Heisenberg relations in the general case

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Abstract

The Heisenberg relations are derived in a quite general setting when the field transformations are induced by three representations of a given group. They are considered also in the fibre bundle approach. The results are illustrated in a case of transformations induced by the Poincaré group.

Contents

1. Heisenberg relations

The (global) origin of the Heisenberg relations is in the equations like

$$\varphi'_i(r) = U \circ \varphi_i(r) \circ U^{-1}. \qquad (1.1)$$

which connect the components φ_i and φ'_i of a quantum field φ with respect to two frames of reference. Here U is an operator acting on the state vectors of the quantum system considered and it is expected that the transformed field operators φ'_i can be expressed explicitly by means of φ_i via some equations.

If the elements U (of the representation) of the group are labeled by $b = (b^1, \ldots, b^s) \in \mathbb{K}^s$ for some $s \in \mathbb{N}$, we may write $U(b)$ for U, and then the corresponding Heisenberg relations are obtained from the above equation with $U(b)$ for U by differentiating it with respect to b^ω, $\omega = 1, \ldots, s$, and then setting $b = b_0$, where $b_0 \in \mathbb{K}^s$ is such that $U(b_0)$ is the identity element.

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The Heisenberg relations are from pure geometric origin and the only physics in them is the motivation leading to the above equation. There are evidences that to them can be given physical sense by identifying in them the generators (of the representation) of the group by the corresponding operators of conserved physical quantities.

2. The Poincaré group

Let us have a quantum field with components φ_i relative to two reference frames connected by a general Poincaré transformation

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The "global" version of the Heisenberg relations is

$$U(\Lambda, a) \circ \varphi_i(x) \circ U^{-1}(\Lambda, a) = D_i^j(\Lambda, a)\varphi_j(\Lambda x + a), \quad (2.2)$$

where U (D) is a representation of the Poincaré group on the space of state vectors (field operators), $U(\Lambda, a)$ $(\boldsymbol{D}(\Lambda, a) = [D_i^j(\Lambda, a)])$ is the mapping (the matrix of the mapping) corresponding via U (D) to (2.1). We have $U(\mathbb{1}, \boldsymbol{0}) = \mathrm{id}$ and $\boldsymbol{D}(\mathbb{1}, \boldsymbol{0}) = \mathbb{1}$.

Define

$$T_\mu := \left.\frac{\partial U(\Lambda, a)}{\partial a^\mu}\right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})} \qquad S_{\mu\nu} := \left.\frac{\partial U(\Lambda, a)}{\partial \Lambda^{\mu\nu}}\right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})}$$

$$H^i_{j\mu} := \left.\frac{\partial D^i_j(\Lambda, a)}{\partial a^\mu}\right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})} \qquad I^i_{j\mu\nu} := \left.\frac{\partial D^i_j(\Lambda, a)}{\partial \Lambda^{\mu\nu}}\right|_{(\Lambda, a) = (\mathbb{1}, \mathbf{0})}.$$

Differentiating (2.2) relative to a^μ and setting after that $(\Lambda, a) = (\mathbb{1}, \mathbf{0})$, we find

$$[T_\mu, \varphi_i(x)]_- = \partial_\mu \varphi_i(x) + H^j_{i\mu} \varphi_j(x), \qquad (2.4)$$

where $[A, B]_- := AB - BA$.

For field theories invariant relative to spacetime translation of the coordinates, i.e. $x \mapsto x + a$, we can suppose that

$$H^i_{j\mu} = 0. \tag{2.5}$$

In this case equation (2.4) reduces to

$$[T_\mu, \varphi_i(x)]_- = \partial_\mu \varphi_i(x). \tag{2.6a}$$

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Similarly, differentiation (2.2) with respect to $\Lambda^{\mu\nu}$ and putting after that $(\Lambda, a) = (\mathbb{1}, \mathbf{0})$, we obtain

$$[S_{\mu\nu}, \varphi_i(x)]_- = x_\mu \partial_\nu \varphi_i(x) - x_\nu \partial_\mu \varphi_i(x) + I^j_{i\mu\nu} \varphi_j(x). \tag{2.6b}$$

As we have mentioned earlier, the Heisenberg relations are from pure geometrical origin. Recalling that the translation (resp. rotation) invariance of a (Lagrangian) field theory results to conservation of system's momentum (resp. angular momentum) operator P_μ (resp. $M_{\mu\nu}$) and the correspondences

$$i\hbar T_\mu \mapsto P_\mu \qquad i\hbar S_{\mu\nu} \mapsto M_{\mu\nu}, \qquad (2.7)$$

with $\hbar$ being the Planck's constant, one may suppose the validity of the Heisenberg relations

$$[P_\mu, \varphi_i(x)]_- = i\hbar \partial_\mu \varphi_i(x) \qquad (2.8a)$$

$$[M_{\mu\nu}, \varphi_i(x)]_- = i\hbar \{ x_\mu \partial_\nu \varphi_i(x) - x_\nu \partial_\mu \varphi_i(x) + I^j_{i\mu\nu} \varphi_j(x) \}. \qquad (2.8b)$$

However, one should be careful when applying the last two equations in the Lagrangian formalism as they are external to it and need a particular proof in this approach; e.g. they hold in the free field theory, but a general proof seems to be missing. In the axiomatic quantum field theory these equations are identically valid as in it the generators of the translations (rotations) are identified up to a constant factor with the components of the (angular) momentum operator, $P_\mu = \mathrm{i}\hbar T_\mu \ (M_{\mu\nu} = \mathrm{i}\hbar S_{\mu\nu})$.

3. Internal transformations

An internal transformation is a change of the reference frame $(u, \{e^i\})$, consisting of a local coordinate system u and a frame $\{e^i\}$ in some vector space V, such that the spacetime coordinates remain unchanged. We suppose that $e^i \colon x \in M \mapsto e^i(x) \in V$, where M is the Minkowski spacetime and the quantum field φ considered takes values in V, i.e.

$$\varphi \colon x \in M \mapsto \varphi(x) = \varphi_i(x)e^i(x) \in V$$

Let G be a group whose elements g_b are labeled by $b \in \mathbb{K}^s$ for some $s \in \mathbb{N}$. Consider two reference frames $(u, \{e^i\})$ and $(u', \{e'^i\})$, with $u' = u$ and $\{e^i\}$ and $\{e'^i\}$ being connected via a matrix $I^{-1}(b)$, where $I \colon G \mapsto \mathrm{GL}(\dim V, \mathbb{K})$ is a matrix representation of G and $I \colon G \ni g_b \mapsto I(b) \in \mathrm{GL}(\dim V, \mathbb{K})$. The components of the fields transform into

$$\varphi'_{u,i}(r) = U(b) \circ \varphi_{u,i}(r) \circ U^{-1}(b) \qquad (3.1)$$

where U is a representation of G on the Hilbert space of state vectors and $U \colon G \ni g_b \mapsto U(b)$.

So that

$$U(b) \circ \varphi_{u,i}(r) \circ U^{-1}(b) = I_i^j(b)\varphi_{u,j}(r). \qquad (3.2)$$

Let g_{b_0} is the identity element of G for some $b_0 \in \mathbb{K}^s$ and for $b = (b^1, \ldots, b^s)$ and $\omega = 1, \ldots, s$ define

$$Q_\omega := \frac{\partial U(b)}{\partial b^\omega}\bigg|_{b=b_0} \qquad I_{i\omega}^j := \frac{\partial I_i^j(b)}{\partial b^\omega}\bigg|_{b=b_0}. \qquad (3.3)$$

Differentiation (3.2) with respect to b^ω and putting in the result $b = b_0$, we get the Heisenberg relation

$$[Q_\omega, \varphi_{u,i}(r)]_- = I_{i\omega}^j \varphi_{u,j}(r) \qquad (3.4)$$

or, if we identify $x \in M$ with $r = u(x)$ and omit the subscript u,

$$[Q_\omega, \varphi_i(x)]_- = I_{i\omega}^j \varphi_j(x). \qquad (3.5)$$

Example. Consider one-dimensional group G, $s = 1$, when $\omega = 1$; so we identify b^1 with $b = (b^1)$. Besides, let

$$I(b) = \mathbb{1} \exp(f(b) - f(b_0)) \qquad (3.6)$$

for some C^1 function f. Then (3.5) reduces to

$$[Q_1, \varphi_i(x)]_- = f'(b_0)\varphi_i(x), \qquad (3.7)$$

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In particular, if we are dealing with phase transformations, i.e.

$$U(b) = \mathrm{e}^{\frac{1}{\mathrm{i}e}bQ_1} \qquad I(b) = \mathbb{1}\mathrm{e}^{-\frac{q}{\mathrm{i}e}b} \qquad b \in \mathbb{R} \tag{3.8}$$

for some constants q and e (having a meaning of charge and unit charge, respectively) and operator Q_1

on system's Hilbert space of states (having a meaning of a charge operator), then we arrive to the familiar equations

$$\varphi'_i(x) = e^{\frac{1}{ie}bQ_1} \circ \varphi_i(x) \circ e^{-\frac{1}{ie}bQ_1} = e^{-\frac{q}{ie}b}\varphi(x) \qquad (3.9)$$

$$[Q_1, \varphi_i(x)]_- = -q\varphi_i(x). \qquad (3.10)$$

on system's Hilbert space of states (having a meaning of a charge operator), then we arrive to the familiar equations

$$\varphi_i'(x) = e^{\frac{1}{ie}bQ_1} \circ \varphi_i(x) \circ e^{-\frac{1}{ie}bQ_1} = e^{-\frac{q}{ie}b}\varphi(x) \qquad (3.9)$$

$$[Q_1, \varphi_i(x)]_- = -q\varphi_i(x). \qquad (3.10)$$

The considerations in the framework of Lagrangian formalism invariant under phase transformations implies conservation of the charge operator Q and suggests the correspondence

$$Q_1 \mapsto Q \qquad (3.11)$$

which in turn suggests the Heisenberg relation

$$[Q, \varphi_i(x)]_- = -q\varphi_i(x). \qquad (3.12)$$

4. The general case

The corner stone of the (global) Heisenberg relations is the equation

$$U \circ \varphi_{u,i}(r) \circ U^{-1} = \frac{\partial (u' \circ u^{-1})(r)}{\partial r} \left(A^{-1}(u^{-1}(r)) \right)_i^j \varphi_{u,j}((u' \circ u^{-1})(r)$$

$$(4.1)$$

representing the components $\varphi'_{u',i}$ of a quantum field φ in a reference frame $(u, \{e'^i = A_j^i e^j\})$ via its components $\varphi_{u,j}$ in a frame $(u, \{e^i\})$ in two different way. Here $A = [A_i^j]$ is a non-degenerate matrix-valued function, $r \in \mathbb{R}^4$ and $\varphi_{u,i} := \varphi_i \circ u^{-1}$.

Let G be s-dimensional, $s \in \mathbb{N}$, Lie group. Let its elements be labeled by $b = (b^1, \ldots, b^s) \in \mathbb{K}^s$ and g_{b_0} is the identity element of G for some fixed $b_0 \in \mathbb{K}^s$. Let there are given three representations H, I and U of G and:

1. $H : G \ni g_b \mapsto H_b : \mathbb{R}^{\dim M} \longrightarrow \mathbb{R}^{\dim M}$ and any change $(U, u) \mapsto (U', u')$ of the charts of M is such that $u' \circ u^{-1} = H_b$ for some $b \in \mathbb{K}^s$.

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2. $I \colon G \ni g_b \mapsto I(b) \in \mathrm{GL}(\dim V, \mathbb{K})$ and any change $\{e_i\} \mapsto \{e'_i = A^j_i e_j\}$ of the frames in V is such that $A^{-1}(x) = I(b)$ for all $x \in M$ and some $b \in \mathbb{K}^s$.

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3. $U \colon G \ni g_b \mapsto U(b)$, where $U(b)$ is an operator on the space of state vectors, and the changes $(u, \{e_i\}) \mapsto (u', \{e'_i\})$ of the reference frames entail (1.1) with $U(b)$ for U.

Under the above hypotheses we have

$$U(b) \circ \varphi_{u,i}(r) \circ U^{-1}(b) = \det\left[\frac{\partial(H_b(r))^i}{\partial r^j}\right] I_i^j(b)\varphi_{u,j}(H_b(r))$$

(4.2)

which can be called global Heisenberg relation in the particular situation. The next step is to differentiate this equation with respect to b^ω, $\omega = 1, \ldots, s$, and then to put $b = b_0$ in the result. In this way we obtain the following (local) Heisenberg relation

$$[U_\omega, \varphi_{u,i}(r)]_- = \Delta_\omega(r)\varphi_{u,i}(r) + I_{i\omega}^j\varphi_{u,j}(r) + (h_{\omega(r)})^k\frac{\partial\varphi_{u,i}(r)}{\partial r^k},$$

(4.3)

where

$$U_\omega := \frac{\partial U(b)}{\partial b^\omega}\bigg|_{b=b_0} \qquad \Delta_\omega(r) := \frac{\partial \det\left[\frac{\partial(H_b(r))^j}{\partial r^j}\right]}{\partial b^\omega}\Bigg|_{b=b_0} \qquad \in \mathbb{R}^{\dim M}$$

$$I^j_{i\omega} := \frac{\partial I^j_i(b)}{\partial b^\omega}\bigg|_{b=b_0} \in \mathbb{K} \quad h_\omega := \frac{\partial H_b}{\partial b^\omega}\bigg|_{b=b_0} : \mathbb{R}^{\dim M} \longrightarrow \mathbb{R}^{\dim M}.$$

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In this setting the Heisenberg relations corresponding to Poincaré transformations are described by $b \mapsto (\Lambda^{\mu\nu}, a^\lambda)$, $H(b) \mapsto \Lambda$, $a(b) \mapsto a$ and $I(b) \mapsto I(\Lambda)$, so that $U_\omega \mapsto (S_{\mu\nu,T_\lambda})$, $\Delta_\omega(r) \equiv 0$, $I^j_{i\omega} \mapsto (I^j_{i\mu\nu}, 0)$ and $(h_\omega(r))^k \frac{\partial}{\partial r^k} \mapsto r_\mu \frac{\partial}{\partial r^\nu} - r_\nu \frac{\partial}{\partial r^\mu}$.

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$$I_{i\omega}^j := \left.\frac{\partial I_i^j(b)}{\partial b^\omega}\right|_{b=b_0} \in \mathbb{K} \qquad h_\omega := \left.\frac{\partial H_b}{\partial b^\omega}\right|_{b=b_0} : \mathbb{R}^{\dim M} \to \mathbb{R}^{\dim M}.$$

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The case of internal transformations, considered in the previous subsection, corresponds to $H_b = \mathrm{id}_{\mathbb{R}^{\dim M}}$ and, consequently, in it $\Delta_\omega(r) \equiv 0$ and $h_\omega = 0$.

5. Fibre bundle approach

Let a physical field is described as a section $\varphi \colon M \to E$ of a vector bundle (E, π, M). Here M is a real differentiable (4-)manifold, serving as a spacetime model, E is the bundle space and $\pi \colon M \to E$ is the projection; the fibres $\pi^{-1}(x)$, $x \in M$.

Let (U, u) be a chart of M and $\{e^i\}$ be a frame in the bundle with domain U, i.e. $e^i \colon x \mapsto e^i(x) \in \pi^{-1}(x)$ with x in the domain of $\{e^i\}$ and $\{e^i(x)\}$ being a basis in $\pi^{-1}(x)$. Below we assume $x \in U \subseteq M$. Thus, we have

$$\varphi \colon M \ni x \mapsto \varphi(x) = \varphi_i(x) e^i(x) = \varphi_{u,i}(\boldsymbol{x}) e^i(u^{-1}(\boldsymbol{x})), \quad (5.1)$$

where $\boldsymbol{x} := u(x)$ $\varphi_{u,i} := \varphi_i \circ u^{-1}$.

The origin of the Heisenberg relations on the background of fibre bundle setting is in the equations

$$U \circ \varphi_i(x) \circ U^{-1} = (A^{-1})^j_i(x)\varphi_j(x) \qquad (5.2)$$

$$U \circ \varphi_{u,i}(\boldsymbol{x}) \circ U^{-1} = (A^{-1})^j_i(x)\varphi_{u,j}(\boldsymbol{x}). \qquad (5.2')$$

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1. $I\colon G \ni g_b \mapsto I(b) \in \mathrm{GL}(\dim V, \mathbb{K})$ and the changes $\{e^i\} \mapsto \{e'^i = A_j^i e^j\}$ of the frames in V are such that $A^{-1}(x) = I(b)$ for all $x \in M$ and some $b \in \mathbb{K}^s$.

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Consider a Lie group G, its representations I and U and reference frames with the following properties:

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2. $U\colon g \ni g_b \mapsto U(b)$, $U(b)$ being operator on the space of state, and the changes $(u, \{e^i\}) \mapsto (u', \{e'^i\})$ of the reference frames entail (1.1) with $U(b)$ for U.

Thus equations (5.2) and (5.2′) transform into

$$U(b) \circ \varphi_i(x) \circ U^{-1}(b) = I_i^j(b)\varphi_j(x) \qquad (5.3)$$

$$U(b) \circ \varphi_{u,i}(\boldsymbol{x}) \circ U^{-1}(b) = I_i^j(b)\varphi_{u,j}(\boldsymbol{x}). \qquad (5.3')$$

Differentiating with respect to b^ω and putting $b = b_0$, we get the Heisenberg relations

$$[U_\omega, \varphi_i(x)]_- = I_{i\omega}^j \varphi_j(x) \qquad (5.4)$$

$$[U_\omega, \varphi_{u,i}(\boldsymbol{x})]_- = I_{i\omega}^j \varphi_{u,j}(\boldsymbol{x}), \qquad (5.4')$$

where $U_\omega := \left.\frac{\partial U(b)}{\partial b^\omega}\right|_{b=b_0} \quad I_{i\omega}^j := \left.\frac{\partial I_i^j(b)}{\partial b^\omega}\right|_{b=b_0}$.

We can rewire the Heisenberg relations obtained as

$$[U_\omega, \varphi]_- = I_{i\omega}^j \varphi_j e^i. \qquad (5.5)$$

The Poncaré transformations are described by $b \mapsto (\Lambda^{\mu\nu}, a^\lambda)$, $U_\omega \mapsto (S_{\mu\nu}, T_\lambda)$ and $I^j_{i\omega} \mapsto (I^j_{i\mu\nu}, 0)$ and, consequently, the equations (5.3) and (5.3′) now read

$$U(\Lambda, a) \circ \varphi_i(x) \circ U^{-1}(\Lambda, a) = I^j_i(\Lambda, a)\varphi_j(x) \qquad (5.6)$$

$$U(\Lambda, a) \circ \varphi_{u,i}(\boldsymbol{x}) \circ U^{-1}(\Lambda, a) = I^j_i(\Lambda, a)\varphi_{u,j}(\boldsymbol{x}). \qquad (5.6')$$

For instance, we have the Heisenberg relations

$$[T_\mu, \varphi_i(x)]_- = 0 \qquad (5.7a)$$

$$[S_{\mu\nu}, \varphi_i(x)]_- = I^j_{i\mu\nu}\varphi_j(x). \qquad (5.7b)$$

The "physical" versions of these relations are

$$[P_\mu, \varphi_i(x)]_- = 0 \tag{5.8a}$$

$$[M_{\mu\nu}, \varphi_i(x)]_- = I^j_{i\mu\nu}\varphi_j(x). \tag{5.8b}$$

These relations contradict to the particle interpretation of quantum field theory which may be retained if one accepts their non-bundle versions. This is possible if the frames used are connected by linear homogeneous transformations with spacetime constant matrices, $A(x) = \text{const}$ or $\partial_\mu A(x) = 0$.

Since the general case is coordinates independent, it describes also the fibre bundle version of internal transformations. This explains why some equations in the both versions are identical up to the meaning $\varphi_{u,i}$ and $I^j_{i\omega}$. In particular, for

$$U(b) = \mathrm{e}^{\frac{1}{\mathrm{i}e}bQ_1} \qquad I(b) = \mathbb{1}\mathrm{e}^{-\frac{q}{\mathrm{i}e}b} \qquad b \in \mathbb{R} \qquad (5.9)$$

the Heisenberg relations (5.4) reduce to

$$[Q_1, \varphi_i(x)]_- = -q\varphi_i(x), \qquad (5.10)$$

which is identical with (3.10), but now φ_i are the components of the section φ in $\{e^i\}$. The invariant form of the last relations is

$$[Q_1, \varphi]_- = -q\varphi. \qquad (5.11)$$

6. Conclusion

We have shown how the Heisenberg equations arise in the general case and in particular situations. They are from pure geometrical origin and one should be careful when applying them to the Lagrangian formalism in which they are subsidiary conditions, like the Lorentz gauge in the electrodynamics. Generally they need not to be consistent with the Lagrangian formalism and their validity should carefully be checked. For instance, if one starts with field operators in the Lagrangian formalism of free fields and adds to it the Heisenberg relations concerning the momentum operator, then the arising scheme is not consistent as in it start to appear distributions, like the Dirac delta function.

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