# Infinitesimals in the Field of Complex Numbers

Todor D. Todorov (Emeritus in Cal Poly, San Luis Obispo)
ttodorov@calpoly.edu
A joint project with Ivan Penkov (Jacobs University, Bremen)

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# Preliminaries: Algebraically Closed and Real Closed Fields

- A field $\mathbb{K}$ is **formally real** or **orderable** if $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ implies $x_1 = x_2 = \cdots = x_n = 0$. The field of real numbers $\mathbb{R}$ is formally real, while $\mathbb{C}$ is not.
- A field $\mathbb{K}$ is **algebraically closed** if every polynomial $P \in \mathbb{K}[x]$ (with coefficients in $\mathbb{K}$) has at least one root in $\mathbb{K}$. The field $\mathbb{C}$ is algebraically closed, but $\mathbb{R}$ is not.
- A formally real field $\mathcal{R}$ is **real closed** if:
  - For every $a \in \mathcal{R}$ either $a$, or $-a$ has a square root.
  - Every polynomial $P \in \mathcal{R}[x]$ (with coefficients in $\mathcal{R}$) of *odd degree* has a root in $\mathcal{R}$.
- **Examples:**
  - The **field $\mathbb{R}$ of real numbers is real closed**. Also the real algebraic numbers is also real closed.
  - More examples follow later in the talk...

# Archimedean and Non-Archimedean Fields

- Every real closed field $\mathcal{R}$ is **orderable in unique way**: $x \geq 0$ if $x = y^2$ for some $y$. We let $|x| = \max\{x, -x\}$.
- **Theorem (Artin-Schreier):** Let $\mathcal{R}$ is a formally real field. Then $\mathcal{R}$ is **real closed** $iff$ $\mathcal{R}(i)$ is **algebraically closed** (Artin & Schreier [2]). Here $\mathcal{R}(i) = \mathcal{R} \oplus i\mathcal{R}$.
- **Definition:** Let $\mathcal{R}$ be a totally ordered field and $x \in \mathcal{R}$. Then:
    1. $x$ is **infinitely large** if $n < |x|$ for all $n \in \mathbb{N}$.
    2. $x$ is **finite** if $|x| \leq n$ for some $n \in \mathbb{N}$.
    3. $x$ is **infinitesimal** if $|x| < 1/n$ for all $n \in \mathbb{N}$.
- We denote by $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{I}$ the **set of the infinitely large, finite and infinitesimal elements of** $\mathcal{R}$, respectively. We write $x \approx 0$ if $x \in \mathcal{I}$.

# Let $\mathcal{R}$ be a Totally-Ordered Field. Then:

- We have $\mathcal{F} \cup \mathcal{L} = \mathcal{R}$, $\mathcal{F} \cap \mathcal{L} = \varnothing$, $\mathcal{I} \subset \mathcal{F}$. Let $x \in \mathcal{R}, x \neq 0$. Then $x \in \mathcal{I}$ $iff$ $1/x \in \mathcal{L}$.
- If $\mathbb{R} \subseteq \mathcal{R}$, then $\mathbb{R} \cap \mathcal{I} = \{0\}$.
- Every finite number $x \in \mathcal{F}$ there exists unique real number $r \in \mathbb{R}$ such that $x \approx r$. Consequently, $\mathcal{I}$ is a maximal ideal in $\mathcal{F}$ and $\mathcal{F}/\mathcal{I} = \mathbb{R}$.
- The field $\mathcal{R}$ is **Archimedean** $iff$ $\mathcal{L} = \varnothing$ $iff$ $\mathcal{R} = \mathcal{F}$ $iff$ $\mathcal{I} = \{0\}$.
- The field $\mathcal{R}$ is **non-Archimedean** $iff$ $\mathcal{L} \neq \varnothing$ $iff$ $\mathcal{R} \neq \mathcal{F}$ $iff$ $\mathcal{I} \neq \{0\}$.

# A Simple Example of a Non-Archimedean Field

- Let $\mathbb{R}(t) = \frac{\mathbb{R}[t]}{\mathbb{R}[t]}$ denotes the **field of the rational functions in one variable with real coefficients**. We have:
- $\mathbb{R} \subset \mathbb{R}(t)$ under the embedding $r \mapsto rt^0$.
- Let $P \in \mathbb{R}[t]$ be a polynomial. We define $P > 0$ if **the coefficient in front of the lowest power of $t$ is positive**. Thus $\mathbb{R}(t)$ converts in a **totally ordered field**.
- For example $2t^2 - 100t^3 > 0$, because $2 > 0$.
- The field $\mathbb{R}(t)$ is a **non-Archimedean, but not real closed field**:
- $t,\ -t,\ 2t - t^2,\ t^2,\ t^2 - 3t^3, \dots$ are **non-zero infinitesimals**.
  - $t^{-1},\ -t^{-1},\ 3t^{-1},\ t^{-2},\ t^{-2} - 3t^3, \dots$ are **infinitely large**.
  - $2,\ 1 + t,\ 2 - t,\ 5 + 2t,\ 6 + t^2,\ \pi + t^2 - 3t^3, \dots$ are **finite, but not infinitesimal**.

# Infinitesimals and Infinitely Large in $\mathbb{R}(t)$

- To show that say, $2t^2 - 100t^3$ is positive infinitesimal, we have to show that $0 < 2t^2 - 100t^3 < 1/n$ for all $n \in \mathbb{N}$. Indeed, $2t^2 - 100t^3 < 1/n$ is equivalent to $1/n - 2t + 100t^3 > 0$, which holds, because $1/n > 0$.
- To show that $3t^{-1}$ is positive infinitely large, we have to show that $n < 3t^{-1}$ for all $n \in \mathbb{N}$. Indeed, $n < 3t^{-1}$is equivalent to $3t^{-1} - n > 0$, which holds, since $3 > 0$.
- To show that $6 + t^2$ is finite, but not infinitesimal, we have to show that $1/m < 6 + t^2 < n$ for some $m, n \in \mathbb{N}$. Indeed, $m = 1$ and $n = 7$ will do.
- The End of the Preliminaries.

# Conventional Definition of Complex Numbers

- $\mathbb{R}(i) = \mathbb{R} \oplus i\mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. We often write $\mathbb{C}$ instead of $\mathbb{R}(i)$.
- Absolute value on $\mathbb{R}(i)$ is defined by $|x + iy|_{\mathbb{R}} = \sqrt{x^2 + y^2}$, where $x, y \in \mathbb{R}$. We often write $|z|$ instead of $|z|_{\mathbb{R}}$.
- **Theorem:** $\mathbb{R}(i)$ is an algebraically closed field of zero-characteristic and of cardinality $\mathfrak{c}$, where $\mathfrak{c} = \mathrm{card}(\mathbb{R})$.
- There is no $z \in \mathbb{R}(i)$ such that $0 < |z|_{\mathbb{R}} < 1/n$ for all $n \in \mathbb{N}$ (there are no non-zero infinitesimals in $\mathbb{R}(i)$, relative to $|z|_{\mathbb{R}}$).

# Abstract Definition of $\mathbb{C}$: Infinitesimals in $\mathbb{C}$

- **Abstract Definition:** The **field $\mathbb{C}$ of complex numbers** is (by definition) an algebraically closed field of zero-characteristic and of cardinality $\mathfrak{c} = \text{card}(\mathbb{R})$.
- **Justification (Steinitz Theorem):** All algebraically closed fields of zero-characteristic and the same cardinality are isomorphic (Steinitz [29]). Thus $\mathbb{C}$ (as defined above) is uniquely determined up to isomorphism.
- **Theorem-Observation:** Let $\mathcal{R}$ be a real closed field of cardinality $\mathfrak{c}$. Then $\mathcal{R}(i)$ is isomorphic to $\mathbb{C}$, i.e.

$$\mathbb{C} \cong \mathcal{R}(i).$$

We define an absolute value $|x + iy|_{\mathcal{R}} = \sqrt{x^2 + y^2}$ with the same formula, but now $x, y \in \mathcal{R}$. We say that $\mathcal{R}$ is a *real part of* $\mathbb{C}$.

# Examples of Real Closed Fields of cardinality $\mathfrak{c} = \text{card}(\mathbb{R})$

- The field $\mathbb{R}$ of real numbers.
- The field $\mathbb{R}\{t\}$ of **Puiseux-Newton series** (used by Newton for calculations algebraic curves and planet's orbits).
- The field $\mathbb{R}\langle t^{\mathbb{R}}\rangle$ of **Levi-Civitá series** (used nowadays for symbolic calculation of derivatives of real functions in computers).
- $\mathbb{R}((t^{\mathbb{R}}))$ is **Hahn field of power series** (developed in connection with Hilbert's seventh problem).
- The field ${}^{\rho}\mathbb{R}$ of **Robinson's asymptotic numbers** (used in the the non-standard version of Colombeau theory).
- The field ${}^{*}\mathbb{R}$ of **Robinson's non-standard real numbers** of cardinality $\mathfrak{c}$ (in the framework of NSA).

# Isomorphic Fields

- Thus (by the above Theorem-Observation) we have:

$$\mathbb{C} \cong \mathbb{R}(i) \cong \mathbb{R}\{t\}(i) \cong \mathbb{R}\langle t^{\mathbb{R}}\rangle(i) \cong$$
$$\cong \mathbb{R}((t^{\mathbb{R}}))(i) \cong {}^{\rho}\mathbb{R}(i) \cong {}^{*}\mathbb{R}(i).$$

- ${}^{\rho}\mathbb{R}(i)$ and ${}^{*}\mathbb{R}(i)$ appear in the literature under the notations ${}^{\rho}\mathbb{C}$ and ${}^{*}\mathbb{C}$, respectively (Oberguggenberger [20], Oberguggenberger &Todorov [21], Todorov &Vernaeve [33], Todorov [34]-[35]). The isomorphisms $\mathbb{C} \cong {}^{\rho}\mathbb{C} \cong {}^{*}\mathbb{C}$ remain so far unnoticed in mathematical community.
- **Theorem (Embeddings):**

$$\mathbb{R}(t) \subset \mathbb{R}\{t\} \subset \mathbb{R}\langle t^{\mathbb{R}}\rangle \subset \mathbb{R}((t^{\mathbb{R}})) \subset {}^{\rho}\mathbb{R} \subset {}^{*}\mathbb{R},$$

where all embeddings are canonical (subfields-subsets) except the last one.

# Infinitesimals in $\mathbb{C}$

- **Infinitesimals in** $\mathbb{C}$: Let $\mathcal{R}$ be one of the real closed non-Archimedean fields: $\mathbb{R}\{t\}$, $\mathbb{R}\langle t^{\mathbb{R}}\rangle$, $\mathbb{R}((t^{\mathbb{R}}))$, ${}^{\rho}\mathbb{R}$ or ${}^{*}\mathbb{R}$.
- Let $\sigma : \mathcal{R}(i) \mapsto \mathbb{C}$ be a field-isomorphism. Then $\sigma[\mathcal{R}]$ is a real closed subfield of $\mathbb{C}$.
- Notice that $\sigma(q) = q$ for all $q \in \mathbb{Q}$. Let $\rho$ be a positive infinitesimal in $\mathcal{R}$, i.e. $0 < \rho < 1/n$ in $\mathbb{R}$ for all $n \in \mathbb{N}$ (for example, $\rho = t$).
- Then $\sigma(\rho)$ is a positive infinitesimal in $\sigma[\mathcal{R}]$ as well, i.e. $0 < \sigma(\rho) < 1/n$ in $\sigma[\mathbb{R}]$ for all $n \in \mathbb{N}$. Thus

$$0 < |\sigma(\rho)|_{\mathcal{R}} < 1/n,$$

for all $n \in \mathbb{N}$, since $|\sigma(\rho)|_{\mathcal{R}} = \sigma(\rho)$.
- The latter means meaning that $\sigma(\rho)$ **is a positive infinitesimal in** $\mathbb{C}$ !!!

# Series Representation of the Fields: $\mathbb{R}$, $\mathbb{R}\{t\}$, $\mathbb{R}\langle t^{\mathbb{R}}\rangle$, $\mathbb{R}((t^{\mathbb{R}}))$, ${}^{\rho}\mathbb{R}$

- $\mathbb{R} = \left\{ \sum_{n=\mu}^{\infty} c_n (\frac{1}{10})^n : \mu \in \mathbb{Z},\ c_n \in \{0, 1, \dots, 9\} \right\}$, the **real numbers**.
- The **field of Puiseux series with real coefficients:**

$$\mathbb{R}\{t\} =: \left\{ \sum_{n=\mu}^{\infty} a_n (\sqrt[m]{t})^n : a_n \in \mathbb{R},\ \mu \in \mathbb{Z},\ m \in \mathbb{N} \right\},$$

  (also known as **Puiseux-Newton series**) (Prestel [23]).
- $\mathbb{R}\{t\}$ is a *real closure of the field of Laurent series* $\mathbb{R}\langle t^{\mathbb{Z}}\rangle$.
- Puiseux-Newton series were introduced by Isaac Newton in 1736 and used for calculation of orbits of planets. Later these series were rediscovered by Victor Puiseux.

# Levi-Civitá and Hahn Series

- The field of **Levi-Civitá series**:

$$\mathbb{R}\langle t^{\mathbb{R}}\rangle =: \Big\{ \sum_{n=0}^{\infty} a_n t^{r_n} : a_n \in \mathbb{R} \text{ and } (r_n) \in \mathbb{R}^{\mathbb{N}}, r_n \nearrow \infty \Big\},$$

  where $(r_n)$ strictly increasing and unbounded.
- For example, $t^{\sqrt{2}} + t^{\pi} + t^4 + t^5 + t^6 + \cdots \in \mathbb{R}\langle t^{\mathbb{R}}\rangle$.
- The field of **Hahn's (generalized) power series with coefficients in $\mathbb{R}$ and valuation group** $(\mathbb{R}, +, <)$ is defined by

$$\mathbb{R}((t^{\mathbb{R}})) = \Big\{ \sum_{r\in\mathbb{R}} a_r t^r : \ a_r \in \mathbb{R},\ \mathrm{supp}\big(\sum_{r\in\mathbb{R}} a_r t^r\big) \text{ is well-ordere}$$

  where $\mathrm{supp}\big(\sum_{r\in\mathbb{R}} a_r t^r\big) = \{r \in \mathbb{R} : a_r \neq 0\}$.
- For example, $1 + t^{1/2} + t^{2/3} + \cdots \in \mathbb{R}((t^{\mathbb{R}})) \setminus \mathbb{R}\langle t^{\mathbb{R}}\rangle$, since $\lim_{n\mapsto\infty} \frac{n}{n+1} = 1 \neq \infty$.

# Non-Standard Numbers ${}^*\mathbb{R}$

- Let $\mathbb{R}^{\mathbb{N}}$ stand for the partially ordered ring of sequences in $\mathbb{R}$ under the pointwise addition and multiplication and order.
- Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We define the **maximal convex ideal** in $\mathbb{R}^{\mathbb{N}}$ by

$$\mathcal{I}_{\mathcal{U}} = \{(a_n) \in \mathbb{R}^{\mathbb{N}} : Z(a_n) \in \mathcal{U}\},$$

  where $Z(a_n) = \{n \in \mathbb{N} : a_n = 0\}$ stands for the zero-set of the sequence $(a_n) \in \mathbb{R}^{\mathbb{N}}$.
- We have $\mathcal{I}_0 \subset \mathcal{I}_{\mathcal{U}}$, where $\mathcal{I}_0 = \{(a_n) \in \mathbb{R}^{\mathbb{N}} : \mathbb{N} \setminus Z(a_n)$ is a finite set$\}$.
- ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{I}_{\mathcal{U}}$ is the **field of non-standard real numbers**. We denote by $\langle a_n \rangle$ the equivalence class of $(a_n)$.

# Basic Properties of ${}^*\mathbb{R}$

- $\mathbb{R} \subset {}^*\mathbb{R}$ under $r \mapsto \langle r, r, \dots \rangle$.
- We define *order* on ${}^*\mathbb{R}$ by $\langle a_n \rangle > 0$ if $\langle a_n \rangle = \langle \varepsilon_n \rangle$ for some sequence $(\varepsilon_n)$ with positive terms $\varepsilon_n > 0$.
- ${}^*\mathbb{R}$ is a **real closed saturated non-Archimedean field**. In particular, $\langle 1/n \rangle$ is a positive infinitesimal and $\langle n \rangle$ is positive infinitely large number.
- Every finite number $x \in \mathcal{F}$ there exists unique real number $r \in \mathbb{R}$ such that $x \approx r$.
- Let $X$ be a subset of $\mathbb{R}$. We define its **non-standard extension** by ${}^*X = \{\langle x_n \rangle \in {}^*\mathbb{R} : \ (x_n) \in X^{\mathbb{N}}\}$.
- Let $X$ be a subset of $\mathbb{R}$ and $f : X \mapsto \mathbb{R}$ be a real function. We define its **non-standard extension**, ${}^*f : {}^*X \mapsto {}^*\mathbb{R}$, by

$$ {}^*f(\langle x_n \rangle) = \langle f(x_n) \rangle. $$

# Robinson's field of asymptotic numbers ${}^{\rho}\mathbb{R}$

- Let $\rho \in {}^*\mathbb{R}$ be a (fixed) positive infinitesimal The **Robinson's field of asymptotic numbers** is defined as ${}^{\rho}\mathbb{R} = \mathcal{M}_\rho / \mathcal{N}_\rho$, where

$$\mathcal{M}_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \rho^{-n} \text{ for some } n \in \mathbb{N}\},$$
$$\mathcal{N}_\rho = \{x \in {}^*\mathbb{R} : |x| < \rho^{n} \text{ for all } n \in \mathbb{N}\}.$$

  Notice that $e^{1/\rho} \notin \mathcal{M}_\rho$ and $e^{-1/\rho} \in \mathcal{N}_\rho$.
- If $x \in \mathcal{M}_\rho$, we let $\widehat{x} = x + \mathcal{N}_\rho$.
- ${}^{\rho}\mathbb{R}$ does not depend on the choice of $\rho$ in the sense that ${}^{\rho_1}\mathbb{R} \cong {}^{\rho_2}\mathbb{R}$ for every two $\rho_1$ and $\rho_2$.

# Hahn Series Representation of ${}^\rho\mathbb{R}$

- ${}^\rho\mathbb{R} \cong {}^*\mathbb{R}((t^{\mathbb{R}}))$, where

$$ {}^*\mathbb{R}((t^{\mathbb{R}})) = \{ \sum_{r\in\mathbb{R}} a_r\, t^r : a_r \in {}^*\mathbb{R},\ \{r \in \mathbb{R}; a_r \neq 0\} \text{ is a well or} $$

  is the Hahn field with coefficients in ${}^*\mathbb{R}$ and valuation group $(\mathbb{R}, +, <)$.
- We have the **field self-embeddings**

$$ {}^\rho\mathbb{R} \hookrightarrow {}^*\mathbb{R},\ {}^*\mathbb{R} \hookrightarrow {}^\rho\mathbb{R},\ {}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R},\ {}^\rho\mathbb{R} \hookrightarrow {}^\rho\mathbb{R}. $$

- ${}^\rho\mathbb{R}$ plays the role of the field of scalars of the algebra of asymptotic functions (a non-standard version of **Colombeau theory**).

# Philosophy: Infinitesimals are Observed or They are Created by Us ?

- **Two philosophical questions** (without answers):
- Infinitesimals exist and are present in $\mathbb{C}$ (regardless of us) and we use the absolute value $|\cdot|_{\mathcal{R}}$ (with non-Archimedean field $\mathcal{R}$) merely to observe them ?
- **or/and**
- We (humans) create infinitesimals within $\mathbb{C}$ by "intruding" into $\mathbb{C}$ with $\mathcal{R}$ and $|\cdot|_{\mathcal{R}}$ ?

# How Leibniz Derived $(x^3)' = 3x^2$

- **Definition:** $(x^3)'$ is the standard function $(x^3)' : \mathbb{R} \mapsto \mathbb{R}$ such that: $(x^3)' \approx \frac{(x+dx)^3 - x^3}{dx}$ for all non-zero infinitesimal $dx$.
- **Deriving the Formula:**

$$\frac{(x+dx)^3 - x^3}{dx} = \frac{x^3 + 3x^2dx + 3xdx^2 + dx^3 - x^3}{dx} = \\ = 3x^2 + 3xdx + dx^2 \approx 3x^2,$$

  because $3x^2$ is standard (real) and $3xdx + dx^2$ is infinitesimal.
- Thus

$$(x^3)' = 3x^2,$$

  as required.

# How Computers Calculate $(x^3)' = 3x^2$

- **Definition:** Left to Humans.
- **Algorithm:**

Step 1: Let $dx \neq 0$ is non-zero real (standard) variable. The algebra is the same:

$$\frac{(x+dx)^3 - x^3}{dx} = \frac{x^3 + 3x^2dx + 3xdx^2 + dx^3 - x^3}{dx} = \\ = 3x^2 + 3xdx + dx^2.$$

Step 2: Let $dx = 0$ and the result is: $(x^3)' = 3x^2$.

- The **logical paradox:** "$dx \neq 0$ and $dx = 0$" is left to Humans.

# The End

- Thank you very much.

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