac{1}{N!}\oint_{\mathbb{U}^N} H\left[\mathcal{A}\boldsymbol{u}\right] H\left[\mathcal{B}\boldsymbol{u}^{-1}\right] |\Delta(\boldsymbol{u})|^2 \frac{d^*\boldsymbol{u}}{\boldsymbol{u}}$$

- The ***character expansion formula*** is

$$\int_{\mathcal{U}_N} H\left[\mathcal{A}U\right] H\left[\mathcal{B}U^{-1}\right] dU = \sum_{\ell(\lambda)\leqslant N} s_\lambda\left[\mathcal{A}\right] s_\lambda\left[\mathcal{B}\right]$$

- It is obtained using the Cauchy formula and the orthogonality of the Schur functions (characters of $\mathcal{U}_N$).

The Schur measure

- Suppose that $s_\lambda[\mathcal{A}]\, s_\lambda[\mathcal{B}] \geqslant 0$. One can define the following measure on partitions of an integer $\mathbb{Y} := \uplus_{n\geqslant 0}\{\lambda : |\lambda| = n\}$ $(|\lambda| := \sum_k \lambda_k)$

$$\mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\{\lambda\}) := \frac{1}{H[\mathcal{A}\mathcal{B}]} s_\lambda[\mathcal{A}]\, s_\lambda[\mathcal{B}]$$

- This is the **Schur measure** introduced in :

A. Okounkov, *Infinite wedge and random partitions*, Selecta Math. New Ser. 7(1):57-81 (**2001**).

- If $\boldsymbol{\lambda}$ is the canonical random variable on $(\mathbb{Y}, \mathcal{F}, \mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})})$, one has

$$\int_{\mathcal{U}_N} H[\mathcal{A}U]\, H\left[\mathcal{B}U^{-1}\right] dU = H[\mathcal{A}\mathcal{B}]\, \mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\ell(\boldsymbol{\lambda}) \leqslant N)$$

- If $f_{\mathcal{A},\mathcal{B}}(z) = \sum_{k\geqslant 1}\left(\frac{p_k[\mathcal{A}]}{k} z^k + \frac{p_k[\mathcal{B}]}{k} z^{-k}\right)$ the **Strong Szegö Theorem** reads

$$\boldsymbol{D}_N\left(e^{f_{\mathcal{A},\mathcal{B}}}\right) = H[\mathcal{A}\mathcal{B}]\, \mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\ell(\boldsymbol{\lambda}) \leqslant N) \xrightarrow[N\to+\infty]{} H[\mathcal{A}\mathcal{B}]$$

The Borodin-Okounkov formula

- This is the fundamental **link Toeplitz-Fredholm** :

$$\boldsymbol{D}_N\left(e^{f_{\mathcal{A},\mathcal{B}}}\right) = H\left[\mathcal{A}\mathcal{B}\right] \det(I - \boldsymbol{\mathcal{K}}_{\mathcal{A},\mathcal{B}})_{\ell^2(N+\mathbb{N}^*)}$$

$$''\Longleftrightarrow'' \qquad \mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\ell(\boldsymbol{\lambda}) \leqslant N) = \det(I - \boldsymbol{\mathcal{K}}_{\mathcal{A},\mathcal{B}})_{\ell^2(N+\mathbb{N}^*)}$$

  where $N + \mathbb{N}^* = \{N, N+1, N+2, \dots\}$, $\boldsymbol{\mathcal{K}}_{\mathcal{A},\mathcal{B}}$ is an operator acting on $\ell^2(N+\mathbb{N}^*)$ with explicit (half-)infinite matrix $\mathcal{K}_{\mathcal{A},\mathcal{B}}[i,j]$.

- The **Fredholm determinant** is defined with $\mathrm{tr}(A) = \sum_m A[m,m]$ and

$$\det(I + x\boldsymbol{\mathcal{K}})_{\ell^2(\mathfrak{X})} := \sum_{k\geqslant 0} x^k \,\mathrm{tr}\left(\wedge^k \boldsymbol{\mathcal{K}}\right)_{\wedge^k \ell^2(\mathfrak{X})}$$

$$= \sum_{k\geqslant 0} \frac{x^k}{k!} \sum_{m_1,\dots,m_k\in\mathfrak{X}} \det(\mathcal{K}[m_i, m_j])_{1\leqslant i,j\leqslant k}$$

A. Borodin, A. Okounkov, *A Fredholm determinant formula for Toeplitz determinants*, Integral Equations Operator Theory 37(4):386-396 (**2000**).

A quick overview of the proof (1/3)

- With the ω involution of symmetric functions (replacing the alphabets $\mathcal{A}, \mathcal{B}$ with their supersymmetric counterparts), one has to prove

$$\mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\boldsymbol{\lambda}_1 \leqslant N) = \det(I - \boldsymbol{\mathcal{K}}_{\mathcal{A},\mathcal{B}})_{\ell^2(N+\mathbb{N}^*)}$$

- The inclusion-Exclusion formula reads

$$\begin{aligned}
\mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\boldsymbol{\lambda}_1 \leqslant N) &= \mathbb{P}(\forall\, k \geqslant 1, \boldsymbol{\lambda}_k \leqslant N) \\
&= \sum_{k \geqslant 0} (-1)^k \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k} \mathbb{P}(\boldsymbol{\lambda}_{i_1} > N, \ldots, \boldsymbol{\lambda}_{i_k} > N) \\
&= \sum_{k \geqslant 0} \frac{(-1)^k}{k!} \sum_{1 \leqslant i_1 \neq i_2 \neq \cdots \neq i_k} \mathbb{P}(\boldsymbol{\lambda}_{i_1} > N, \ldots, \boldsymbol{\lambda}_{i_k} > N) \\
&= \sum_{k \geqslant 0} \frac{(-1)^k}{k!} \sum_{m_1, \ldots, m_k > N} \\
&\qquad \sum_{1 \leqslant i_1 \neq i_2 \neq \cdots \neq i_k} \mathbb{P}(\boldsymbol{\lambda}_{i_1} = m_1, \ldots, \boldsymbol{\lambda}_{i_k} = m_k)
\end{aligned}$$

A quick overview of the proof (2/3)

- The ***correlation functions*** are defined by

$$\rho_k(m_1, \dots, m_k) := \sum_{1 \leqslant i_1 \neq i_2 \neq \cdots \neq i_k} \mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\boldsymbol{\lambda}_{i_1} = m_1, \dots, \boldsymbol{\lambda}_{i_k} = m_k)$$
$$= \mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}\left(\boldsymbol{\lambda} \supset \{m_1, \dots, m_k\}\right)$$

  and one wants to write them as a determinant.
- The Schur functions are also given by the Jacobi-Trudi formula in vertex form as scalar products on the fermionic Fock space $\oplus_\lambda \mathbb{C}|\lambda\rangle$:

$$s_\lambda[\mathcal{A}] = \langle \lambda | \mathcal{T}_\mathcal{A} | \varnothing \rangle , \qquad \mathcal{T}_\mathcal{A} := e^{\sum_{k \geqslant 1} \frac{p_k[\mathcal{A}]}{k} \alpha_k}$$

- The bosons (α_k) satisfy $[\alpha_k, \alpha_\ell] = k\delta_{k,-\ell}$ and are *quadratic forms* in the fermions $(\psi_k)_k$ (***Boson-Fermion correspondance***) :

$$\alpha_n = \sum_{k \in \mathbb{Z}} \psi_{k-n} \psi_k^*$$

A quick overview of the proof (3/3)

- The 1-dimensional projection $\psi_k\psi_k^*$ acts on $|\lambda\rangle$ with eigenvalue

$$\psi_k\psi_k^*|\lambda\rangle = \mathbb{1}_{\{k\in\lambda\}}|\lambda\rangle$$

- The correlation functions $\mathbb{P}_{\mathrm{Schur}(\mathcal{A},\mathcal{B})}(\boldsymbol{\lambda} \supset \{m_1, \dots, m_k\})$ thus read

$$\begin{aligned}
\rho_k(m_1,\dots,m_k) &= \frac{1}{H[\mathcal{A}\mathcal{B}]}\sum_{\lambda\supset M} \langle\lambda|\mathcal{T}_\mathcal{A}|\varnothing\rangle\,\langle\varnothing|\mathcal{T}_\mathcal{B}^*|\lambda\rangle \\
&= \frac{\left\langle\varnothing\middle|\mathcal{T}_\mathcal{B}^*\sum_{\lambda\supset M}\langle\lambda|\mathcal{T}_\mathcal{A}|\varnothing\rangle\,\middle|\lambda\right\rangle}{H[\mathcal{A}\mathcal{B}]} \\
&= \frac{\left\langle\varnothing\middle|\mathcal{T}_\mathcal{B}^*\sum_{\lambda}\langle\lambda|\mathcal{T}_\mathcal{A}|\varnothing\rangle\prod_{k\in M}\psi_k\psi_k^*\middle|\lambda\right\rangle}{H[\mathcal{A}\mathcal{B}]} \\
&= \frac{\left\langle\varnothing\middle|\mathcal{T}_\mathcal{B}^*\sum_{\lambda}\langle\lambda|\mathcal{T}_\mathcal{A}|\varnothing\rangle\prod_{k\in M}\psi_k\psi_k^*\middle|\lambda\right\rangle}{H[\mathcal{A}\mathcal{B}]} \\
&= \frac{\left\langle\varnothing\middle|\mathcal{T}_\mathcal{B}^*\prod_{k\in M}\psi_k\psi_k^*\mathcal{T}_\mathcal{A}\middle|\varnothing\right\rangle}{H[\mathcal{A}\mathcal{B}]}, \quad \mathcal{T}_\mathcal{A} = e^{\sum_k \frac{p_k[\mathcal{A}]}{k}\alpha_k(\psi,\psi^*)}
\end{aligned}$$

- One concludes with the (fermionic) Wick formula.

Outline

Definition

- The Ising model we consider is the *scalar one* defined by 2-dimensional "spins" $\sigma_{i,j} \in \{\pm 1\}$ with $1 \leqslant i, j \leqslant N$. One can consider Pauli matrices instead (Ising Spin chain) but we keep the analysis simple (historically, this is the model first considered by Ising and Lenz).
- The model is defined by a probability measure on the set of configurations $\boldsymbol{\sigma} = (\sigma_{i,j})_{i,j \in [N]^2} \in \{\pm 1\}^{[N]^2}$ given by

$$\mathbb{P}_\beta(\boldsymbol{\sigma}) := \frac{1}{\mathcal{Z}_N(\beta)} e^{-\beta \mathcal{H}(\boldsymbol{\sigma})}, \qquad \mathcal{Z}_N(\beta) := \sum_{\boldsymbol{\sigma} \in \{\pm 1\}^{[N]^2}} e^{-\beta \mathcal{H}(\boldsymbol{\sigma})}$$

- $\beta = (k_B T)^{-1}$ is the *inverse temperature* and

$$\mathcal{H}(\boldsymbol{\sigma}) := -\sum_{a,b \in [N]^2} J_{a,b} \sigma_a \sigma_b, \qquad J_{a,b} := \mathbb{1}_{\{a \sim b\}} = \mathbb{1}_{\{|a-b|=1\}}$$

- One can take periodic boundary conditions (replacing $[N]^2$ by the torus $(\mathbb{Z}/N\mathbb{Z})^2$) or 0, etc. There is no external field.

The Onsager formula

- One is interested in computing the Spin-Spin diagonal and horizontal correlation

$$\mathbb{E}_\beta\left(\boldsymbol{\sigma}_{1,1}\boldsymbol{\sigma}_{N,N}\right) \equiv \left\langle \boldsymbol{\sigma}_{1,1}\boldsymbol{\sigma}_{N,N} \right\rangle_\beta = \frac{1}{\mathcal{Z}_N(\beta)} \sum_{\boldsymbol{\sigma}\in\{\pm 1\}^{[N]^2}} \boldsymbol{\sigma}_{1,1}\boldsymbol{\sigma}_{N,N}\, e^{-\beta\mathcal{H}(\boldsymbol{\sigma})}$$

$$\mathbb{E}_\beta\left(\boldsymbol{\sigma}_{1,1}\boldsymbol{\sigma}_{1,N}\right) \equiv \left\langle \boldsymbol{\sigma}_{1,1}\boldsymbol{\sigma}_{1,N} \right\rangle_\beta$$

- The link with Random Unitary Matrices is provided by the

Theorem (Onsager (1942), Onsager-Kauffman (1948))

For $\kappa = \sinh(\beta)^{-2}$ *and* $\gamma_1 = \tanh(\beta)$, γ_2 *explicit,*

$$\left\langle \boldsymbol{\sigma}_{1,1}\boldsymbol{\sigma}_{N,N} \right\rangle_\beta = \int_{\mathcal{U}_{N-1}} \det(\varphi_{\mathrm{diag}}(U))\,dU, \quad \varphi_{\mathrm{diag}}(z) = \left(\frac{1-\kappa z^{-1}}{1-\kappa z}\right)^{1/2}$$

$$\left\langle \boldsymbol{\sigma}_{1,1}\boldsymbol{\sigma}_{1,N} \right\rangle_\beta = \int_{\mathcal{U}_{N-1}} \det(\varphi_{\mathrm{Ons}}(U))\,dU, \quad \varphi_{\mathrm{Ons}}(z) = \left(\frac{1-\gamma_1 z}{1-\frac{\gamma_1}{z}}\,\frac{1-\frac{\gamma_2}{z}}{1-\gamma_2 z}\right)^{1/2}$$

- This is equivalent to a Toeplitz determinant identity (original formula).
- Proven with numerous methods (transfer matrix, mapping to a polygon problem, mapping to a dimer covering problem, Kac-Ward identities, etc.). A good survey is given by :

P. Deift, A. Its, I. Krasovsky, *Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model: some history and some recent results*, CMP 66(9):1360-1438 (**2013**).

- One wants to compute the limiting correlation (when $N \to +\infty$) to see if a phase transition happens in β (difference of behaviour depending on the value of β, e.g. a break in analyticity). This is done by using the Strong Szegö theorem if the conditions on the symbol are fullfilled (this is the case if $\kappa, \gamma < 1$).
- The Borodin-Okounkov formula is the “best” way to prove the result.

Citation from Deift-Its-Krasovskv :

The mathematical challenge that is posed by SSLT can be viewed as follows. In functional analysis one commonly considers continuous maps F, say, from one fixed topological space X to another fixed topological space Y. Then in order to verify that $\lim_{n\to\infty} F(x_n) = F(x)$ all one has to do is verify that $x_n \to x$ as $n \to \infty$ in the topology of X. In particular, if X is the space of trace class operators on a separable Hilbert space $\mathcal{H}$, and $F(x) = \det(1+x)$, then $F : X \to Y = \mathbb{C}$ and $\lim_{n\to\infty} \det(1+x_n) = \det(1+x)$ if $x_n \to x$ in trace norm. But SSLT is not of this standard type. Instead, one has $F : M(n,\mathbb{C}) \to \mathbb{C}$ taking $A_n \in M(n,\mathbb{C})$ to $F(A_n) = \det A_n$. So the initial space $X = X_n = M(n,\mathbb{C})$ is changing with n and it is not at all clear how to topologize the situation so that "$\lim_n F(A_n) = F(\lim A_n)$". If, for example, we let $X = M(\infty,\mathbb{C})$ be the inductive limit of the $M(n,\mathbb{C})$'s, then $T_n(\varphi)$ converges to the Toeplitz operator $T(\varphi)$ in the associated topology, but $T(\varphi)$ is not of the form $1+$ trace class, and so $F(T(\varphi)) = \det(T(\varphi))$ is not defined. In such situations when $X = X_n$, or $Y = Y_n$, is varying, there is no general procedure to follow and each problem must be addressed on an ad hoc basis. And when there is no preferred path to the solution of a problem, there are many paths, as we see from the six very different methods above.

In this regard, the method of Borodin-Okounkov-Geronimo-Case is singled out: after factoring out $e^{n(\log\varphi)_0 + E(\varphi)}$, one has

(72)
$$\frac{\det(T_n(\varphi))}{e^{n(\log\varphi)_0 + E(\varphi)}} = \det(1 - B_n)$$

where $B_n = Q_n\, H(b)\, H(\widetilde{c})\, Q_n$ is trace class in ℓ_2^+ and $B_n \to 0$ in trace norm as $n \to \infty$. In other words, one of the paths is to renormalize the problem so that it assumes the standard form.

Outline

- Another important tool that constitutes the main alternative to the Toeplitz-Fredholm analysis concerns representation theory.
- The Strong Szegö Theorem can be probabilistically recast into a convergence in distribution for particular *linear statistics*, i.e. functionals of the form $\mathrm{tr}(f(U_N)) = \sum_{k=1}^n f(e^{i\boldsymbol{\theta}_k})$. In particular

$$\int_{\mathcal{U}_N} H\left[\mathcal{A}U\right] H\left[\mathcal{B}U^{-1}\right] dU = \int_{\mathcal{U}_N} e^{\sum_{k\geqslant 1} \frac{p_k[\mathcal{A}]}{k} \mathrm{tr}(U^k) + \sum_{k\geqslant 1} \frac{p_k[\mathcal{B}]}{k} \mathrm{tr}(U^{-k})} dU$$

$$\xrightarrow[N\to+\infty]{} H\left[\mathcal{A}\mathcal{B}\right] = e^{\sum_{k\geqslant 1} \frac{1}{k} p_k[\mathcal{A}] p_k[\mathcal{B}]}$$

- If $p_k\left[\mathcal{A}\right] = p_k\left[\mathcal{B}\right] = k\frac{x_k}{2}$, one has

$$\int_{\mathcal{U}_N} e^{\sum_{k\geqslant 1} x_k \mathfrak{Re}(\mathrm{tr}(U^k))} dU \xrightarrow[N\to+\infty]{} e^{\sum_{k\geqslant 1} k\frac{x_k^2}{2}} = \mathbb{E}\left(e^{\sum_{k\geqslant 1} x_k\sqrt{k}G_k}\right)$$

where $(G_k)_k$ are independent Gaussians $\mathscr{N}(0,1)$.

Schur-Weyl duality

- One can bypass completely the analysis of Toeplitz-Fredholm with representation theory.
- The link between power functions and Schur functions is given by the following result :

Theorem (Schur-Weyl duality for $(\mathfrak{S}_N, \mathcal{U}_N)$)

One has

$$p_\mu = \sum_{\lambda \vdash N} \chi^\lambda_\mu s_\lambda, \qquad p_\mu := \prod_{k \geqslant 1} p_{\mu_k} \qquad \Longrightarrow \qquad \chi^\lambda_\mu = \langle s_\lambda, p_\mu \rangle_{L^2(\mathcal{U}_N)}$$

- The χ^λ are the irreducible characters of the symmetric group $\mathfrak{S}_N$. They are evaluated in any permutation σ having the *cycle type* μ.
- This is the character restatement of a more general result on the decomposition of the bimodule $\mathfrak{S}_k \curvearrowright (\mathbb{C}^N)^{\otimes k} \curvearrowleft \mathcal{U}_N$.

The Diaconis-Shashahani theorem

- Using the *orthogonality of characters* of $\mathfrak{S}_N$, one has

$$\int_{\mathcal{U}_N} \prod_{k\geqslant 1} \operatorname{tr}(U^{\mu_k}) \operatorname{tr}\left(U^{-\lambda_k}\right) dU = \frac{1}{N!} \oint_{\mathbb{U}^k} p_\mu(\boldsymbol{u}) p_\lambda(\boldsymbol{u}^{-1}) \left|\Delta(\boldsymbol{u})\right|^2 \frac{d^*\boldsymbol{u}}{\boldsymbol{u}}$$

$$= \sum_{\nu,\alpha\vdash N} \chi_\mu^\nu \chi_\lambda^\alpha \int_{\mathcal{U}_N} s_\nu(U) s_\alpha(U^{-1}) dU$$

$$= \sum_{\nu\vdash N} \chi_\mu^\nu \chi_\lambda^\nu$$

$$= \delta_{\lambda,\mu} z_\lambda \mathbb{1}_{\{\ell(\lambda)\leqslant N\}}, \quad z_\lambda := \prod_{k\geqslant 1} k^{m_k(\lambda)} m_k(\lambda)!$$

- Write $\lambda = (\lambda_1, \lambda_2, \dots) = \langle 1^{m_1}, 2^{m_2}, \dots\rangle$ where $m_k \equiv m_k(\lambda)$ are the multiplicities of k in λ. Then

$$p_\lambda = \prod_k p_{\lambda_k} = \prod_j p_j^{m_j(\lambda)}$$

- One can thus rephrase the last result into

$$\mathbb{E}\left(\prod_{j\geqslant 1} \operatorname{tr}\left(U_N^j\right)^{m_j} \overline{\operatorname{tr}\left(U_N^j\right)^{m'_j}}\right) = \delta_{\boldsymbol{m},\boldsymbol{m}'} \prod_{j\geqslant 1} j^{m_j} m_j! \, \mathbb{1}_{\{\sum_k m_k \leqslant N\}}$$

$$= \mathbb{E}\left(\prod_{j\geqslant 1} \left(\sqrt{j} Z_j\right)^{m_j} \overline{\left(\sqrt{j} Z'_j\right)^{m'_j}}\right) \mathbb{1}_{\{|\boldsymbol{m}|\leqslant N\}}$$

and $(Z_1, \dots, Z_N)$, $(Z'_1, \dots, Z'_N)$ are independent complex Gaussians.

- The algebraic equivalent of the Strong Szegö Theorem is :

Theorem (Diaconis-Shashahani, 1994)

$$\mathbb{E}\left(\prod_{j\geqslant 1} \operatorname{tr}\left(U_N^j\right)^{m_j} \overline{\operatorname{tr}\left(U_N^j\right)^{m'_j}}\right) \xrightarrow[N\to+\infty]{} \mathbb{E}\left(\prod_{j\geqslant 1} \left(\sqrt{j} Z_j\right)^{m_j} \overline{\left(\sqrt{j} Z'_j\right)^{m'_j}}\right)$$

P. Diaconis, M. Shahshahani, *On the Eigenvalues of Random Matrices*, J. Appl. Prob. 31:49-62 (**1994**).

Outline

Characters of the Hecke algebra (1/3)

- A q-deformation of the Schur-Weyl duality was implemented in :

A. Ram, *A Frobenius formula for the characters of the Hecke algebra,* Invent. Math. 106(1):61-88 (**1991**).

Theorem (Ram, 1991)

(2.7) Theorem. Let $V = \mathbb{C}(q)^n$. *As an* $H_f \otimes U_q(\mathfrak{sl}(n))$ *representation*

$$V^{\otimes f} \cong \bigoplus_{\substack{\lambda \vdash f \\ \ell(\lambda) \leq n}} H_\lambda \otimes V_\lambda ,$$

where H_λ *is a irreducible* H_f *representation and* V_λ *is an irreducible* $U_q(\mathfrak{sl}(n))$ *representation.*

- $H_f \equiv \mathcal{H}_f(q)$ is the Hecke algebra, and $U_q(\mathfrak{sl}_n)$ a quantum group.

Characters of the Hecke algebra (2/3)

- Ram computes moreover the character of $(\mathbb{C}(q)^n)^{\otimes k}$ and of V^λ :

Theorem (Ram, 1991)

$$\tilde{q}_r(x;q) = \frac{1}{q-1}\, h_r(X(q-1)),$$

$$\tilde{q}_\mu(x_1,\dots,x_n;q) = \prod_{i=1}^{\ell(\mu)} \tilde{q}_{\mu_i}(x;q).$$

We have the following Frobenius type formula for the characters of H_f.

(4.14) Theorem. *For each $\mu \vdash f$,*

$$\tilde{q}_\mu(x;q) = \sum_{\lambda \vdash f} \chi^\lambda(T_{\gamma_\mu}) s_\lambda(x),$$

where χ^λ denotes the irreducible character of H_f corresponding to λ and $s_\lambda(x)$ denotes the Schur function in the variables $x_1,\dots,x_n$.

- We thus have $\widetilde{q}_\lambda(X;q) = \frac{1}{(1-q)^{\ell(\lambda)}} h_\lambda\left[(q-1)X\right] \equiv \frac{1}{(1-q)^{\ell(\lambda)}} h_\lambda(qX/X)$ (supersymmetric h_λ in the notations of Macdonald).

Characters of the Hecke algebra (3/3)

- Finally, one gets

Theorem (Ram, 1991)

The irreducible character $\chi^{\lambda,q}$ of $\mathcal{H}_N(q)$ evaluated in a "minimal word with descent type μ" noted $M_\mu := T_{\sigma_\mu}$ (an equivalent of the conjugacy class in $\mathcal{H}_N(q)$)

$$\chi^{\lambda,q}(M_\mu) = \int_{\mathcal{U}_N} s_\lambda(U) \frac{1}{(1-q)^{\ell(\lambda)}} h_\mu\left[(q-1)\, U^{-1}\right] dU$$

- To compare with the Frobenius formula in $\mathfrak{S}_n$

$$\chi^\lambda(\sigma_\mu) = \int_{\mathcal{U}_N} s_\lambda(U) p_\mu\left(U^{-1}\right) dU$$

- This formula was also discovered (independently) by King-Wybourne (1990) and Kerov-Vershik (1989).

The Ocneanu trace/Jones polynomial

- For all $z \in \mathbb{C}$, one defines the trace τ_z on $\mathcal{H}_\infty(q) := \uplus_{k \geqslant 0} \mathcal{H}_k(q)$ by

$$\tau_z(ab) = \tau_z(ba) \qquad \text{(trace)}$$

$$\tau_z(\mathrm{id}) = 1$$

$$\tau_z(s_i h) = z\tau_z(h) \ \ \forall\, h \in \mathcal{H}_N(q), \ \ s_i = \text{generator of } \mathcal{H}_N(q)$$

- One can prove that

Theorem (Ocneanu, 1985)

$$\tau_z = \sum_{\lambda \vdash N} W_\lambda(q, z) \chi^{\lambda, q}$$

where, in λ-ring notation

$$W_\lambda(q, z) = s_\lambda\left[\frac{w - z}{1 - q}\right] \equiv s_\lambda\left(\tfrac{w}{1-q} / \tfrac{z}{1-q}\right), \qquad w := 1 - q + z$$

- As a result, the Ocneanu trace evaluated in a minimal word of descent type μ as the matrix integral

$$\begin{aligned}
\tau_z(M_\mu) &= \sum_{\lambda \vdash N} s_\lambda\left[\frac{w-z}{1-q}\right] \int_{\mathcal{U}_N} s_\lambda(U) \frac{h_\mu\left[(q-1)\,U^{-1}\right]}{(1-q)^{\ell(\mu)}}\, dU \\
&= \int_{\mathcal{U}_N} h_N\left[\frac{w-z}{1-q}\,U\right] \frac{h_\mu\left[(q-1)\,U^{-1}\right]}{(1-q)^{\ell(\mu)}}\, dU \\
&= \frac{h_\mu\left[(q-1)\frac{w-z}{1-q}\right]}{(1-q)^{\ell(\mu)}} \\
&= \frac{h_\mu\left[z-w\right]}{(1-q)^{\ell(\mu)}} \\
&= \prod_{k \geqslant 1} \frac{z^{\mu_k}(1-w/z)}{1-q} \\
&= z^{|\mu|-\ell(\mu)}
\end{aligned}$$

- This is a result of Ocneanu/King-Wybourne critically used by Jones.

Outline

Conclusion

- We have only partially touched some of the use of random unitary matrices/unitary matrix integrals with the previous topics (random partitions, Ising model, Toeplitz correspondance, evaluation of characters of $\mathfrak{S}_n$ and $\mathcal{H}_n(q)$, etc.).
- Lots of other applications : realisations of tau-functions of some classical integrable systems (KP, Calogero-Moser with $\beta = 2$, etc.), eigenvectors of some vertex operators, models of partition functions of quantum gravity, toy-models of the Riemann Zeta function, etc.

Thank you for your attention.