Tsallis distribution as a Λ-deformation of the Maxwell-Jüttner distribution

$$q = 1 + \frac{\hbar\sqrt{\Lambda}}{mc\mathfrak{n}}$$

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Further senses along which S_q, or other entropic functionals, would be unique are certainly welcome.

Constantino Tsallis in "Senses along Which the Entropy S_q Is Unique", **2023** Entropy *25*, 743.

This work started this September and was inspired by

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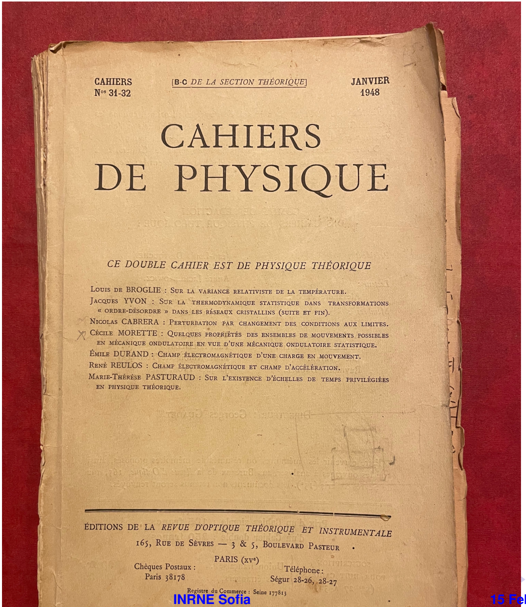
OUTLINE

1. Temperature, Heat, Entropy, that Obscure Objects of Desire
2. Maxwell-Jüttner distribution (from Synge)
3. de Sitterian material
4. Tsallis distribution as a Λ-deformation of the Maxwell-Jüttner distribution

PREAMBLE:

Temperature, Heat, Entropy, that Obscure Objects of Desire

When Physics was written in French in France (found in my personal library)



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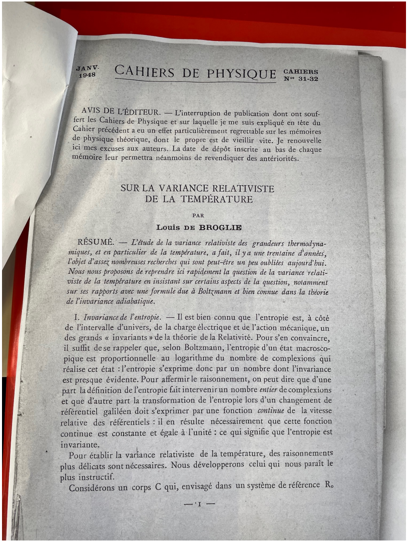
Entropy invariance and relativistic variance of temperature according to de Broglie (1948)



JANV. 1948 — CAHIERS DE PHYSIQUE — CAHIERS N^{os} 31-32

AVIS DE L'ÉDITEUR. — L'interruption de publication dont ont souffert les Cahiers de Physique et sur laquelle je me suis expliqué en tête du Cahier précédent a eu un effet particulièrement regrettable sur les mémoires de physique théorique, dont le propre est de vieillir vite. Je renouvelle ici mes excuses aux auteurs. La date de dépôt inscrite au bas de chaque mémoire leur permettra néanmoins de revendiquer des antériorités.

SUR LA VARIANCE RELATIVISTE
DE LA TEMPÉRATURE

PAR

Louis DE BROGLIE

RÉSUMÉ. — *L'étude de la variance relativiste des grandeurs thermodynamiques, et en particulier de la température, a fait, il y a une trentaine d'années, l'objet d'assez nombreuses recherches qui sont peut-être un peu oubliées aujourd'hui. Nous nous proposons de reprendre ici rapidement la question de la variance relativiste de la température en insistant sur certains aspects de la question, notamment sur ses rapports avec une formule due à Boltzmann et bien connue dans la théorie de l'invariance adiabatique.*

I. *Invariance de l'entropie.* — Il est bien connu que l'entropie est, à côté de l'intervalle d'univers, de la charge électrique et de l'action mécanique, un des grands « invariants » de la théorie de la Relativité. Pour s'en convaincre, il suffit de se rappeler que, selon Boltzmann, l'entropie d'un état macroscopique est proportionnelle au logarithme du nombre de complexions qui réalise cet état : l'entropie s'exprime donc par un nombre dont l'invariance est presque évidente. Pour affermir le raisonnement, on peut dire que d'une part la définition de l'entropie fait intervenir un nombre *entier* de complexions et que d'autre part la transformation de l'entropie lors d'un changement de référentiel galiléen doit s'exprimer par une fonction *continue* de la vitesse relative des référentiels : il en résulte nécessairement que cette fonction continue est constante et égale à l'unité : ce qui signifie que l'entropie est invariante.

Pour établir la variance relativiste de la température, des raisonnements plus délicats sont nécessaires. Nous développerons celui qui nous paraît le plus instructif.

Considérons un corps C qui, envisagé dans un système de référence R_0

— 1 —

It is well known that entropy, alongside the spacetime interval, electric charge, and mechanical action, is one of the fundamental "invariants" of the theory of relativity. To convince oneself of this, it is enough to recall that, according to Boltzmann, the entropy of a macroscopic state is proportional to the logarithm of the number of microstates that realize that state. To strengthen this reasoning, one can argue that, on the one hand, the definition of entropy involves a *integer* number of microstates, and, on the other hand, the transformation of entropy during a Galilean reference frame change must be expressed as a continuous function of the relative velocity of the reference frames. Consequently, this continuous function is necessarily constant and equal to unity, which means that entropy is constant.

"Relativistic thermodynamics": what it could be

In relativistic thermodynamics (i.e., in accordance with special relativity) there are 3 points of view (see Wu, EPL 2009)[1], distinguished from the way heat ΔQ and temperature T transform under a Lorentz boost from frame $\mathcal{R}_0$ (e.g., laboratory) to comoving frame $\mathcal{R}$ with velocity $\mathbf{v} = v\hat{\mathbf{n}}$ relative to $\mathcal{R}_0$ and Lorentz factor

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

- Point of view (**a**) (Einstein, Planck, de Broglie ...), the covariant one,

$$\Delta Q = \Delta Q_0 \gamma^{-1}\,, \quad T = T_0\gamma^{-1}\,.$$

- Point of view (**b**) (Ott, Arzelies, ...), the anti-covariant one,

$$\Delta Q = \Delta Q_0 \gamma\,, \quad T = T_0\gamma\,.$$

- Point of view (**c**) (Landsberg, 1966, ...), "nothing changes",

$$\Delta Q = \Delta Q_0\,, \quad T = T_0\,.$$

- Also note that for some authors (Landsbergh, Sewell, ...) "there is no meaningful law of temperature under boosts"

[1] Z. C. Wu, Inverse Temperature 4-vector in Special Relativity, **2009** EPL *88*, 20005

Relativistic covariance of Temperature according to de Broglie (1948)

- Consider a body $\mathcal{B}$, its proper frame $\mathcal{R}_0$, total proper mass M_0, in thermodynamical equilibrium with temperature T_0 and fixed volume V_0 (e.g., a gas enclosed with surrounding rigid wall)
- Then consider $\mathcal{B}$ from an inertial frame $\mathcal{R}$ in which $\mathcal{B}$ has constant velocity $\mathbf{v} = v\hat{\mathbf{n}}$ relative to $\mathcal{R}_0$.
- Suppose that a source in $\mathcal{R}$ provides $\mathcal{B}$ with heat ΔQ. In order to keep the velocity $\mathbf{v}$ of $\mathcal{B}$ constant a work W has to be done on $\mathcal{B}$ and the proper mass of the latter is consequently modified $M_0 \to M_0'$. Then, from energy conservation :

$$(M_0' - M_0)\gamma c^2 = \Delta Q + W\,, \quad \gamma = \gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}\,,$$

and relativistic 2d Newton law

$$\Delta P = M_0'\gamma v - M_0\gamma v = \int F \mathrm{d}t = \frac{1}{v}\int F v \mathrm{d}t = \frac{W}{v}$$

we derive

$$\Delta Q = \frac{c^2}{v^2}\gamma^{-2} W = (M_0' - M_0)c^2\gamma^{-2}$$

- In the frame $\mathcal{R}_0$ there is no work done (volume is constant), there is just transmitted heat $\Delta Q_0 = (M_0' - M_0)c^2$.
- Hence heat transforms as

$$\Delta Q = \Delta Q_0 \gamma^{-1}$$

and since the entropy $S = \int \frac{\mathrm{d}Q}{T}$ is relativistic invariant, $S = S_0$, temperature transforms as

$$T = T_0\gamma^{-1}$$

Maxwell-Jüttner distribution (from Synge)

Derivation for simple gases (Synge)-I

- Notations for Minkowskian event and 4-momenta 4-vectors,

$$\mathbb{M}_{1,3} \ni \underline{x} = (x^\mu) = (x^0 = x_0, x^i = -x_i\,,\ i = 1,2,3) \equiv (x^0, \mathbf{x})\,,\ \underline{x}\cdot\underline{x}' = x^\mu x'_\mu = x^0 x'^0 - \mathbf{x}\cdot\mathbf{x}'\,,$$

$$\underline{k} = (k^\mu) = (k^0, \mathbf{k})\,.$$

- Let $\underline{k}$ be a 4-momentum pointing toward $A \in$ mass shell hyperboloid $\mathcal{V}_m^+ = \{\underline{k}\,,\ \underline{k}\cdot\underline{k} = m^2c^2\}$, and an infinitesimal hyperbolic interval at A, with length $\mathrm{d}\sigma = mc\,\mathrm{d}\omega$ ($\mathrm{d}\omega = \frac{\mathrm{d}^3\mathbf{k}}{k_0}$ is the Lorentz-invariant element on $\mathcal{V}_m^+$),
- given a time-like unit vector $\underline{n}$,
- given a straight line Δ passing through the origin and orthogonal (in $\mathbb{M}_{1,3}$ sense) to $\underline{n}$,
- denote by $\mathrm{d}\Omega$ the length of the projection of $\mathrm{d}\sigma$ on Δ along $\underline{n}$. One proves that

$$\mathrm{d}\Omega = |\underline{k}\cdot\underline{n}|\,\mathrm{d}\omega \quad (= \mathrm{d}^3\mathbf{k} \text{ if } \underline{n} = (1,\mathbf{0}))$$

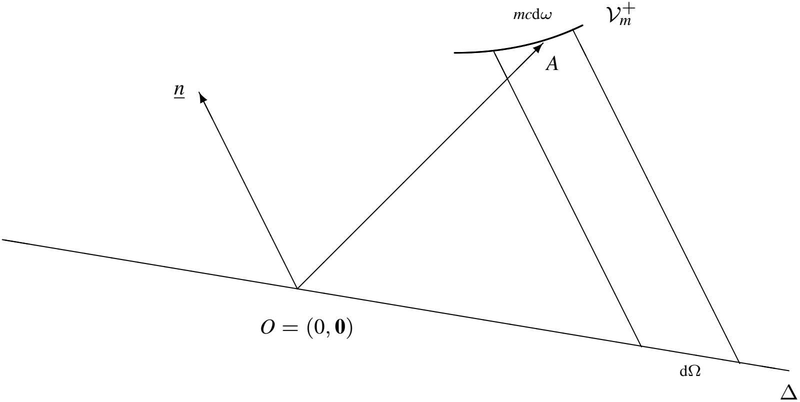


Figure: $\underline{n}$ is a time-like unit vector, Δ is a straight line passing through the origin and orthogonal (in the Minkowskian metric sense) to $\underline{n}$. The 4-momentum $\underline{k} = (k^\mu) = (k^0, \mathbf{k})$ points toward a point A of the mass shell hyperboloid $\mathcal{V}_m^+ = \{\underline{k}\,,\ \underline{k} \cdot \underline{k} = m^2c^2\}$. $\mathrm{d}\Omega$ is the length of the projection, along $\underline{n}$, of an infinitesimal hyperbolic interval at A of length $mc\mathrm{d}\omega$.

Derivation for simple gases (Synge)-II

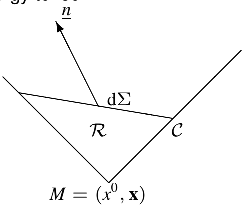
- The sample population consists of those particles with world lines cutting the infinitesimal space-like segment $\mathrm{d}\Sigma$ orthogonal to the time-like unit vector $\underline{n}$.
- Every particle that crosses the portion $\mathcal{C}$ of the null cone between M and $\mathrm{d}\Sigma$ **must** (causal cone) also cross $\mathrm{d}\Sigma$. Therefore the following population number is preassigned

$$\nu = \mathrm{d}\Sigma \int_{\mathcal{R}} \mathcal{N}(\underline{x}, \underline{k})\, \mathrm{d}\Omega$$

where $\mathcal{N}(\underline{x}, \underline{k})$ is the distribution function, and $\mathcal{R}$ the region delimited by M and $\mathrm{d}\Sigma$.
- By the conservation of 4-momentum at each collision in a simple gas, the flux of 4-momentum across $\mathrm{d}\Sigma$ is predetermined as the flux across $\mathcal{C}$,

$$T_{\mu} \cdot \underline{n}\, \mathrm{d}\Sigma = \mathrm{d}\Sigma \int_{\mathcal{R}} \mathcal{N}(\underline{x}, \underline{k})\, ck_{\mu} \mathrm{d}\Omega$$

where $T = (T_{\mu\nu})$ is the energy tensor.



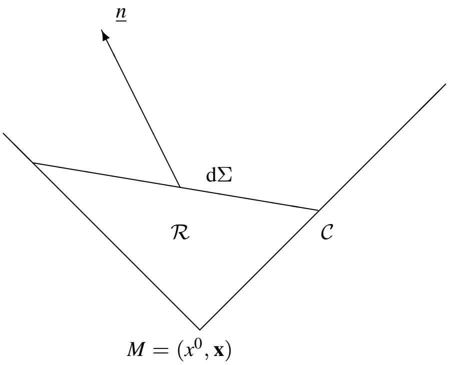


Figure: $\mathcal{C}$ is the portion of the null cone starting at the event $M = (x^0, \mathbf{x})$ and limited by the infinitesimal space-like segment $\mathrm{d}\Sigma$ orthogonal to the time-like unit vector $\underline{n}$. $\mathcal{R}$ is the region delimited by M, the portion of the light cone $\mathcal{C}$, and $\mathrm{d}\Sigma$.

Derivation for simple gases (Synge)-III

- The most probable distribution function $\mathcal{N}$ at M is that which maximizes the following entropy integral

$$F = -\mathrm{d}\Sigma \int_{\mathcal{R}} \mathcal{N} \, \log \mathcal{N} \, \mathrm{d}\Omega$$

- Variational calculus with constraints on ν and $T_\mu \cdot \underline{n}$ leads to the solution

$$\mathcal{N}(\underline{x}, \underline{k}) = C(\underline{x}) \, \exp(-\underline{\eta}(\underline{x}) \cdot \underline{k})$$

- Scalar C and 4-vector $\underline{\eta}$ (dimension of inverse momentum) are connected with Lagrange multipliers and determined by the constraints on $\nu = \underline{N} \cdot \underline{n} \, \mathrm{d}\Sigma$ ($\underline{\mathrm{N}}$ is the numerical-flux 4-vector) and $T_\mu \cdot \underline{n} \, \mathrm{d}\Sigma$:

$$C \int_{\mathcal{V}_m^+} k_\mu \, e^{-\underline{\eta} \cdot \underline{k}} \, \mathrm{d}\omega = N_\mu \, , \qquad C \int_{\mathcal{V}_m^+} c k_\mu k_\nu \, e^{-\underline{\eta} \cdot \underline{k}} \, \mathrm{d}\omega = T_{\mu\nu} \, .$$

established by taking into account that $\underline{n}$ is arbitrary.

- With the equations of conservation

$$\underline{\partial} \cdot \underline{N} = 0 \, , \quad \underline{\partial} \cdot T_\mu = 0 \, ,$$

we finally get as many equations as functions of $\underline{x}$: $C, \underline{\eta}, \underline{N}, T$.

Derivation for simple gases (Synge)-IV

- For instance, if we deal with a simple gas consisting of material particles of proper mass m, we introduce the mean 4-velocity of the fluid, $\underline{\lambda} = (\lambda_\mu = c\eta_\mu / \sqrt{\underline{\eta} \cdot \underline{\eta}})$, $\underline{\lambda} \cdot \underline{\lambda} = c^2$, so that

$$\mathcal{N}(\underline{x}, \underline{k}) := \frac{\mathcal{N}_0}{mcK_1\left(mc^2/k_B T_a\right)} \exp\left(-\frac{\underline{\lambda} \cdot \underline{k}}{k_B T_a}\right) .$$

- $T_a := c/(k_B\sqrt{\underline{\eta} \cdot \underline{\eta}})$ is a "relativistic" absolute temperature. It is precisely the relativistic invariant, which might fit pointview (**c**).
- The appearance of the Bessel functions K_1 comes from the constraint

$$N_\mu = C \int_{\mathcal{V}_m^+} k_\mu\, e^{-\underline{\eta} \cdot \underline{k}} \frac{\mathrm{d}^3\mathbf{k}}{k_0} = -C\frac{\partial Z}{\partial \eta^\mu} = C\frac{\eta_\mu mc}{\sqrt{\underline{\eta} \cdot \underline{\eta}}} K_1\left(mc\sqrt{\underline{\eta} \cdot \underline{\eta}}\right) ,$$

where $Z = \int_{\mathcal{V}_m^+} e^{-\underline{\eta} \cdot \underline{k}} \frac{\mathrm{d}^3\mathbf{k}}{k_0} \propto K_0\left(mc\sqrt{\underline{\eta} \cdot \underline{\eta}}\right)$ is the partition function.

- The invariant quantity $\mathcal{N}_0 = \underline{N} \cdot \underline{\lambda}/c$ is the number of particles per unit length ("numerical density") in the rest frame of the fluid ($\lambda_0 = c$).
- Note that the Maxwell-Boltzmann non relativistic distribution is recovered by considering the limit at $k_B T_a \ll mc^2$ in the rest frame of the fluid:

$$K_1\left(\frac{mc^2}{k_B T_a}\right) \approx \sqrt{\frac{\pi k_B T_a}{2mc^2}}\, e^{-\frac{mc^2}{k_B T_a}}$$

$$\Rightarrow \mathcal{N}(\underline{x}, \underline{k}) \approx N_0\sqrt{\frac{2}{\pi m k_B T_a}} \exp\left(-\frac{k_0 c - mc^2}{k_B T_a}\right) \approx N_0\sqrt{\frac{2}{\pi m k_B T_a}} \exp\left(-\frac{\mathbf{k}^2}{2mk_B T_a}\right) .$$

Inverse temperature 4-vector

- The found distribution on the Minkowskian mass shell for a simple gas consisting of particles of proper mass m

$$\mathcal{N}(\underline{x},\underline{k}) = \frac{\mathcal{N}_0}{mcK_1\left(mc^2/k_B T_a\right)} \exp\left(-\frac{\underline{\lambda}\cdot\underline{k}}{k_B T_a}\right)$$

leads us to introduce the relativistic thermodynamic, future directed, time-like 4-coldness vector $\underline{\beta}$, as the 4-version of the reciprocal of the thermodynamic temperature (see also Wu 2009):

$$\frac{\underline{\lambda}}{k_B T_a} \equiv \underline{\beta} = \left(\beta^0 = \beta_0 > 0, \beta^i = -\beta_i\right) = \left(\beta_0, \boldsymbol{\beta}\right),$$

with relativistic invariant

$$\sqrt{\underline{\beta}\cdot\underline{\beta}} = \frac{c}{k_B T_a} \equiv \beta_a\,.$$

- It is precisely the way as the component β_0 transforms under a Lorentz boost, $\beta_0' = \gamma(v)(\beta_0 - \mathbf{v}\cdot\boldsymbol{\beta}/c)$, which explains the way the temperature transforms à la de Broglie, $T \mapsto T' = T\gamma^{-1}$.
- So, in the sequel, we call Maxwell-Jüttner distribution the following relativistic invariant:

$$\mathcal{N}(\underline{\beta},\underline{k}) = \frac{\mathcal{N}_0}{mcK_1\left(mc\beta_a\right)} \exp\left(-\underline{\beta}\cdot\underline{k}\right)$$

Maxwell-Jüttner distribution in a nutshell

- Distribution

$$\mathcal{N}(\underline{x},\underline{k}) = \frac{\mathcal{N}_0}{mcK_1\left(mc^2/k_B T_a\right)} \exp\left(-\underline{\beta}(\underline{x})\cdot\underline{k}\right)$$

- Momentum 4-vector

$$\underline{k} = (k^0, \mathbf{k}) \in \mathcal{V}_m^+ \,, \quad \underline{k}\cdot\underline{k} = m^2c^2$$

- Coldness 4-vector field $\sim$ 4-version of the reciprocal of the thermodynamic temperature in terms of the mean 4-velocity $\underline{\lambda}$ of the fluid

$$\underline{\beta} = (\beta_0 > 0, \boldsymbol{\beta}) = \frac{\underline{\lambda}}{k_B T_a}\,, \quad \sqrt{\underline{\beta}\cdot\underline{\beta}} = \frac{c}{k_B T_a}$$

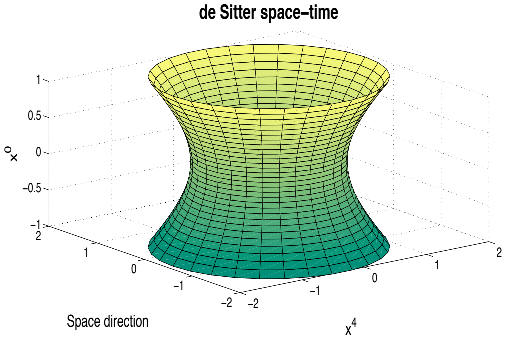
de Sitterian material

de Sitter geometry

- de Sitter space is viewed as a hyperboloid embedded in a five-dimensional Minkowski space $\mathbb{M}_{1,4}$ with metric $g_{\alpha\beta}$ =diag$(1, -1, -1, -1, -1)$ (but keep in mind that all points are physically equivalent)

$$\mathrm{M}_R \equiv \{x \in \mathbb{R}^5;\ x^2 = g_{\alpha\beta}\, x^\alpha x^\beta = -R^2\}, \quad \alpha, \beta = 0, 1, 2, 3, 4,$$

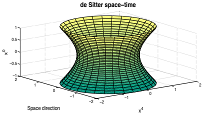
  where the pseudo-radius R (or inverse of curvature) is given by $R = \sqrt{\dfrac{3}{\Lambda}}$ within the cosmological ΛCDM standard model.
- de Sitter symmetry group is SO$_0(1, 4)$ with ten (Killing) generators $K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha$.



Flat Minkowskian limit of de Sitter geometry

- Example of global coordinates on M_R, $ct \in \mathbb{R}, \mathbf{n} \in \mathbb{S}^2, r/R \in [0,\pi]$:

$$\begin{aligned} M_R \ni x &= (x^0, x^1, x^2, x^3, x^4) \equiv (x^0, \mathbf{x}, x^4) \\ &= (R\sinh(ct/R),\ R\cosh(ct/R)\sin(r/R)\mathbf{n}, R\cosh(ct/R)\cos(r/R)) \equiv x(t,\mathbf{x})\,. \end{aligned}$$



- $\lim_{R\to\infty} M_R = \mathbb{M}_{1,3}$, the Minkowski spacetime tangent to M_R at, say, the de Sitter origin point $O_{dS} = (0,\mathbf{0},R)$, since then

$$M_R \ni x \underset{R\to\infty}{\approx} (ct, \mathbf{r} = r\,\mathbf{n}, R) \equiv (\underline{\ell}, R)\,, \quad \underline{\ell} \in \mathbb{M}_{1,3}$$

- $\lim_{R\to\infty} \mathrm{SO}_0(1,4) = \mathcal{P}_+^\uparrow(1,3) = \mathbb{M}_{1,3} \rtimes \mathrm{SO}_0(1,3)$, the Poincaré group.

- The ten de Sitter Killing generators contract (in the Wigner-Inonü sense) to their Poincaré counterparts $K_{\mu\nu}$, Π_μ, $\mu = 0,1,2,3$, after rescaling the four $K_{4\mu} \longrightarrow \Pi_\mu = K_{4\mu}/R$.

de Sitter plane waves as binomial deformations of Minkowskian plane waves

- de Sitter (scalar) plane waves have the form
$$\phi_{\tau,\xi}(x) = \left(\frac{x\cdot\xi}{R}\right)^{\tau}\,,\quad x\in \mathrm{M}_R\,,\quad \xi\in\mathcal{C}_{1,4}\,,$$
where $\mathcal{C}_{1,4} = \{\xi\in\mathbb{R}^5\,,\ \xi\cdot\xi = 0\}$ is the null cone in $\mathbb{M}_{1,4}$.
- They are solutions of the Klein-Gordon-like equation
$$\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta}\phi_{\tau,\xi}(x) = R^2\square_R\phi_{\tau,\xi}(x) = \tau(\tau+3)\phi_{\tau,\xi}(x)\,,$$
where $M_{\alpha\beta} = -\mathrm{i}\,(x_\alpha\partial_\beta - x_\beta\partial_\alpha)$ is the quantum representation of the Killing $K_{\alpha\beta}$, and $\square_R$ stands for the d'Alembertian operator on M_R.
- For $\tau = -\frac{3}{2}+\mathrm{i}\nu$, $\nu\in\mathbb{R}$, they describe free quantum motions of "massive" scalar particles on M_R.
- The term "massive" is justified by the flat Minkowskian limit $R\to\infty$, i.e. $\Lambda\to 0$:
  - First one has the Garidi relation between proper mass m (curvature independent) of the particle and the parameter $\nu\geq 0$
  $$m = \frac{\hbar}{Rc}\left[\nu^2+\frac{1}{4}\right]^{1/2} \Leftrightarrow \nu = \sqrt{\frac{R^2m^2c^2}{\hbar^2}-\frac{1}{4}} \underset{R\text{ large}}{\approx} \frac{Rmc}{\hbar} = \frac{mc}{\hbar}\sqrt{\frac{3}{\Lambda}}\,,$$
  (the quantity $\frac{\hbar c\nu}{R}$ is a kind of *at rest desitterian energy*, which is distinct of the proper mass energy mc^2 if $\Lambda\neq 0$).
  - Then with the mass shell parametrisation $\xi = \left(\xi^0 = \frac{k_0}{mc}, \boldsymbol{\xi} = \frac{\mathbf{k}}{mc}, \xi^4 = 1\right)\in\mathcal{C}^+_{1,4}$:
  $$\phi_{\tau,\xi}(x) = (x\cdot\xi/R)^{-3/2+\mathrm{i}\nu} \underset{R\to\infty}{\to} e^{\mathrm{i}\underline{k}\cdot\underline{\ell}/\hbar}\,,\quad \underline{\ell} = (ct,\mathbf{r})\,.$$

Analytic extension of dS plane waves for dS QFT

- Actually dS plane waves $\phi_{\tau,\xi}(x) = \left(\frac{x\cdot\xi}{R}\right)^{\tau}$, $\tau = -3/2 + \mathrm{i}\nu$, are not defined on all M_R due to the possible change of sign of $x\cdot\xi$.
- A solution is found through extension to tubular domains in complexified M_R^c

$$\mathcal{T}^{\pm} := \mathrm{T}^{\pm} \cap \mathrm{M}_R\,, \quad \mathrm{T}^{\pm} := \mathbb{M}_{1,4} + \mathrm{i}\mathrm{V}^{\pm}\,,$$

where the forward and backward light cones $\mathrm{V}^{\pm} := \{x \in \mathbb{M}_{1,4}\,,\, x^0 \gtrless \sqrt{\mathbf{x}^2 + (x^4)^2}\}$ allow for a causal ordering in $\mathbb{M}_{1,4}$.
- Then the extended plane waves $\phi_{\tau,z}(x) = \left(\frac{z\cdot\xi}{R}\right)^{\tau}$ are globally defined for $z \in \mathcal{T}^{\pm}$ and $\xi \in \mathcal{C}^{+}_{1,4}$.
- These analytic extensions allow for a consistent QFT for free scalar fields on M_R: the two-point Wightman function $\mathcal{W}_\nu(x,x') = \langle\Omega, \phi(x)\phi(x')\Omega\rangle$ can be extended to the complex covariant, maximally analytic, two-point function having the spectral representation in terms of these extended plane waves:

$$W_\nu(z,z') = c_\nu \int_{\mathcal{V}_m^+ \cup \mathcal{V}_m^-} (z\cdot\xi)^{-3/2+\mathrm{i}\nu}(\xi\cdot z')^{-3/2-\mathrm{i}\nu}\,\frac{\mathrm{d}\mathbf{k}}{k_0}\,, \quad z \in \mathcal{T}^-\,,\, z' \in \mathcal{T}^+\,.$$

- Details are found in J. Bros, J.P. G., and U. Moschella, Quantum Field Theory in the de Sitter Universe, **1994** Phys. Rev. Lett. *73* 1746-1749. See references in M. Enayati, J.P. G., H. Pejhan, and A. Wang, *The de Sitter (dS) Group and its Representations; An Introduction to Elementary Systems and Modeling the Dark Energy Universe*, Springer Nature (2022)..

KMS interpretation of $W_\nu(z, z')$ analyticity

- For the analyticity of $W_\nu(z, z')$ we deduce that $\mathcal{W}_\nu(x, x')$ defines a $2\mathrm{i}\pi R/c$ periodic analytic function of t, whose domain is the periodic cut plane

$$\mathbb{C}^{\mathrm{cut}}_{x,x'} = \{t \in \mathbb{C}\,,\ \mathrm{Im}(t) \neq 2n\pi R/c\,,\ n \in \mathbb{Z}\} \cup \{t\,,\ t - 2\mathrm{i}n\pi R/c \in I_{x,x'}\,,_{n\in\mathbb{Z}}\}\,,$$

where $I_{x,x'}$ is the real interval on which $(x - x')^2 < 0$.

- Hence $W_\nu(z, z')$ is analytic in the strip

$$\{t \in \mathbb{C}\,,\ 0 < \mathrm{Im}(t) < 2\mathrm{i}\pi R/c\}\,,$$

and satisfies:

$$\mathcal{W}_\nu(x'(t + t', \mathbf{x}), x) = \lim_{\epsilon\to 0+} W_\nu\left(x, x'(t + t' + 2\mathrm{i}\pi R/c - \mathrm{i}\epsilon, \mathbf{x})\right)\,, \quad t' \in \mathbb{R}\,.$$

- This is a KMS relation at ($\sim$ Hawking) temperature

$$T_\Lambda = \frac{\hbar c}{2\pi k_B R} := \frac{\hbar c}{2\pi k_B}\sqrt{\frac{\Lambda}{3}}\,.$$

de Sitter (dS) plane waves in a nutshell

- dS (scalar) plane waves for "massive" scalar particles on dS manifold $\mathrm{M}_R = \left\{ x = (\underline{x}, x^4)\,,\ x \cdot x = \underline{x} \cdot \underline{x} - (x^4)^2 = -R^2 = -\frac{3}{\Lambda} \right\}$

$$\phi_{\tau,\xi}(x) = \left(\frac{x \cdot \xi}{R}\right)^{3/2 - \mathrm{i}\nu}\,, \quad x = (\underline{x}, x^4) \in \mathrm{M}_R \subset \mathbb{M}_{1,4}$$

- de Sitterian "momentum" on the null cone $\mathcal{C}_{1,4} = \{\xi \in \mathbb{R}^5\,,\ \xi \cdot \xi = 0\}$ in $\mathbb{M}_{1,4}$

$$\xi = \left(\underline{\xi} = \frac{\underline{k}}{mc}, \xi^4 = 1\right) \in \mathcal{C}_{1,4}$$

- $3+1$ Minkowskian limit at $R \to \infty$ (i.e. $\Lambda \to 0$)

$$(x \cdot \xi / R)^{-3/2 + \mathrm{i}\nu} \underset{R \to \infty}{\to} e^{\mathrm{i}\underline{k} \cdot \underline{\ell}/\hbar}\,, \quad \underline{\ell} = (ct, \mathbf{r})\,, \quad \nu = \frac{mc}{\hbar}\sqrt{\frac{3}{\Lambda}} = \frac{mc^2}{2\pi k_B T_\Lambda}$$

Tsallis distribution as a Λ-deformation of the Maxwell-Jüttner distribution

Tsallis entropy and distribution

- Given a discrete (resp. continuous) set of probabilities $\{p_i\}$ (resp. continuous $x \mapsto p(x)$) with $\sum_i p_i = 1$ (resp. $\int p(x)\mathrm{d}x = 1$) , and a real q, the *Tsallis entropy* is defined as

$$S_q(p_i) = k\frac{1}{q-1}\left(1 - \sum_i p_i^q\right) \quad \text{resp.} \quad S_q[p] = \frac{1}{q-1}\left(1 - \int (p(x))^q \mathrm{d}x\right)$$

- As $q \to 1$, $S_q(p_i) \to S_{\mathrm{BG}}(p) = -k\sum_i p_i \ln p_i$ (Boltzmann-Gibbs)
- Tsallis entropy is non additive : For two independent systems A and B, for which $p(A \cup B) = p(A)\,p(B)$, $S_q(A \cup B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$.
- A *Tsallis distribution* is a probability distribution derived from the maximization of the Tsallis entropy under appropriate constraints.
- The so-called q-exponential Tsallis distribution has the probability density function

$$(2-q)\lambda[1-(1-q)\lambda x]^{1/(1-q)} \equiv (2-q)\lambda e_q(-\lambda x)\,,$$

where $q < 2$ and $\lambda > 0$ (rate), arises from the maximization of the Tsallis entropy under appropriate constraints, including constraining the domain to be positive [2].

[2] Tsallis, C. Nonadditive entropy and nonextensive statistical mechanics-an overview after 20 years. Braz. J. Phys. 2009, 39, 337-356

Coldness in de Sitter

- In analogy with the de Sitter plane waves, let us introduce the distributions on subset $\sim \mathcal{V}_m^+ \subset \mathcal{C}_{1,4}^+ = \{\xi \in \mathbb{M}_{1,4}\,,\ \xi\cdot\xi = 0\,,\ \xi^0 > 0\}$:
$$\phi_{\tau,\xi}(x) = \left(\frac{\mathfrak{b}\cdot\xi}{B}\right)^\tau\,,\quad \mathfrak{b}\in \mathsf{M}_B\,,\quad \xi = \left(\frac{k^0}{mc} > 0, \frac{\mathbf{k}}{mc}, -1\right)\,,$$
where one should note the negative value -1 for ξ_4, and
$\mathsf{M}_B \equiv \{\mathfrak{b}\in\mathbb{M}_{1,4}\,,\ \mathfrak{b}^2 = g_{\alpha\beta}\,\mathfrak{b}^\alpha\mathfrak{b}^\beta = -B^2\}$, $\quad\alpha,\beta = 0,1,2,3,4$, is the manifold of the "deSitterian coldnesses".
- Like for M_R we use global coordinates on M_B: $\beta^0\in\mathbb{R}$, $\boldsymbol{\beta} = \|\boldsymbol{\beta}\|\mathbf{n}\in\mathbb{R}^3$, $\|$, $\|\boldsymbol{\beta}\|/B \in [0,\pi]$, with
$$\mathsf{M}_B \ni \mathfrak{b} = (\mathfrak{b}^0, \mathfrak{b}^1, \mathfrak{b}^2, \mathfrak{b}^3, \mathfrak{b}^4) \equiv (\mathfrak{b}^0, \boldsymbol{\mathfrak{b}}, \mathfrak{b}^4)$$
$$= \Big(B\sinh(\beta^0/B),\ B\cosh(\beta^0/B)\sin(\|\boldsymbol{\beta}\|/B)\mathbf{n}, B\cosh(\beta^0/B)\cos(\|\boldsymbol{\beta}\|/B)\Big) \equiv \mathfrak{b}(\underline{\beta})\,,$$
- in such a way that at large B we recover the Minkowskian coldness $\underline{\beta}$:
$$\mathsf{M}_B \ni \mathfrak{b} \underset{B\to\infty}{\approx} (\underline{\beta}, B)\,.$$
- We now need to connect the desitterian coldness scale B with Λ. Inspired by relativistic invariant $\beta_a = \frac{c}{k_B T_a}$ and the KMS temperature $T_\Lambda = \dfrac{\hbar c}{2\pi k_B}\sqrt{\dfrac{\Lambda}{3}}$ we write
$$B \propto \frac{2\pi}{\hbar}\sqrt{\frac{3}{\Lambda}}\,,\quad \text{i.e.}\quad B = \frac{\mathfrak{n}}{\hbar\sqrt{\Lambda}}\,,\quad \Lambda_{\text{current}} = 1.1056\times 10^{-52}\,\mathrm{m}^{-2}\,,\quad \hbar = 1.054571817\ldots\times 10^{-34}\,\mathrm{J\,s}$$
where $\mathfrak{n}$ is a numerical factor. $B \approx 0.9\times 10^{60}\,\mathfrak{n}$ SI is the inverse of a momentum.

A de Sitterian Tsallis distribution

- Now consider the distribution on $\mathrm{M}_B \times \mathcal{V}_m^+$ with $B = \frac{\mathfrak{n}}{\hbar\sqrt{\Lambda}}$:

$$\mathcal{N}(\mathfrak{b},\underline{k}) = C_B \left(\frac{\mathfrak{b}\cdot\xi}{B}\right)^{-mcB} = C_B \left(\frac{\mathfrak{b}^0}{B}\frac{k^0}{mc} - \frac{\mathfrak{b}}{B}\cdot\frac{\mathbf{k}}{mc} + \frac{\mathfrak{b}^4}{B}\right)^{-mcB} .$$

$$\mathfrak{b} \in \mathrm{M}_B\,, \quad \xi = \left(\frac{k^0}{mc} > 0,\, \frac{\mathbf{k}}{mc}, -1\right) ,$$

where $\mathrm{M}_B \equiv \{\mathfrak{b} \in \mathbb{M}_{1,4}\,,\ \mathfrak{b}^2 = g_{\alpha\beta}\ \mathfrak{b}^\alpha \mathfrak{b}^\beta = -B^2\}$, $\quad \alpha,\beta = 0,1,2,3,4$, is the manifold of the “deSitterian coldnesses”, and constant C_B involves Legendre function of $mc^2/k_B T_a$ (?!).

- With global coordinates

$$\mathrm{M}_B \ni \mathfrak{b} = \left(B\sinh(\beta^0/B),\, B\cosh(\beta^0/B)\sin(\|\boldsymbol{\beta}\|/B)\mathbf{n}, -B\cosh(\beta^0/B)\cos(\|\boldsymbol{\beta}\|/B)\right) ,$$

with the constraint $\beta^0/B \in [0,\pi/2)$, $\mathcal{N}(\mathfrak{b},\underline{k})$ reads

$$\mathcal{N}(\mathfrak{b},\underline{k}) = C_B \left(\cosh(\beta^0/B)\cos(\|\boldsymbol{\beta}\|/B) + \sinh(\beta^0/B)\,\frac{k^0}{mc} - \cosh(\beta^0/B)\sin(\|\boldsymbol{\beta}\|/B)\,\frac{\mathbf{n}\cdot\mathbf{k}}{mc}\right)^{-mcB}$$

$$= C_B e^{\left[-mcB\log\left(\cosh(\beta^0/B)\cos(\|\boldsymbol{\beta}\|/B)+\sinh(\beta^0/B)\,\frac{k^0}{mc}-\cosh(\beta^0/B)\sin(\|\boldsymbol{\beta}\|/B)\,\frac{\mathbf{n}\cdot\mathbf{k}}{mc}\right)\right]}$$

$$= C_B \exp\left[-mcB\log\left(\cosh(\beta^0/B)\cos(\|\boldsymbol{\beta}\|/B)\right)\right] \times$$

$$\times \exp -mcB\log\left[1 + \frac{\sinh(\beta^0/B)\,\frac{k^0}{mc} - \cosh(\beta^0/B)\sin(\|\boldsymbol{\beta}\|/B)\,\frac{\mathbf{n}\cdot\mathbf{k}}{mc}}{\cosh(\beta^0/B)\cos(\|\boldsymbol{\beta}\|/B)}\right]$$

A de Sitterian Tsallis distribution

- At large B this expression becomes the Maxwell-Jüttner distribution:

$$\mathcal{N}(\mathfrak{b}, \underline{k}) \approx C_B e^{-\underline{\beta}\cdot\underline{k}}\,.$$

- So, going back to the original expression

$$\mathcal{N}(\mathfrak{b}, \underline{k}) = C_B \left(\frac{\mathfrak{b}\cdot\xi}{B}\right)^{-mcB} = C_B \left(\frac{\mathfrak{b}^0}{B}\frac{k^0}{mc} - \frac{\boldsymbol{\mathfrak{b}}}{B}\cdot\frac{\mathbf{k}}{mc} + \frac{\mathfrak{b}^4}{B}\right)^{-mcB}$$
$$= C_B \left(\frac{\mathfrak{b}^4}{B}\right)^{-mcB} \left(1 + \frac{\underline{\mathfrak{b}}\cdot\underline{k}}{\mathfrak{b}^4 mc}\right)^{-mcB}\,, \quad \underline{\mathfrak{b}} := (\mathfrak{b}^0, \boldsymbol{\mathfrak{b}})\,.$$

- Introducing

$$q = 1 + \frac{1}{mcB} = 1 + \frac{\hbar\sqrt{\Lambda}}{mc\mathfrak{n}}\,,$$

we get the Tsallis-type distribution

$$\mathcal{N}(\mathfrak{b}, \underline{k}) = C_B \left(1 - (1-q)\frac{B}{\mathfrak{b}^4}\,\underline{\mathfrak{b}}\cdot\underline{k}\right)^{\frac{1}{1-q}}\,.$$

de Sitterian Tsallis distribution in a nutshell

- de Sitterian coldness manifold with $B = \frac{\mathfrak{n}}{\hbar\sqrt{\Lambda}}$, $\mathfrak{n}$: numerical,

$$\mathrm{M}_B \ni \mathfrak{b} = (\mathfrak{b}^0, \boldsymbol{\mathfrak{b}}, \mathfrak{b}^4)\,, \quad \mathfrak{b}\cdot\mathfrak{b} = (\mathfrak{b}^0)^2 - \boldsymbol{\mathfrak{b}}\cdot\boldsymbol{\mathfrak{b}} - (\mathfrak{b}^4)^2 = -B^2$$

- de Sitterian distribution on $\mathrm{M}_B \times \mathcal{V}_m^+$:

$$\mathcal{N}(\mathfrak{b}, \underline{k}) = C_B \left(\frac{\mathfrak{b}\cdot\xi}{B}\right)^{-mcB}\,, \quad \xi = \left(\frac{k^0}{mc} > 0, \frac{\mathbf{k}}{mc}, -1\right)$$

- de Sitterian Tsallis-type distribution

$$\mathcal{N}(\mathfrak{b}, \underline{k}) = C_B \left(1 - (1-q)\frac{B}{\mathfrak{b}^4}\,\underline{\mathfrak{b}}\cdot\underline{k}\right)^{\frac{1}{1-q}}$$

- With

$$q = 1 + \frac{1}{mcB} = 1 + \frac{\hbar\sqrt{\Lambda}}{mc\mathfrak{n}}$$

THANK FOR YOUR INDULGENT ATTENTION