Infinitesimals in the Field of Complex Numbers

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Abstract

We show that the field of complex numbers $\mathbb{C}$ contains non-zero infinitesimals. They (infinitesimals) become "visible" only under absolute values on $\mathbb{C}$ different from the usual absolute value. The talk is intended as a "mathematical entertainment", but it can be used for a more serious discussion as well. After all, any new property observed about $\mathbb{C}$ should be treated as a surprise, since the concept of complex numbers is one of the most popular and beloved ones in mathematics. An additional intrigue arises from the fact that infinitesimals are often an object of dislike, hate and obscurity in modern mathematics in spite of (and perhaps, exactly because of) their irreplaceable role for the creation of infinitesimal calculus by Leibniz, Newton, Euler and others.

Key Words: Totally ordered field, real closed field, Archimedean field, non-Archimedean field, algebraically closed field, Steinitz theorem, Artin-Schreier theorem, Puiseux-Newton series, Levi-Civitá series, Robinson's asymptotic numbers, Robinson's non-standard real numbers.

Contents:

1 Preliminaries

1. A field $\mathbb{K}$ is **formally real** (or **orderable**) if $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ always implies $x_1 = x_2 = \cdots = x_n = 0$. The field of real numbers $\mathbb{R}$ is formally real, while $\mathbb{C}$ is not.

2. A field $\mathbb{K}$ is **algebraically closed** if every polynomial $P \in \mathbb{K}[x]$ (with coefficients in $\mathbb{K}$) has at least one root in $\mathbb{K}$. The field $\mathbb{C}$ is algebraically closed, but $\mathbb{R}$ is not.

3. A formally real field $\mathcal{R}$ is **real closed** if

   (a) For every $a \in \mathcal{R}$ either a, or $-a$ has a square root.

   (b) Every polynomial $P \in \mathcal{R}[x]$ (with coefficients in $\mathcal{R}$) of *odd degree* has a root in $\mathcal{R}$.

4. **Examples:**

   - The **field $\mathbb{R}$ of real numbers is real closed**. Also the real algebraic numbers is also real closed. More examples follow later in the talk.
   - The **field $\mathbb{Q}$ of rational numbers is not real closed**, because neither $\sqrt{3}$, nor $\sqrt{-3}$ are in $\mathbb{Q}$.

5. **Theorem (Artin-Schreier):** Let $\mathcal{R}$ is a formally real field. Then $\mathcal{R}$ is *real closed* iff $\mathcal{R}(i)$ is *algebraically closed* (Artin & Schreier [2]). Here $\mathcal{R}(i) = \mathcal{R} \oplus i\mathcal{R}$.

6. A totally ordered field $\mathcal{R}$ is **Archimedean** if $\mathcal{R}$ has no infinitely large elements (or equivalently, has no non-zero infinitesimals): for every $x \in \mathcal{R}$ there exists $n \in \mathbb{N}$ such that $|x| \leq n$. Here $|x| = \max\{x, -x\}$.

7. $\mathcal{R}$ is **non-Archimedean** if it is not Archimedean, i.e. if there exists $x \in \mathcal{R}$ such that $n < x$ for all $n \in \mathbb{N}$.

8. **Definition:** Let $\mathcal{R}$ be a totally ordered field extension of $\mathbb{R}$. Let $x \in \mathcal{R}$. Then:

   (a) x is **finite** if $|x| \leq n$ for some $n \in \mathbb{N}$.

   (b) x is **infinitesimal** if $|x| < 1/n$ for all $n \in \mathbb{N}$.

(c) x is **infinitely large** if $n < |x|$ for all $n \in \mathbb{N}$.

We denote by $\mathcal{F}, \mathcal{I}$ and $\mathcal{L}$ the set of the finite, infinitesimal and infinitely large numbers in $\mathcal{R}$. We write $x \approx y$ if $x - y \in \mathcal{I}$. Notice that every totally ordered field contains $\mathbb{Q}$ as a subfield (thus n, $1/n \in \mathcal{R}$).

9. We have $\mathcal{F} \cup \mathcal{L} = \mathcal{R}$, $\mathcal{F} \cap \mathcal{L} = \varnothing$, $\mathcal{I} \subset \mathcal{F}$, $\mathbb{R} \cap \mathcal{I} = \{0\}$. Let $x \in \mathcal{R}, x \neq 0$. Then $x \in \mathcal{I}$ iff $1/x \in \mathcal{L}$.

10. Every finite number $x \in \mathcal{F}$ there exists unique real number $r \in \mathbb{R}$ such that $x \approx r$. Consequently, $\mathcal{I}$ is a maximal ideal in $\mathcal{F}$ and $\mathcal{F}/\mathcal{I} = \mathbb{R}$.

11. The field $\mathcal{R}$ is Archimedean iff $\mathcal{I} = \{0\}$ iff $\mathcal{R} = \mathcal{F}$ iff $\mathcal{L} = \varnothing$.

12. The field $\mathcal{R}$ is non-Archimedean iff $\mathcal{I} \neq \{0\}$ iff $\mathcal{R} \neq \mathcal{F}$ iff $\mathcal{L} \neq \varnothing$.

13. **Example:** Let $\mathbb{R}(t) = \frac{\mathbb{R}[t]}{\mathbb{R}[t]}$ denotes the **field of the rational functions in one variable with real coefficients**. We have:

   - $\mathbb{R} \subset \mathbb{R}(t)$ under the embedding $r \mapsto rt^0$.
   - Let $P \in \mathbb{R}[t]$ be a polynomial. We define $P > 0$ if **the coefficient in front of the lowest power of t is positive**. For example $2t^2 - 100t^3 > 0$, because $2 > 0$. Thus $\mathbb{R}(t)$ converts in a **totally ordered field**.
   - The field of rational functions $\mathbb{R}(t)$ **is not real closed**, because neither $\sqrt{t}$, nor $\sqrt{-t}$ are in $\mathbb{R}(t)$. However, the field $\mathbb{R}(t)$ is a **non-Archimedean field**:

     (a) $t,\ -t,\ 2t - t^2,\ t^2,\ t^2 - 3t^3, \ldots$ are **non-zero infinitesimals**.

     (b) $t^{-1},\ -t^{-1},\ 3t^{-1},\ t^{-2},\ t^{-2} - 3t^3, \ldots$ are **infinitely large**.

     (c) $2,\ 1+t,\ 2-t,\ 5+2t,\ 6+t^2,\ \pi + t^2 - 3t^3, \ldots$ are **finite, but not infinitesimal**.
   - To show that say, $2t^2 - 100t^3$ is positive infinitesimal, we have to show that $0 < 2t^2 - 100t^3 < 1/n$ for all $n \in \mathbb{N}$. Indeed, $2t^2 - 100t^3 < 1/n$ is equivalent to $1/n - 2t + 100t^3 > 0$, which holds, because $1/n > 0$.
   - To show that $3t^{-1}$ is positive infinitely large, we have to show that $n < 3t^{-1}$ for all $n \in \mathbb{N}$. Indeed, $n < 3t^{-1}$is equivalent to $3t^{-1} - n > 0$, which holds, since $3 > 0$.

- To show that $6+t^2$ is finite, but not infinitesimal, we have to show that $1/m < 6+t^2 < n$ for some $m, n \in \mathbb{N}$. Indeed, $m = 1$ and $n = 7$ will do.

2 Conventional Definition of Complex Numbers

1. $\mathbb{R}(i) = \mathbb{R} \oplus i\mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. We often write $\mathbb{C}$ instead of $\mathbb{R}(i)$.

2. Absolute value on $\mathbb{R}(i)$ is defined by $|x+iy|_{\mathbb{R}} = \sqrt{x^2+y^2}$, where $x, y \in \mathbb{R}$. We often write $|z|$ instead of $|z|_{\mathbb{R}}$.

3. **Theorem:** $\mathbb{R}(i)$ is an algebraically closed field of zero-characteristic and of cardinality $\mathfrak{c}$, where $\mathfrak{c} = \text{card}\,(\mathbb{R})$.

4. There is no $z \in \mathbb{R}(i)$ such that $0 < |z|_{\mathbb{R}} < 1/n$ for all $n \in \mathbb{N}$ (there are no non-zero infinitesimals in $\mathbb{R}(i)$, relative to $|z|_{\mathbb{R}}$).

3 Abstract Definition of Complex Numbers: Infinitesimals in $\mathbb{C}$

1. **Abstract Definition:** $\mathbb{C}$ is an algebraically closed field of zero-characteristic and of cardinality $\mathfrak{c} = \text{card}\,(\mathbb{R})$.

2. Notice $\mathbb{R}$ is not a part of the above definition (except throughout its cardinality, $\mathfrak{c}$).

3. **Justification (Steinitz Theorem):** All algebraically closed fields of zero-characteristic and the same cardinality are isomorphic (Steinitz [29]). Thus $\mathbb{C}$ (as defined above) is uniquely determined up to isomorphism.

4. **Theorem-Observation:** Let $\mathcal{R}$ be a real closed field of cardinality $\mathfrak{c}$. Then $\mathcal{R}(i)$ is isomorphic to $\mathbb{C}$, i.e.

$$\mathbb{C} \cong \mathcal{R}(i).$$

   We define an absolute value $|x+iy|_{\mathcal{R}} = \sqrt{x^2+y^2}$ with the same formula, but now $x, y \in \mathcal{R}$. We say that $\mathcal{R}$ is a *real part of* $\mathbb{C}$.

5. There are plenty of mutually non-isomorphic real closed fields $\mathcal{R}$ of cardinality $\mathfrak{c} = \operatorname{card}(\mathbb{R})$ (Archimedean and non-Archimedean alike). Here are some of them:

   (a) $\mathbb{R}$ is an Archimedean real closed field. The rest of the fields below are non-Archimedean of cardinality $\mathfrak{c} = \operatorname{card} \mathbb{R}$:

   (b) The field $\mathbb{R}\{t\}$ of **Puiseux-Newton series** used by Newton for calculations algebraic curves and planet's orbits (Prestel [23]).

   (c) The field $\mathbb{R}\langle t^{\mathbb{R}}\rangle$ of **Levi-Civitá series** used nowadays for symbolic calculation of derivatives of real functions in computers (Levi-Civitá [12]).

   (d) $\mathbb{R}((t^{\mathbb{R}}))$ is **Hahn field of power series** developed in connection with Hilbert's seventh problem (Hahn [8]).

   (e) The field ${}^{\rho}\mathbb{R}$ of **Robinson's asymptotic numbers** used in the the non-standard version of Colombeau theory (Robinson [27], Lightstone & Robinson [13]).

   (f) The field ${}^{*}\mathbb{R}$ of **Robinson's non-standard real numbers** of cardinality $\mathfrak{c}$ in the framework of non-standard analysis (Robinson [26]).

   (g) An many others...

6. Thus (by the above Theorem-Observation) we have:

$$\mathbb{C} \cong \mathbb{R}(i) \cong \mathbb{R}\{t\}(i) \cong \mathbb{R}\langle t^{\mathbb{R}}\rangle(i) \cong \mathbb{R}((t^{\mathbb{R}}))(i) \cong {}^{\rho}\mathbb{R}(i) \cong {}^{*}\mathbb{R}(i).$$

   3.1 Remark. ${}^{\rho}\mathbb{R}(i)$ and ${}^{*}\mathbb{R}(i)$ appear in the literature under the notations ${}^{\rho}\mathbb{C}$ and ${}^{*}\mathbb{C}$ (Oberguggenberger [20], Oberguggenberger &Todorov [21], Todorov &Vernaeve [33], Todorov [34]-[35]). We refer to ${}^{\rho}\mathbb{C}$ and ${}^{*}\mathbb{C}$ as the **fields of the complex asymptotic and complex non-standard numbers**, respectively. As far as we can detect, the isomorphisms $\mathbb{C} \cong {}^{\rho}\mathbb{C} \cong {}^{*}\mathbb{C}$ remained so far unnoticed in the mathematical community.

7. **Theorem (Embeddings):**

$$\mathbb{R}(t) \subset \mathbb{R}\{t\} \subset \mathbb{R}\langle t^{\mathbb{R}}\rangle \subset \mathbb{R}((t^{\mathbb{R}})) \subset {}^{\rho}\mathbb{R} \subset {}^{*}\mathbb{R},$$

   where all embeddings are canonical (subfields-subsets) except the last one. Here $\mathbb{R}(t)$ is the field of rational functions mentioned earlier.

8. **Corollary:** The fields $\mathbb{R}\{t\}$, $\mathbb{R}\langle t^{\mathbb{R}}\rangle$, $\mathbb{R}((t^{\mathbb{R}}))$, ${}^{\rho}\mathbb{R}$, ${}^{*}\mathbb{R}$ **are all non-Archimedean**.

   Proof: $\mathbb{R}(t)$ is non-Archimedean.

9. **Example:** The series $t + 2!\, t^2 + 3!\, t^3 + \dots$ is a positive infinitesimal in $\mathbb{R}\langle t^{\mathbb{R}}\rangle$, $\mathbb{R}((t^{\mathbb{R}}))$, ${}^{\rho}\mathbb{R}$ and ${}^{*}\mathbb{R}$, but not in $\mathbb{R}(t)$.

10. **Infinitesimals in** $\mathbb{C}$: Let $\mathcal{R}$ be one of the real closed non-Archimedean fields: $\mathbb{R}\{t\}$, $\mathbb{R}\langle t^{\mathbb{R}}\rangle$, $\mathbb{R}((t^{\mathbb{R}}))$, ${}^{\rho}\mathbb{R}$ or ${}^{*}\mathbb{R}$. Let $\sigma : \mathcal{R}(i) \mapsto \mathbb{C}$ be a field-isomorphism. Then $\sigma[\mathcal{R}]$ is a real closed subfield of $\mathbb{C}$. Notice that $\sigma(q) = q$ for all $q \in \mathbb{Q}$. Let ρ be a positive infinitesimal in $\mathcal{R}$, i.e. $0 < \rho < 1/n$ in $\mathbb{R}$ for all $n \in \mathbb{N}$ (for example, $\rho = t$). Then $\sigma(\rho)$ is a positive infinitesimal in $\sigma[\mathcal{R}]$ as well, i.e. $0 < \sigma(\rho) < 1/n$ in $\sigma[\mathbb{R}]$ for all $n \in \mathbb{N}$. Thus $0 < |\sigma(\rho)|_{\mathcal{R}} < 1/n$ for all $n \in \mathbb{N}$, since $|\sigma(\rho)|_{\mathcal{R}} = \sigma(\rho)$. The latter means meaning that $\sigma(\rho)$ is a positive infinitesimal in $\mathbb{C}$!!!

4 Series Representation of the Fields: $\mathbb{R}$, $\mathbb{R}\{t\}$, $\mathbb{R}\langle t^{\mathbb{R}}\rangle$, $\mathbb{R}((t^{\mathbb{R}}))$, ${}^{\rho}\mathbb{R}$

1. $\mathbb{R} = \left\{ \sum_{n=\mu}^{\infty} c_n (\frac{1}{10})^n : \mu \in \mathbb{Z},\ c_n \in \{0, 1, \dots, 9\} \right\}$, the **real numbers**.

2. $\mathbb{R}\{t\} =: \left\{ \sum_{n=\mu}^{\infty} a_n (\sqrt[m]{t})^n : \mu \in \mathbb{Z},\ m \in \mathbb{N},\ a_n \in \mathbb{R} \right\}$ is the **field of Puiseux series** with real coefficients (also known as **Puiseux-Newton series**) (Prestel [23]).

   - $\mathbb{R}\{t\}$ is a *real closure of the field of Laurent series* $\mathbb{R}\langle t^{\mathbb{Z}}\rangle$.
   - Puiseux series were introduced by Isaac Newton [19] and later rediscovered by Victor Puiseux [24]-[25].

3. The field of **Levi-Civita series** with real coefficients is defined by

$$\mathbb{R}\langle t^{\mathbb{R}}\rangle =: \left\{ \sum_{r \in \mathbb{R}} a_r t^r : a_r \in \mathbb{R} \text{ and } \{r \in \mathbb{R} : a_r \neq 0\} \text{ is a left-finite set} \right\}.$$

   Recall that S is a *left-finite* subset of $\mathbb{R}$ if for every $r \in \mathbb{R}$ the set $\{s \in S : s \leq r\}$ is finite. Here is a characterization: $\sum_{r \in \mathbb{R}} a_n t^r$ is a Levi-Civitá series with real coefficients iff $\sum_{r \in \mathbb{R}} a_r t^r$ can be presented in the form

$\sum_{n=0}^{\infty} b_n t^{r_n}$ for some sequence (b_n) in $\mathbb{R}$ and some strictly increasing unbounded sequence (r_n) in $\mathbb{R}$ (Levi-Civitá [12]). For example, $t^{\sqrt{2}} + t^{\pi} + t^4 + t^5 + t^6 + \cdots \in \mathbb{R}\langle t^{\mathbb{R}}\rangle$.

4. The field of **Hahn's (generalized) series with coefficients in** $\mathbb{R}$ **and valuation group** $(\mathbb{R}, +, <)$ is defined by

$$\mathbb{R}((t^{\mathbb{R}})) =: \Big\{ \sum_{r\in\mathbb{R}} a_r t^r : a_r \in \mathbb{R} \text{ and } \{r \in \mathbb{R} : a_r \neq 0\} \text{ is a well ordered set} \Big\},$$

(Hahn [8]). For example, $1 + t^{1/2} + t^{2/3} + t^{3/4} + \cdots \in \mathbb{R}((t^{\mathbb{R}})) \setminus \mathbb{R}\langle t^{\mathbb{R}}\rangle$, since $\lim_{n\mapsto\infty} \frac{n}{n+1} = 1 \neq \infty$.

5. Non-Standard Numbers ${}^*\mathbb{R}$ (Robinson [26]):

   - Let $\mathbb{R}^{\mathbb{N}}$ stand for the partially ordered ring of sequences in $\mathbb{R}$ under the pointwise addition and multiplication and order.
   - Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We define the **maximal convex ideal** in $\mathbb{R}^{\mathbb{N}}$ by

     $$\mathcal{I}_{\mathcal{U}} = \{(a_n) \in \mathbb{R}^{\mathbb{N}} : Z(a_n) \in \mathcal{U}\},$$

     where $Z(a_n) = \{n \in \mathbb{N} : a_n = 0\}$ stands for the zero-set of the sequence $(a_n) \in \mathbb{R}^{\mathbb{N}}$.
   - We have $\mathcal{I}_0 \subset \mathcal{I}_{\mathcal{U}}$, where $\mathcal{I}_0 = \{(a_n) \in \mathbb{R}^{\mathbb{N}} : \mathbb{N} \setminus Z(a_n) \text{ is a finite set}\}$.
   - ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{I}_{\mathcal{U}}$ is the field of *non-standard real numbers*. We denote by $\langle a_n\rangle$ the equivalence class of (a_n).
   - $\mathbb{R} \subset {}^*\mathbb{R}$ under $r \mapsto \langle r, r, \ldots\rangle$.
   - We define *order* on ${}^*\mathbb{R}$ by $\langle a_n\rangle > 0$ if $\langle a_n\rangle = \langle \varepsilon_n\rangle$ for some sequence (ε_n) with positive terms $\varepsilon_n > 0$.
   - ${}^*\mathbb{R}$ is a **real closed saturated non-Archimedean field**. In particular, $\langle 1/n\rangle$ is a positive infinitesimal and $\langle n\rangle$ is positive infinitely large number.
   - Every finite number $x \in \mathcal{F}$ there exists unique real number $r \in \mathbb{R}$ such that $x \approx r$.
   - Let X be a subset of $\mathbb{R}$. We define its **non-standard extension** by ${}^*X = \{\langle x_n\rangle \in {}^*\mathbb{R} :\ (x_n) \in X^{\mathbb{N}}\}$.

- Let X be a subset of $\mathbb{R}$ and $f : X \mapsto \mathbb{R}$ be a real function. We define its **non-standard extension**, ${}^*f : {}^*X \mapsto {}^*\mathbb{R}$, by

$$ {}^*f(\langle x_n \rangle) = \langle f(x_n) \rangle. $$

6. Let $\rho \in {}^*\mathbb{R}$ be a (fixed) positive infinitesimal. The field **Robinson's field of asymptotic numbers** is defined as ${}^\rho\mathbb{R} = \mathcal{M}_\rho / \mathcal{N}_\rho$, where

$$ \mathcal{M}_\rho = \{x \in {}^*\mathbb{R} : |x| \le \rho^{-n} \text{ for some } n \in \mathbb{N}\}, $$
$$ \mathcal{N}_\rho = \{x \in {}^*\mathbb{R} : |x| < \rho^{n} \text{ for all } n \in \mathbb{N}\}, $$

(Robinson [27], Lightstone & Robinson [13]). Notice that $e^{1/\rho} \notin \mathcal{M}_\rho$ and $e^{-1/\rho} \in \mathcal{N}_\rho$. If $x \in \mathcal{M}_\rho$, we let $\widehat{x} = x + \mathcal{N}_\rho$.

7. ${}^\rho\mathbb{R}$ does not depend on the choice of ρ in the sense that ${}^{\rho_1}\mathbb{R} \cong {}^{\rho_2}\mathbb{R}$ for every two positive infinitesimals, ρ_1 and ρ_2.

8. ${}^\rho\mathbb{R} \cong {}^*\mathbb{R}((t^{\mathbb{R}}))$, where ${}^*\mathbb{R}((t^{\mathbb{R}}))$ is the Hahn field with coefficients in ${}^*\mathbb{R}$ and valuation group $(\mathbb{R}, +, <)$, i.e.

$$ {}^*\mathbb{R}((t^{\mathbb{R}})) = \Big\{ \sum_{r \in \mathbb{R}} a_r\, t^r : \; a_r \in {}^*\mathbb{R},\; \{r \in \mathbb{R}; a_r \ne 0\} \text{ is a well ordered set} \Big\}, $$

(Todorov & Wolf [31]).

9. We have the **field self-embeddings** ${}^\rho\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ and ${}^*\mathbb{R} \hookrightarrow {}^\rho\mathbb{R}$.

10. ${}^\rho\mathbb{R}$ plays the role of the field of scalars of the algebra of asymptotic functions in the non-standard version of **Colombeau theory** (Todorov &Vernaeve [33]).

5 Philosophy: Infinitesimals are Observed or They are Created by Us ?

Two philosophical questions (without answers):

1. Infinitesimals exist and are present in $\mathbb{C}$ (regardless of us) and we use the absolute value $|\cdot|_{\mathcal{R}}$ (with non-Archimedean field $\mathcal{R}$) merely to observe them ?

   or/and

2. We (humans) create infinitesimals within $\mathbb{C}$ by "intruding" into $\mathbb{C}$ with $\mathcal{R}$ and $|\cdot|_{\mathcal{R}}$?

6 How Leibniz Derived $(x^3)' = 3x^2$

1. **Definition:**
   (a) $(x^3)'$ is the standard function $(x^3)' : \mathbb{R} \mapsto \mathbb{R}$ such that:
   (b) $(x^3)' \approx \frac{(x+dx)^3 - x^3}{dx}$ for every non-zero infinitesimal dx.

2. **Deriving the Formula:**

$$\frac{(x+dx)^3 - x^3}{dx} = \frac{x^3 + 3x^2dx + 3xdx^2 + dx^3 - x^3}{dx} = 3x^2 + 3xdx + dx^2 \approx 3x^2,$$

   because $3x^2$ is standard (real) and $3xdx + dx^2$ is infinitesimal. Thus

$$(x^3)' = 3x^2,$$

   as required.

7 How Computers Calculate $(x^3)' = 3x^2$

1. **Definition:** Left to Humans.

2. **Algorithm:**
   (a) **Step 1:** Let $dx \neq 0$ is non-zero real (standard) variable. The algebra is the same:

$$\frac{(x+dx)^3 - x^3}{dx} = \frac{x^3 + 3x^2dx + 3xdx^2 + dx^3 - x^3}{dx} = 3x^2 + 3xdx + dx^2.$$

   (b) **Step 2:** Let $dx = 0$ and the result is:

$$(x^3)' = 3x^2.$$

3. The **logical paradox:** "$dx \neq 0$ and $dx = 0$" is left to Humans.

4. The above logical paradox is exactly the reason for the **revolution against infinitesimals** starting from Bolzano and Weirstrass in 19-th century.

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