

# A mathematical base for fibre bundle formulation of Lagrangian quantum field theory

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# Abstract

This report presents a differential-geometric foundations for an attempt to formulate Lagrangian (canonical) quantum field theory on fibre bundles. In it the standard Hilbert space of quantum field theory is replaced with a Hilbert bundle; the former playing a role of a (typical) fibre of the latter one. Suitable sections of that bundle replace the ordinary state vectors and the operators on the system's Hilbert space are transformed into morphisms of the same bundle. In particular, the field operators are mapped into corresponding field morphisms.

# 1. Introduction

The purpose of this work is to be presented grounds for a consistent formulation of quantum field theory in terms of fibre bundles. The ideas for that goal are shared from [1-5], where the quantum mechanics is formulated on the geometrical language of fibre bundle theory.

## The plan:

At first we present some basic definitions. Special attention will be paid on the Hilbert bundles, which will replace the Hilbert spaces of the ordinary quantum field theory, and the metric structure in them.

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Then a motivation is presented why the (Hilbert) fibre bundles are a natural scene for a mathematical formulation of quantum field theory.

## 2. Fibre Bundles. Hilbert bundles

A bundle is a triple  $(E, \pi, B)$  of sets  $E$  (bundle space) and  $B$  (base) and surjective mapping  $\pi: E \rightarrow B$  (projection). If  $b \in B$ ,  $\pi^{-1}(b)$  is the fibre over  $b$  and, if  $Q \subseteq B$ ,  $(E, \pi, B)|_Q := (\pi^{-1}(Q), \pi|_{\pi^{-1}(Q)}, Q)$  is the restriction on  $Q$  of a bundle  $(E, \pi, B)$ . A section of  $(E, \pi, B)$  is a mapping  $\sigma: B \rightarrow E$  such that  $\pi \circ \sigma = \text{id}_B$ , where  $\text{id}_Z$  is the identity mapping of a set  $Z$ , and their set is denoted by  $\text{Sec}(E, \pi, B)$ . The set of morphisms of  $(E, \pi, B)$  is

$$\text{Mor}(E, \pi, B) := \{(\varphi, f) \mid \varphi: E \rightarrow E, f: B \rightarrow B, \pi \circ \varphi = f \circ \pi\}.$$

The set of all  $B$ -morphisms (strong morphisms) of  $(E, \pi, B)$  is  $\text{Mor}_B(E, \pi, B) := \{\varphi \mid \varphi: E \rightarrow E, \pi \circ \varphi = \pi\}$ .

The bundle of point-restricted morphisms is defined by

$$\begin{aligned} E_0 &:= \{(\varphi_b, f) \mid \varphi_b = \varphi|_{\pi^{-1}(b)}, b \in B, (\varphi, f) \in \text{Mor}(E, \pi, B)\} \\ &= \{(\varphi_b, f) \mid \varphi_b: \pi^{-1}(b) \rightarrow \pi^{-1}(f(b)), b \in B, f: B \rightarrow B\} \end{aligned}$$

and  $\pi_0: E_0 \rightarrow B$  with  $\pi_0(\varphi_b, f) := b$  for  $(\varphi_b, f) \in E_0$ .



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The bundle  $\text{mor}_B(E, \pi, B)$  of (point-) restricted morphisms over  $B$  of  $(E, \pi, B)$  has a bundle space

$$\begin{aligned} E_0^B &:= \{\varphi_b \mid \varphi_b = \varphi|_{\pi^{-1}(b)}, b \in B, \varphi \in \text{Mor}_B(E, \pi, B)\} \\ &= \{\varphi_b \mid \varphi_b: \pi^{-1}(b) \rightarrow \pi^{-1}(b), b \in B\} \end{aligned}$$

and projection  $\pi_0^B: E_0^B \rightarrow B$  with  $\pi_0^B(\varphi_b) := b, \varphi_b \in E_0^B$ .

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and projection  $\pi_0^B: E_0^B \rightarrow B$  with  $\pi_0^B(\varphi_b) := b, \varphi_b \in E_0^B$ .

There is a natural bijection

$$\text{Mor}_B(E, \pi, B) \xrightarrow{\chi} \text{Sec}(\text{mor}_B(E, \pi, B)).$$

A map  $\text{Sec}(E, \pi, B) \rightarrow \text{Sec}(E, \pi, B)$  is called morphism of  $\text{Sec}(E, \pi, B)$ . Their set is  $\text{MorSec}(E, \pi, B)$ .

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**Definition 2.1.** A Hilbert (fibre) bundle is a vector bundle whose fibres over the base are isomorphic Hilbert spaces.

Let  $(F, \pi, M, \mathcal{F})$  be a Hilbert bundle and  $F_x := \pi^{-1}(x)$ .

Let  $l_x: F_x \rightarrow \mathcal{F}$ ,  $x \in M$ , be the point-trivializing isomorphisms defined by the decomposition functions via  $\phi_W|_x(x, \psi) =: l_x^{-1}(\psi) \in \pi^{-1}(x)$  for every  $\psi \in \mathcal{F}$ .

Let  $\langle \cdot | \cdot \rangle: \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$  be the scalar product in  $\mathcal{F}$  and

$\langle \cdot | \cdot \rangle_x: F_x \times F_x \rightarrow \mathbb{R}$  be the scalar product in  $F_x$ . A

Hilbert bundle with compatible vector and metric

structure,  $\langle \varphi_x | \psi_x \rangle_x = \langle l_x(\varphi_x) | l_x(\psi_x) \rangle$  for every

$\varphi_x, \psi_x \in F_x$ , is called compatible Hilbert bundle.

In such a bundle we have

$$\langle \cdot | \cdot \rangle_x = \langle l_x \cdot | l_x \cdot \rangle, \quad x \in M \quad (2.1)$$

$$\langle \cdot | \cdot \rangle = \langle l_x^{-1} \cdot | l_x^{-1} \cdot \rangle_x, \quad x \in M. \quad (2.1')$$

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Defining the Hermitian conjugate mapping  $A_x^\dagger: \mathcal{F} \rightarrow F_x$  of  $A_x: F_x \rightarrow \mathcal{F}$  by  $\langle A_x^\dagger \varphi | \chi_x \rangle_x := \langle \varphi | A_x \chi_x \rangle$ ,  $\varphi \in \mathcal{F}$ ,  $\chi_x \in F_x$ , we find ( $\dagger$  denotes conjugation in  $\mathcal{F}$ )

$$A_x^\dagger = l_x^{-1} \circ \left( A_x \circ l_x^{-1} \right)^\dagger. \quad (2.2)$$

We call a mapping  $A_x: F_x \rightarrow \mathcal{F}$  unitary if it has an inverse  $A_x^{-1}$  and  $A_x^\dagger = A_x^{-1}$ .

The isometric isomorphisms  $l_x: F_x \rightarrow \mathcal{F}$  are unitary in this sense:

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The Hermitian conjugate mapping to  $A_{x \rightarrow y} \in \{C_{x \rightarrow y}: F_x \rightarrow F_y, x, y \in M\}$  is  $A_{x \rightarrow y}^\dagger: F_x \rightarrow F_y$  such that  $\langle A_{x \rightarrow y}^\dagger \Phi_x | \Psi_y \rangle_y := \langle \Phi_x | A_{y \rightarrow x} \Psi_y \rangle_x$ ,  $\Phi_x \in F_x$ ,  $\Psi_y \in F_y$ . Its explicit form is

$$A_{x \rightarrow y}^\dagger = l_y^{-1} \circ \left( l_x \circ A_{y \rightarrow x} \circ l_y^{-1} \right)^\dagger \circ l_x. \quad (2.4)$$

As  $(\mathcal{A}^\dagger)^\dagger \equiv \mathcal{A}$  for any  $\mathcal{A}: \mathcal{F} \rightarrow \mathcal{F}$ , we have

$$\left( A_{x \rightarrow y}^\dagger \right)^\dagger = A_{x \rightarrow y}. \quad (2.5)$$

A mapping  $A_{x \rightarrow y}$  is called Hermitian if

$$A_{x \rightarrow y}^\dagger = A_{x \rightarrow y}. \quad (2.6)$$

The mappings  $l_{x \rightarrow y} := l_y^{-1} \circ l_x$  are Hermitian. A mapping  $A_{x \rightarrow y}: F_x \rightarrow F_y$  is called unitary if it has a left inverse mapping  $A_{x \rightarrow y}^{-1}: F_y \rightarrow F_x$  and

$$A_{x \rightarrow y}^\dagger = A_{y \rightarrow x}^{-1}. \quad (2.7)$$

The last equation is equivalent with

$$\langle A_{y \rightarrow x} \cdot \mid A_{y \rightarrow x} \cdot \rangle_x = \langle \cdot \mid \cdot \rangle_y: F_y \times F_y \rightarrow \mathbb{C}, \quad (2.7')$$

i.e. the unitary mappings are fibre-isometric. The following mappings are unitary

$$l_{x \rightarrow y} := l_y^{-1} \circ l_x: \pi^{-1}(x) \rightarrow \pi^{-1}(y). \quad (2.8)$$

Let  $A \in \text{Mor}(F, \pi, M, \mathcal{F})$ . The Hermitian conjugate bundle morphism  $A^\dagger$  to  $A$  is defined by

$$\langle A^\dagger \Phi_x | \Psi_x \rangle_x := \langle \Phi_x | A \Psi_x \rangle_x, \quad \Phi_x, \Psi_x \in F_x. \text{ Thus}$$

$$A_x^\dagger := A^\dagger|_{F_x} = l_x^{-1} \circ \left( l_x \circ A_x \circ l_x^{-1} \right)^\dagger \circ l_x. \quad (2.9)$$

A bundle morphism  $A$  is Hermitian if  $A_x^\dagger = A_x$ , i.e. if

$$A^\dagger = A, \quad (2.10)$$

and it is called unitary if  $A_x^\dagger = A_x^{-1}$ , i.e. if

$$A^\dagger = A^{-1}. \quad (2.11)$$

which is equivalent to  $\langle A \cdot | A \cdot \rangle_x = \langle \cdot | \cdot \rangle_x : F_x \times F_x \rightarrow \mathbb{C}$ .

Hence the unitary morphisms are fibre-metric compatible, i.e. they are isometric .

### 3. The bundle transport

Let  $(E, \pi, B, \mathcal{E})$  be a  $\mathbb{K}$ -vector  $C^1$  bundle,  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ .

**Definition 3.1.** The bundle transport in  $(E, \pi, B, \mathcal{E})$  is a mapping  $l: (x, y) \mapsto l_{x \rightarrow y}$ ,  $x, y \in B$ , where the mapping

$$l_{x \rightarrow y}: \pi^{-1}(x) \rightarrow \pi^{-1}(y), \quad (3.1)$$

called (bundle) transport from  $x$  to  $y$ , is defined by

$$l_{x \rightarrow y} := l_y^{-1} \circ l_x \quad (3.2)$$

with  $l_x: \pi^{-1}(x) \rightarrow \mathcal{E}$ ,  $x \in B$ , being the point-trivializing isomorphisms of  $(E, \pi, B, \mathcal{E})$ .

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Some of the properties of the bundle transport:

$$l_{x \rightarrow y} \circ l_{z \rightarrow x} = l_{z \rightarrow y}, \quad x, y, z \in B, \quad (3.3)$$

$$l_{x \rightarrow x} = \text{id}_{\pi^{-1}(x)}, \quad x \in B, \quad (3.4)$$

$$l_{x \rightarrow y}(\lambda u + \mu v) = \lambda l_{x \rightarrow y}u + \mu l_{x \rightarrow y}v, \quad \lambda, \mu \in \mathbb{K}, \quad u, v \in \pi^{-1}(x), \quad (3.5)$$

$$(l_{x \rightarrow y})^{-1} = l_{y \rightarrow x}. \quad x, y \in B. \quad (3.6)$$

The bundle transport is Hermitian and unitary in a sense that such are  $l_{x \rightarrow y}$ ,  $x, y \in B$ .

A section  $X \in \text{Sec}(E, \pi, B, \mathcal{E})$  is  $l$ -transported if for some (and hence any)  $x \in B$  and every  $y \in B$  is fulfilled

$$X(y) = l_{x \rightarrow y}(X(x)) \text{ with } X(x) := X_x, \quad X: x \mapsto X_x. \quad (3.7)$$

Such a section is uniquely defined by specifying its value at a single point. If  $\mathcal{X}_0$  is a fixed vector in the fibre  $\mathcal{E}$ , the section  $X$  given via

$$X(x) = l_x^{-1}(\mathcal{X}_0) \quad (3.8)$$

is  $l$ -transported. Sections of this kind will represent the states of quantum fields in our approach.

Let  $(U, \kappa)$ , be a chart in a neighborhood  $U$  of  $x \in B$  and  $x(\varepsilon, \mu) \in U$ ,  $\varepsilon \in \mathbb{R}$  and  $\mu = 1, \dots, \dim B$ , is such that  $\kappa^\nu(x(\varepsilon, \mu)) = \kappa^\nu(x) + \varepsilon \delta_\mu^\nu$  with  $\delta_\mu^\nu = 1$  for  $\nu = \mu$  and  $\delta_\mu^\nu = 0$  for  $\nu \neq \mu$ . The bundle transport generates derivations

$$D_\mu: \text{Sec}^1(E, \pi, B) \rightarrow \text{Sec}^0(E, \pi, B), \quad \mu = 1, \dots, \dim B$$

$$(D_\mu Y)(x) := D_\mu|_x(Y) := \lim_{\varepsilon \rightarrow 0} \frac{l_{x(\varepsilon, \mu) \rightarrow x}(Y(x(\varepsilon, \mu))) - Y(x)}{\varepsilon}.$$

The bundle transport also generates the derivations

$$\hat{D}_\mu: \{ \hat{A} \in \text{MorSec}^1(E, \pi, B) \text{ is of class } C^1 \text{ and } \hat{A}(\cdot) = A \circ \kappa \} \\ \rightarrow \{ \hat{A} \in \text{MorSec}^0(E, \pi, B) \text{ is of class } C^0 \text{ and } \hat{A}(\cdot) = A \circ (\cdot) \}$$

$$\hat{D}_\mu(\hat{A}) := [D_\mu, \hat{A}]_- = D_\mu \circ \hat{A} - \hat{A} \circ D_\mu$$

$\hat{A} \in \text{MorSec}^1(E, \pi, B)$  generated by  $A \in \text{Mor}_B^1(E, \pi, B)$ .



$D_\mu$  and  $\hat{D}_\mu$  are linear and ( $f$  is a  $C^1$  function on  $B$ )

$$D_\mu(f Y) = \frac{\partial f}{\partial x^\mu} Y + f D_\mu(Y), \quad (3.9)$$

$$\hat{D}_\mu(\hat{A} \circ \hat{C}) = (\hat{D}_\mu(\hat{A})) \circ \hat{C} + \hat{A} \circ (\hat{D}_\mu(\hat{C})). \quad (3.10)$$

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**Proposition 3.1.** *If  $Y$  is a  $C^1$  section,  $A$  a  $C^1$  morphism of  $(E, \pi, B)$ , and  $\hat{A}(\cdot) = A \circ (\cdot)$  is the morphisms of  $\text{Sec}(E, \pi, B)$  generated by  $A$ , then*

$$(D_\mu Y)(x) = l_x^{-1} \left( \frac{\partial(l_x(Y(x)))}{\partial x^\mu} \right) = (l_x^{-1} \circ \frac{\partial}{\partial x^\mu} \circ l_x)(Y(x)) \quad (3.11)$$

$$((\hat{D}_\mu \hat{A})(Y))(x) = \left( l_x^{-1} \circ \frac{\partial(l_x \circ A_x \circ l_x^{-1})}{\partial x^\mu} \circ l_x \right)(Y(x)). \quad (3.12)$$

If  $\{x^\mu\} \mapsto \{x'^\mu\}$  is a coordinate changes, then  $D_\mu$  and  $\hat{D}_\mu$  behave like basic (tangent) vector fields over  $B$ :

$$D_\mu \mapsto D'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} D_\nu, \quad \hat{D}_\mu \mapsto \hat{D}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \hat{D}_\nu. \quad (3.13)$$

'Covariant derivatives' along a vector field  $V = V^\mu \frac{\partial}{\partial x^\mu}$  tangent to  $B$ :  $D_V := V^\mu D_\mu$  and  $\hat{D}_V := V^\mu \hat{D}_\mu$ .

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The transport  $l$  induces a ‘transport along the identity mapping of  $B$ ’ in the bundle of point-restricted

morphisms  $\text{mor}_B(E, \pi, B) = (E_0, \pi_0, B)$ . This is a map

$\circlearrowleft l: (x, y) \mapsto \circlearrowleft l_{x \rightarrow y}$ ,  $x, y \in B$ , with  $\circlearrowleft l_{x \rightarrow y}: \pi_0^{-1}(x) \rightarrow \pi_0^{-1}(y)$ , (transport from  $x$  to  $y$  in  $\text{mor}_B(E, \pi, B)$ ) defined by

$$\circlearrowleft l_{x \rightarrow y}(\chi_x) := l_{x \rightarrow y} \circ \chi_x \circ l_{y \rightarrow x} = l_y^{-1} \circ (l_x \circ \chi_x \circ l_x^{-1}) \circ l_y: \pi^{-1}(y) \rightarrow \pi^{-1}(x)$$

for  $\chi_x: \pi^{-1}(x) \rightarrow \pi^{-1}(x)$ .

The transports  $\circlearrowleft l_{x \rightarrow y}$  are  $\mathbb{K}$ -linear and satisfy:

$$\circlearrowleft l_{x \rightarrow y} \circ \circlearrowleft l_{z \rightarrow x} = \circlearrowleft l_{z \rightarrow y}, \quad x, y, z \in B, \quad (3.14)$$

$$\circlearrowleft l_{x \rightarrow x} = \text{id}_{\{\pi^{-1}(x) \rightarrow \pi^{-1}(x)\}}, \quad x \in B, \quad (3.15)$$

$$\left(\circlearrowleft l_{x \rightarrow y}\right)^{-1} = \circlearrowleft l_{y \rightarrow x}, \quad x, y \in B. \quad (3.16)$$

$\circlearrowleft l$  is the transport associated to the bundle transport  $l$ . A morphism  $A \in \text{Mor}_B(E, \pi, B)$  is  $\circlearrowleft l$ -transported if the restrictions  $A_x := A|_{\pi^{-1}(x)}$  satisfy the equation

$$A_y = \circlearrowleft l_{x \rightarrow y}(A_x) = l_{x \rightarrow y} \circ A_x \circ l_{y \rightarrow x}, \quad x, y \in B. \quad (3.17)$$

For fixed  $x \in B$  and  $\chi_0: \pi^{-1}(x) \rightarrow \pi^{-1}(x)$ ,

$$\circlearrowleft l_x(\chi_0)|_{\pi^{-1}(y)} := \circlearrowleft l_{x \rightarrow y}(\chi_0) = l_{x \rightarrow y} \circ \chi_0 \circ l_{y \rightarrow x} \quad (3.18)$$

is  $\circlearrowleft l$ -transported  $B$ -morphism.

For fixed  $x$ , any morphism  $A \in \text{Mor}_B(E, \pi, B)$  defines the  $\circlearrowleft$ -transported morphism

$$\circlearrowleft_x(A) := \circlearrowleft_x(A_x): y \mapsto (\circlearrowleft_x A)_y := \circlearrowleft_{x \rightarrow y} A_x = l_{x \rightarrow y} \circ A_x \circ l_{y \rightarrow x}.$$

The transport  $\circlearrowleft$  induces derivations  $\circlearrowleft D_\mu$  on the set  $\text{Sec}^1(\text{mor}_B(E, \pi, B))$  via

$$(\circlearrowleft D_\mu A)(x) := \lim_{\varepsilon \rightarrow 0} \frac{\circlearrowleft_{x(\varepsilon, \mu) \rightarrow x} (A_{x(\varepsilon, \mu)}) - A_x}{\varepsilon} \quad (3.19)$$

with  $x(\varepsilon, \mu)$  defined above and  $A \in \text{Sec}^1(\text{mor}_B(E, \pi, B))$ .

We have

$$(\circlearrowleft D_\mu A)(x) = l_x^{-1} \circ \frac{\partial (l_x \circ A(x) \circ l_x^{-1})}{\partial x^\mu} \circ l_x. \quad (3.20)$$

Now we shall present some expressions in (local) bases which may be a little more familiar to the physicists.

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Let  $\{e_i(x)\}$  be a basis in  $\pi^{-1}(x)$ ,  $\{f_i\}$  a basis in  $\mathcal{F}$  with  $\partial_\mu f_i = 0$ , and  $\mathbf{l}_x$  and  $\mathbf{l}(y, x)$  be the matrices of respectively  $l_x$  and  $l_{x \rightarrow y}$  in them, i.e. if  $l_x(e_i(x)) =: l_x^j{}_i f_j$  and  $l_{x \rightarrow y}(e_i(x)) =: l^j{}_i(y, x) e_i(y)$ . Then  $\mathbf{l}_x := [l_x^j{}_i]$  and  $\mathbf{l}(y, x) := [l^j{}_i(y, x)] = \mathbf{l}^{-1}(y) \mathbf{l}(x)$  and, for  $Y = Y^i e_i$ ,

$$(D_\mu Y)(x) = \left( \frac{\partial Y^i(x)}{\partial x^\mu} + \Gamma^i{}_{j\mu}(x) Y^j(x) \right) e_i(x) \quad (3.21)$$

$$(D_\mu e_j)(x) =: \Gamma^i{}_{j\mu}(x) e_i(x) \quad (3.22)$$

$$\Gamma_\mu(x) := [\Gamma^i{}_{j\mu}(x)] = \left. \frac{\partial \mathbf{l}(y, x)}{\partial x^\mu} \right|_{y=x} = \mathbf{l}_x^{-1} \frac{\partial \mathbf{l}(x)}{\partial x^\mu} \quad (3.23)$$

$\Gamma^i{}_{j\mu}$  are the coefficients (components) of  $l$  (resp.  $D_\mu$ ).



Under the changes  $e_i \mapsto e'_i = C_i^j e_j$  and  $x^\mu \mapsto x'^\mu$ ,  $C = [C_i^j]$  being non-degenerate matrix-valued  $C^1$  function, the matrices  $\Gamma_\mu$  transform into

$$\Gamma'_\mu(x) := \left( C^{-1}(x) \Gamma_\nu(x) C(x) + C^{-1}(x) \frac{\partial C(x)}{\partial x^\nu} \right) \frac{\partial x^\nu}{\partial x'^\mu}. \quad (3.24)$$

For a  $C^1$  section  $A$ , we find the matrix of  $\hat{D}_\mu \hat{A}$  in  $\{e_i\}$  as

$$[(\hat{D}_\mu \hat{A})_i^j] \Big|_x = \left( \frac{\partial \mathbf{A}}{\partial x^\mu} + [\Gamma_\mu(x), \mathbf{A}]_- \right) \Big|_x = \mathbf{l}_x^{-1} \frac{\partial \mathbf{A}(x)}{\partial x^\mu} \mathbf{l}_x \quad (3.25)$$

with  $\mathbf{A}$  being the matrix of  $A$  in  $\{e_i\}$ .

## 4. On bundle formulation of quantum field theory

A quantum field  $\varphi$  is an operator-valued vector distribution,  $\varphi = (\varphi_1, \dots, \varphi_n)$  with  $n \in \mathbb{N}$  and  $\varphi_i$ ,  $i = 1, \dots, n$ , being operator-valued distributions called components of  $\varphi$ , such that

$$\varphi(f) = \int_M d^4y \sum_i \varphi_i(y) f^i(y) = \sum_i \varphi_i(f^i),$$
$$\varphi_i(g) := \int_M d^4y \varphi_i(y) g(y) \quad (4.1)$$

for a vector test function  $f = (f^1, \dots, f^n)$ ,  $\varphi_i$  – the non-smearred fields acting in  $\mathcal{F}$  and  $f^i, g: M \rightarrow \mathbb{K}$  – test functions. We adopt the Heisenberg picture of motion.

In the bundle description the system's Hilbert space  $\mathcal{F}$  of states is replaced with a Hilbert fibre bundle  $(F, \pi, M, \mathcal{F})$  (of states) with Minkowski spacetime  $M$  as a base, projection  $\pi: F \rightarrow M$ , fibres  $\pi^{-1}(x)$  with  $x \in M$ , and the system's ordinary Hilbert space  $\mathcal{F}$  as a (standard, typical) fibre. Let  $\{l_x : x \in M\}$  be a set of (linear) isomorphisms  $l_x: F_x \rightarrow \mathcal{F}$ . Given an observer  $O_x$  at  $x \in M$ , if a system is characterized by a state vector  $\mathcal{X} \in \mathcal{F}$ , the vector

$$X(x) := l_x^{-1}(\mathcal{X}) \quad (4.2)$$

should be considered as its state vector relative to  $O_x$ . Hereof the state vector  $\mathcal{X}$  is replaced with a state section  $X \in \text{Sec}(F, \pi, M, \mathcal{F})$  such that  $X: x \mapsto X(x) = l_x^{-1}(\mathcal{X})$ .

At a point  $x \in M$ , the bundle analogue  $A_x$  of an operator  $\mathcal{A}(x): \mathcal{F} \rightarrow \mathcal{F}$  can be defined by requiring the transition  $\mathcal{A}(x) \mapsto A_x$  to preserve the scalar products:

$$\prec \mathcal{X} | \mathcal{A}(x)(\mathcal{Y}) \succ = \langle X(x) | A_x(Y(x)) \rangle_x, \quad (4.3)$$

where  $\mathcal{X}, \mathcal{Y} \in \mathcal{F}$ ,  $x \in M$ ,  $X(x) = l_x^{-1}(\mathcal{X})$ ,  $Y(x) = l_x^{-1}(\mathcal{Y})$ , and  $\prec \cdot | \cdot \succ$  and  $\langle \cdot | \cdot \rangle_x$  are the scalar products in the Hilbert spaces  $\mathcal{F}$  and  $\pi^{-1}(x)$ , which are connected by

$$\langle \cdot | \cdot \rangle_x = \prec l_x \cdot | l_x \cdot \succ, \quad \prec \cdot | \cdot \succ = \langle l_x^{-1} \cdot | l_x^{-1} \cdot \rangle_x. \quad (4.4)$$

If  $\mathcal{A}(x)$  represents a dynamical variable  $\mathbf{A}$ , the observed (mean) value of  $\mathbf{A}$  is independent of the way we compute (or describe) it and

$$A_x = l_x^{-1} \circ \mathcal{A}(x) \circ l_x. \quad (4.5)$$

In this way, we see that to an operator  $\mathcal{A}(x): \mathcal{F} \rightarrow \mathcal{F}$  there corresponds a morphism

$$A \in \text{Mor}_M(F, \pi, M, \mathcal{F}) \quad (4.6)$$

of the system's Hilbert bundle with  $A|_{\pi^{-1}(x)} = A_x$ .

In particular, the non-smearred components  $\varphi_i$  of a quantum field  $\varphi$  change to  $\Phi_i \in \text{Mor}_M(F, \pi, M, \mathcal{F})$  with

$$\Phi_i|_{\pi^{-1}(x)} := l_x^{-1} \circ \varphi_i(x) \circ l_x. \quad (4.7)$$

To the smeared fields  $\varphi_i$  should correspond

$$\Phi_i : x \mapsto \Phi_i|_x := l_x^{-1} \circ \varphi_i(\cdot) \circ l_x. \quad (4.8)$$

so that the smeared field  $\Phi$  becomes a mapping

$$\Phi : x \mapsto \Phi|_x = l_x^{-1} \circ \varphi(\cdot) \circ l_x, \quad (4.9)$$

$$\Phi_x(f) = \int_M d^4y \sum_i \circ l_{y \rightarrow x}(\Phi_i(y)) f^i(y), \quad (4.10)$$

the transport  $\circ l$  in  $\text{mor}_M(F, \pi, M, \mathcal{F})$  is defined via (3.14).

The morphisms, like  $A$ , are not the exact objects we need. Since in the ordinary theory the operators like,  $\mathcal{A}$ , act on state vectors, like  $\mathcal{X}$ , we should expect the bundle analogue  $\hat{A}$  of  $\mathcal{A}$  to act on a state section  $X$  producing again a section  $\hat{A}(X) \in \text{Sec}(F, \pi, M, \mathcal{F})$ . As in the module  $\text{Sec}(F, \pi, M, \mathcal{F})$  there is a natural scalar product  $\langle \cdot | \cdot \rangle$  with values in the  $\mathbb{C}$ -valued functions, the r.h.s. of (4.3) should be identified with the value at  $x \in M$  of the scalar product  $\langle X | \hat{A}(Y) \rangle$ . Combining these ideas, we conclude that

$$\hat{A}(X) = A \circ X. \quad (4.11)$$

Hereof, the  $\hat{A}$  is the morphism in  $\text{MorSec}(F, \pi, M, \mathcal{F})$  generated by  $A \in \text{Mor}_M(F, \pi, M, \mathcal{F})$  whose restriction on  $\pi^{-1}(x)$  is  $A_x = l_x^{-1} \circ \mathcal{A}(x) \circ l_x$ .

Assuming  $\hat{A}$  to be the ‘right’ bundle analogue of  $\mathcal{A}$ , we conclude that

$$\hat{A}(X): x \mapsto \hat{A}(X)|_x = (A \circ X)(x) = A_x(X(x)) = l_x^{-1}(\mathcal{A}(x)(\mathcal{X})) \quad (4.12)$$

i.e. the image of  $\mathcal{A}(x)(\mathcal{X})$  according to (4.1) is exactly the value at  $x$  of  $\hat{A}(X)$  and

$$\langle \mathcal{X} | \mathcal{A}(x)(\mathcal{Y}) \rangle = \langle X | \hat{A}(Y) \rangle|_x \quad (4.13)$$

which expresses the invariance of the scalar products when we replace state vectors with state sections.

Since the scalar products define the observed (expectation, mean) values of physically observable quantities, the last equality expresses the independence of these values of the way we calculate them.



## 5. Conclusion

The present investigation is a continuation of the fibre bundle formulation of quantum physics begun in [1-5]. Here we have applied a slightly different approach to the quantum field theory. The basic idea is the Hilbert space of states to be replaced with a Hilbert bundle with it as a typical fibre, then all quantities of the ordinary quantum field theory are mapped into their bundle analogues by means of the bundle transport, which is an internal object of any particular bundle, or other mappings build from it.

## 5. Conclusion

The present investigation is a continuation of the fibre bundle formulation of quantum physics begun in [1-5]. Here we have applied a slightly different approach to the quantum field theory. The basic idea is the Hilbert space of states to be replaced with a Hilbert bundle with it as a typical fibre, then all quantities of the ordinary quantum field theory are mapped into their bundle analogues by means of the bundle transport, which is an internal object of any particular bundle, or other mappings build from it.

The realization of such a procedure is not unique. The main reason being that the quantum field theory has not a unique generally accepted formulation.

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