Chapter VI

Normal frames for connections on differentiable fibre bundles

8. Links between connections and transports along paths in fibre bundles

As the title of this section indicates, its content is partially outside of the main topic of the present book. It generalizes part of section IV.14 and investigates relations between some axiomatic approaches to the general theories of connections, parallel transports, and transports along paths. We hope that the material below will clarify some problems that may have arisen in chapter IV and will be useful for readers interested in the axiomatization of the concept of a 'parallel transport'.

The widespread approach to the concept of a "parallel transport" is it to be considered as a secondary one and defined on the basis of the connection theory [6, 7, 10–13, 16, 28, 60, 98, 106, 107, 117, 141–145]. However, the opposite approach, in which the parallel transport is axiomatically defined and from it the connection theory is constructed, is also known [17, 23, 30–33, 91, 148–151] and goes back to 1949 [1]; e.g. it is systematically realized in [23], where the connection theory on vector bundles is investigated. In [114] the concept of a "parallel transport" was generalize to the one of "transport along paths". The relations between both concepts were analyzed in [115]. The aim of the present section is to be investigated

[1] It seems that the earliest written accounts on this approach are the ones due to Ü. G. Lumiste [30, sec. 2.2] and C. Teleman [17, chapter IV, sec. B.3] (both published in 1964), the next essential steps being made by P. Dombrowski [31, § 1] and W. Poor [23]. Besides, the author of [31] states that his paper is based on unpublished lectures of prof. Willi Rinow (1907–1979) in 1949; see also [23, p. 46] where the author claims that the first axiomatical definition of a parallel transport in the tangent bundle case is given by prof. W. Rinow in his lectures at the Humboldt University in 1949. Some heuristic comments on the axiomatic approach to parallel transport theory can be found in [8, sec. 2.1] too.

some links between general connections on fibre bundles and transport along paths in them. Recall that similar problems, but in the linear case in vector bundles, were explored in section IV.14.

The bundle and base spaces of the bundles in this section, with an exception of subsection 8.5, are supposed to be of differentiable of class C^1; however, some parts of the text below, like definition 8.1, are valid in more general situations, e.g. in topological bundles.

Definition 8.1. A *transport along paths* in a (topological) bundle (E, π, B) is a mapping I assigning to every path $\gamma\colon J \to M$ a mapping I^γ, termed *transport along* γ, such that $I^\gamma\colon (s,t) \mapsto I^\gamma_{s\to t}$ where the mapping

$$I^\gamma_{s\to t}\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \qquad s,t \in J, \tag{8.1}$$

called *transport along* γ *from* s *to* t, has the properties:

$$I^\gamma_{s\to t} \circ I^\gamma_{r\to s} = I^\gamma_{r\to t} \qquad r,s,t \in J \tag{8.2}$$

$$I^\gamma_{s\to s} = \mathsf{id}_{\pi^{-1}(\gamma(s))} \qquad s \in J, \tag{8.3}$$

where $\circ$ denotes composition of mappings and id_X is the identity mapping of a set X.

Remark 8.1. If $(E, \pi, M))$ is a vector bundle and the mappings (8.1) are linear, definition 8.1 reduces to definition IV.3.1.

An analysis and various comments on this definition can be found in [87, 102, 114, 115]; see also Sect IV.3.

As we shall see below, an important role is played by transports along paths satisfying some additional conditions, in particular

$$I^{\gamma|J'}_{s\to t} = I^\gamma_{s\to t} \qquad s,t \in J' \tag{8.4}$$

$$I^{\gamma\circ\chi}_{s\to t} = I^\gamma_{\chi(s)\to\chi(t)} \qquad s,t \in J'', \tag{8.5}$$

where $J' \subseteq J$ is a subinterval, $\gamma|J'$ is the restriction of γ to J', and $\chi\colon J'' \to J$ is a bijection of a real interval J'' onto J.

Putting $r = t$ in (8.2) and using (8.3), we see that the mappings (8.1) are invertible and

$$(I^\gamma_{s\to t})^{-1} = I^\gamma_{t\to s}. \tag{8.6}$$

8.1. Transports along paths and connections

The following result describes how a transport along paths generates a connection. [2]

[2] The author thanks Petko Nikolov (Physical department of the Sofia University "St. Kliment Okhridski") for a discussion which led to a correct formulation of theorem 8.1 as well as to pointing his attention to some peculiarities in [33].

Theorem 8.1. *Let I be a transport along paths in a bundle (E, π, M). Let $\gamma\colon J \to M$ be a path and, for any $s_0 \in J$ and $p \in \pi^{-1}(\gamma(s_0))$, the lift $\bar{\gamma}_{s_0,p}\colon J \to E$ of γ be defined by*

$$\bar{\gamma}_{s_0,p}(t) = I^{\gamma}_{s_0 \to t}(p) \qquad t \in J. \tag{8.7}$$

Suppose the transport I is such that:

(a) (C^1 smoothness) The path

$$\bar{\gamma}_{s_0,p} \text{ is of class } C^1 \tag{8.8a}$$

for every C^1 path γ, s_0 and p.

(b) (Initial uniqueness – cf. definition IV.14.1(v)*) If $\gamma_i\colon J_i \to M$, $i = 1, 2$, are two C^1 paths and for some $s_i \in J_i$ is fulfilled $\gamma_1(s_1) = \gamma_2(s_2)$ and $\dot{\gamma}_1(s_1) = \dot{\gamma}_2(s_2)$, then the lifted paths $\bar{\gamma}_{i;s_i,p}$, defined via (*8.7*) with $p \in \pi^{-1}(\gamma_1(s_1)) = \pi^{-1}(\gamma_2(s_2))$, have equal tangent vectors at p,*

$$\dot{\bar{\gamma}}_{1;s_1,p}(s_1) = \dot{\bar{\gamma}}_{2;s_2,p}(s_2). \tag{8.8b}$$

*(c) (Linearization) If $\gamma_i\colon J_i \to M$, $i = 1, 2$, are two C^1 paths and for some $s_i \in J_i$ is fulfilled $\gamma_1(s_1) = \gamma_2(s_2)$, then for every $a_1, a_2 \in \mathbb{K}$ there exists a C^1 path $\gamma_3\colon J_3 \to M$ (generally depending on a_1, a_2, γ_1 and γ_2) such that $\gamma_3(s_3) = \gamma_1(s_1)$ ($= \gamma_2(s_2)$) for some $s_3 \in J_3$ and the vector tangent to the lifted path $\bar{\gamma}_{3;s_3,p}$, defined via (*8.7*) with $p \in \pi^{-1}(\gamma_3(s_3))$, at s_3 is*

$$\dot{\bar{\gamma}}_{3;s_3,p}(s_3) = a_1 \dot{\bar{\gamma}}_{1;s_1,p}(s_1) + a_2 \dot{\bar{\gamma}}_{2;s_2,p}(s_2). \tag{8.8c}$$

Then

$$\begin{aligned}\Delta^I\colon p \mapsto \Delta^I_p := \Big\{ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s_0} \big(\bar{\gamma}_{s_0,p}(t)\big) & \\ : \gamma\colon J \to M \text{ is } C^1 \text{ and injective},\ s_0 \in J,\ \gamma(s_0) = \pi(p) \Big\} &\subseteq T_p(E),\end{aligned} \tag{8.9}$$

with $p \in E$, is a distribution which is a connection on (E, π, M), i.e.

$$\Delta^v_p \oplus \Delta^I_p = T_p(E) \qquad p \in E, \tag{8.10}$$

with Δ^v being the vertical distribution on E, $\Delta^v_p = T_p(\pi^{-1}(\pi(p)))$.

Proof. To begin with, we shall write the following result.

Lemma 8.1. *If a C^1 path $\bar{\gamma}\colon J \to E$ is a lift of a C^1 path $\gamma\colon J \to M$, $\pi \circ \bar{\gamma} = \gamma$, then*

$$\pi_*(\dot{\bar{\gamma}}(t)) = \dot{\gamma}(t). \tag{8.11}$$

Proof of Lemma 8.1 Let $\{u^\mu = x^\mu \circ \pi, u^a\}$ be bundle coordinate system on E and p be a point in its domain. Then (8.11) follows from $\pi_*\big(\frac{\partial}{\partial u^I}\big|_p\big) = \frac{\partial(x^\mu \circ \pi)}{\partial u^I}\big|_p \frac{\partial}{\partial x^\mu}\big|_{\pi(p)}$ whose consequence is (3.33), and

$$\dot{\bar{\gamma}}^\mu = \dot{\gamma}^\mu \tag{8.12}$$

which is a corollary of $\dot{\bar{\gamma}}^\mu(t) = \frac{\mathrm{d}(u^\mu \circ \bar{\gamma}(t))}{\mathrm{d}t} = \frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t} = \dot{\gamma}^\mu(t)$ for all $t \in J$. $\square$

From (8.9) and (8.11), we get

$$\pi_*(\Delta_p^I) = \big\{\dot{\gamma}(s_0) : \gamma \colon J \to M \text{ is } C^1 \text{ and injective}, s_0 \in J,\ \gamma(s_0) = \pi(p)\big\} \\ = T_{\pi(p)}(M)$$

as $\dot{\gamma}(s_0)$ is an arbitrary vector in $T_{\pi(p)}(M) = T_{\gamma(s_0)}(M)$. Besides, the condition (c) ensures that Δ_p^I is a vector subspace in $T_p(E)$. Thus $\pi_*|_{\Delta_p^I} \colon \Delta_p^I \to T_{\pi(p)}(M)$ is a surjective mapping between vector spaces. It is also linear as it is a restriction of the tangent mapping π_*, which in tern is a linear mapping [7, sec. 1.22], on the vector subspace Δ_p^I. At last, we shall prove that $\pi_*|_{\Delta_p^I}$ is injective, from where it follows that $\pi_*|_{\Delta_p^I}$ is a vector space isomorphism for every $p \in E$ which, in its turn, implies that $\Delta^I \colon p \to \Delta_p^I$ is a distribution satisfying (8.10) as $\pi_*(\Delta_p^v) = 0_{\pi(p)} \in T_{\pi(p)}(M)$.

If $G_i \in \Delta_p^I$, $i = 1, 2$, then there exist paths $\gamma_i \colon J_i \to M$ such that $\gamma_i(s_i) = \pi(p)$ for some $s_i \in J_i$ and $G_i = \dot{\bar{\gamma}}_{i;s_i,p}(s_i)$, with $i = 1, 2$ and the lifted paths in the r.h.s. being given by (8.7). Then $\pi_*(G_i) = \dot{\gamma}_i(s_i)$, due to (8.11). Suppose that $\dot{\gamma}_1(s_1) = \dot{\gamma}_2(s_2)$. Then condition (b) entails $\dot{\bar{\gamma}}_{1;s_1,p}(s_1) = \dot{\bar{\gamma}}_{2;s_2,p}(s_2)$ so that $G_1 = G_2$, which means that $\pi_*|_{\Delta_p^I} \colon \Delta_p^I \to T_{\pi(p)}(M)$ is injective. $\square$

Remark 8.2. The condition (a) ensures that all our constructions have a sense. If the condition (b) is not valid, than $\pi_*|_{\Delta_p^I}$ is linear and surjective, but we cannot prove that it is also injective; precisely, $\pi_*|_{\Delta_p^h}$ is injective iff the property (b) is valid. At last, the condition (c) guarantees that Δ_p^I is a vector subspace of $T_p(E)$; in fact, Δ_p^I is a vector space iff the property (c) is valid. This is quite an essential moment as the direct complement B to a vector subspace A of a vector space V (in our case $B = \Delta_p^h$, $A = \Delta_p^v$ and $V = T_p(E)$) is generally *not* a vector subspace; i.e. the equality $V = A \oplus B$, with V and A being vector spaces, does not generally imply that B is a vector space; for example, if $A = \{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$, then $\mathbb{R}^2 = A \oplus B$ with $B = \{(x, y) \in \mathbb{R}, y \neq 0\} \cup \{(0, 0)\}$, but B is not a vector subspace in $\mathbb{R}^2$, as $(a, -c), (b, +c) \in B$ for $a, b, c \in \mathbb{R}$ and $c \neq 0$ but $(a, -c) + (b, +c) = (a + b, 0) \notin B$ for $a + b \neq 0$.

Remark 8.3. Since (8.8c) and (8.11) entail

$$\dot{\gamma}_3(s_3) = a_1 \dot{\gamma}_1(s_1) + a_2 \dot{\gamma}_2(s_2), \tag{8.13a}$$

the path γ_3 is uniquely fixed by the conditions that it passes through the point $\pi(p)$,

$$\gamma_3(s_3) = \pi(p). \tag{8.13b}$$

Therefore, conditions (c) is equivalent to the requirement the vector tangent at s_3 to the lift, given via (8.7), of the path defined via (8.13) to be (8.8c).

Remark 8.4. The role of the transport I along paths in theorem 8.1 is on its base to be constructed a lifting of the paths in M to paths in E with appropriate properties. Namely, such a lifting should assign to a path $\gamma\colon J \to M$ a unique path $\bar{\gamma}_{s_0,p}\colon J \to E$ passing through a given point $p \in \pi^{-1}(\gamma(s_0))$, for some $s_0 \in J$, and such that $\pi \circ \bar{\gamma}_{s_0,p} = \gamma$ and, if $q = \bar{\gamma}_{s_0,p}(t_0)$ for some $t_0 \in J$, then $\bar{\gamma}_{t_0,q} = \bar{\gamma}_{s_0,p}$. On this ground one can generalize theorem 8.1 as well as some of the next considerations and results.

Definition 8.2. The connection Δ^I, defined in theorem 8.1, will be called *assigned to* (*defined by, generated by*) the transport I along paths.

Remark 8.5. Note that the transport I in this definition must have the properties (8.8) described in theorem 8.1.

Exercise 8.1. Consider a *linear* transport L along paths in a *vector* bundle (E, π, M), E and M being C^1 manifolds. Let $\{u^I\} = \{u^\mu = x^\mu \circ \pi, u^a\}$ be vector bundle coordinates in E and $\{e_a\}$ be the frame generating $\{u^a\}$ (see the introduction to section 4) and $\boldsymbol{L}$ and $\mathbf{\Gamma}$ be respectively the matrix and the matrix of the coefficients on L in $\{e_a\}$. Then

$$\bar{\gamma}_{s_0,p}(t) = L^a{}_b(t, s_0; \gamma)p^b e_a(\gamma(t)) \tag{8.14}$$

due to (IV.3.12). Using (IV.3.15) and (IV.3.25), prove that

$$\dot{\bar{\gamma}}^\mu_{s_0,p}(t) = \dot{\gamma}^\mu(t) \qquad \dot{\bar{\gamma}}^a_{s_0,p}(t) = -\Gamma^a{}_b(t;\gamma)\bar{\gamma}^b_{s_0,p}(t). \tag{8.15}$$

In particular, we have

$$\dot{\bar{\gamma}}^\mu_{s_0,p}(s_0) = \dot{\gamma}^\mu(s_0) \qquad \dot{\bar{\gamma}}^a_{s_0,p}(s_0) = -\Gamma^a{}_b(s_0;\gamma)p^b \tag{8.16}$$

where $p^b = u^b(p)$, i.e. $p = p^b e_b(\pi(p))$. The first equalities in (8.15) and (8.16) are consequences of (8.11) or (8.12) and, in this sense, are inessential for the following.

Example 8.1a. In a case of a linear transport along paths in a vector bundle, the condition (a) from proposition 8.1 means that the transport is of class C^1 (along C^1 paths) – see the beginning of subsection IV.3.3. Said otherwise, it is equivalent to C^1 dependence of the transport's matrix on path's parameter due to (8.14). It follows from (8.15) that, if the linear transport is of class C^k, $k \geq 1$, relative to all C^m paths in M for some m such that $1 \leq m \leq k$, then the lifted path $\bar{\gamma}_{s_0,p}$ is of class C^m for all C^m paths in M; usually $m = k$.

Example 8.1b. In a case of a C^1 linear transport along paths in a vector bundle, the equations (8.16) entail that the condition (b) from proposition 8.1 is equivalent to

$$\mathbf{\Gamma}(s_1;\gamma_1) = \mathbf{\Gamma}(s_2;\gamma_2) \tag{8.17}$$

for the particular values $s_1 \in J_1$ and $s_2 \in J_2$ for which $\gamma_1(s_1) = \gamma_2(s_2)$ and $\dot{\gamma}_1(s_1) = \dot{\gamma}_2(s_2)$. This is an equation for (the matrix of the coefficients of) the transport L which must be valid for any C^1 paths γ_1 and γ_2 with the properties just mentioned. Its solutions is

$$\mathbf{\Gamma}(s;\gamma) = G(\gamma(s), \dot{\gamma}(s)) \tag{8.18}$$

for every C^1 path $\gamma\colon J \to M$, $s \in J$ and some matrix-valued function G. In particular, $\mathbf{\Gamma}(s;\gamma)$ may be given by (IV.6.1),

$$\mathbf{\Gamma}(s;\gamma) = \Gamma_\mu(\gamma(s))\dot{\gamma}^\mu(s)) \tag{8.19}$$

in which case the transport may admit normal frames (see proposition (IV.6.1)).

Example 8.1c. In a case of a C^1 linear transport along paths in a vector bundle, the equations (8.16) entail that the condition (c) from proposition 8.1 is equivalent to

$$\mathbf{\Gamma}(s_3;\gamma_3) = a_1\mathbf{\Gamma}(s_1;\gamma_1) + a_2\mathbf{\Gamma}(s_2;\gamma_2) \tag{8.20}$$

where $a_1, a_2 \in \mathbb{K}$. The C^1 paths involved are such that $\gamma_1(s_1) = \gamma_2(s_2) = \gamma_3(s_3) = \pi(p)$ and, according to remark 8.3, the path γ_3 is uniquely defined via the initial-value problem (8.13). Consequently, equation (8.20) is an equation for (the matrix of the coefficients of) the transport in a sense that it should be an identity relative to a_1, a_2, γ_1 and γ_2 provided the path γ_3 is the solution of

$$\dot{\gamma}_3(s_3) = a_1\dot{\gamma}_1(s_1) + a_2\dot{\gamma}_2(s_2), \tag{8.21a}$$
$$\gamma_3(s_3) = \pi(p). \tag{8.21b}$$

In particular, the combination of (8.21a) and (8.18) immediately results into (8.19) which provides a solution of (8.20) due to (8.21a). Going some pages ahead (see corollary 8.5 on page 405 and theorem 8.9 on page 405), we can formulate this result as: the linear connections on vector bundles are the only ones that can be generated by linear transports in them and, conversely, the linear parallel transports along paths in vector bundles are the only ones that are generated by the parallel transports corresponding to linear connections.

It is worth recording the simple fact that, if equation (8.4) or (8.5) holds, then the path $\bar{\gamma}_{s_0,p}$, defined via (8.7) is such that respectively

$$\bar{\gamma}_{s_0,p}|J' = (\overline{\gamma|J'})_{s_0,p} \qquad s_0 \in J' \tag{8.22}$$
$$\bar{\gamma}_{\chi(s_0),p}\circ\chi = (\overline{\gamma\circ\chi})_{s_0,p} \qquad s_0 \in J''. \tag{8.23}$$

Exercise 8.2. Prove that for C^1 paths equations (8.4) and (8.5) are consequences of the suppositions (a)–(c) from theorem 8.1 but the converse is generally not true, i.e. (8.4) and (8.5) are necessary but generally not sufficient conditions for (8.9) to be a connection. To prove (8.4), use assumption (b) and the local existence of a unique C^1 path passing through a given point and having a fixed tangent vector at it. The proof of (8.5) is more complicated. For the purpose show that the paths $\bar{\beta}_{s_0,p}\colon t \to I^\beta_{s_0\to t}(p)$ and $\bar{\delta}_{s_0,p}\colon t \to I^\gamma_{\chi(s_0)\to\chi(t)}(p) = \bar{\gamma}_{\chi(s_0),p}\circ\chi(t)$, for $s_0, t \in J''$, in E are liftings of the path $\beta = \gamma\circ\chi$ and are Δ^I-horizontal, i.e. $\dot{\bar{\beta}}_{s_0,p}(t) \in \Delta^I_{\bar{\beta}_{s_0,p}(t)}$ and $\dot{\bar{\delta}}_{s_0,p}(t) \in \Delta^I_{\bar{\delta}_{s_0,p}(t)}$. (Here use that $\dot{\beta}(t) = \frac{\mathrm{d}\chi(t)}{\mathrm{d}t}\dot{\gamma}(\chi(t))$ and similarly for $\bar{\delta}_{s_0,p}$.) Then the existence of a unique horizontal lift of any C^1 path (see also proposition 8.4 below) implies $\bar{\beta}_{s_0,p} = \bar{\delta}_{s_0,p}$.

8.2. Parallel transports and parallel transports along paths

For the further exploration of the relations between transports along paths and connections (or parallel transports generated by them), we shall need the notion of an inverse path and of a product of paths. There are not 'natural' definitions of these concepts, but this is not important for us as the (parallel) transports we shall consider bellow are parametrization invariant in some sense, like (8.5). For that reason, the concepts mentioned will be defined only for *canonical paths* $[0,1] \to M$, whose domain is the real interval $[0,1] := \{r \in \mathbb{R} : 0 \leq r \leq 1\}$. The path *inverse* to $\gamma\colon [0,1] \to M$ is $\gamma_{\text{-}} := \gamma \circ \tau_{\text{-}}\colon [0,1] \to M$, with $\tau_{\text{-}}\colon [0,1] \to [0,1]$ being given by $\tau_{\text{-}}(t) := 1 - t$ for $t \in [0,1]$. If $\gamma_1, \gamma_2\colon [0,1] \to M$ and $\gamma_1(1) = \gamma_2(0)$, the *product* $\gamma_1\gamma_2$ of γ_1 and γ_2 is a canonical path $\gamma_1\gamma_2\colon [0,1] \to M$ such that $(\gamma_1\gamma_2)(t) := \gamma_1(2t)$ for $t \in [0,1/2]$ and $(\gamma_1\gamma_2)(t) := \gamma_2(2t-1)$ for $t \in [1/2,1]$. For more details on this item, see [108, 132].

Recall now the basic properties of the parallel transports generated by connections.

Proposition 8.1. *Let*

$$\mathsf{P}\colon \gamma \mapsto \mathsf{P}^{\gamma}\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma,\tau] \to M \tag{8.24}$$

be the parallel transport generated by a connection on some bundle (E,π,M). The mapping P has the following properties:

(i) *The parallel transport P is invariant under orientation preserving changes of the paths' parameters. Precisely, if $\gamma\colon [\sigma,\tau] \to M$ and $\chi\colon [\sigma',\tau'] \to [\sigma,\tau]$ is an orientation preserving C^1 diffeomorphism, then*

$$\mathsf{P}^{\gamma\circ\chi} = \mathsf{P}^{\gamma}. \tag{8.25}$$

(ii) *If $\gamma\colon [0,1] \to M$ and $\gamma_{\text{-}}\colon [0,1] \to M$ is its canonical inverse, $\gamma_{\text{-}}(t) = \gamma(1-t)$ for $t \in [0,1]$, then*

$$\mathsf{P}^{\gamma_{\text{-}}} = \big(\mathsf{P}^{\gamma}\big)^{-1}. \tag{8.26}$$

(iii) *If $\gamma_1, \gamma_2\colon [0,1] \to M$, $\gamma_1(1) = \gamma_2(0)$, and $\gamma_1\gamma_2\colon [0,1] \to M$ is their canonical product, then*

$$\mathsf{P}^{\gamma_1\gamma_2} = \mathsf{P}^{\gamma_2} \circ \mathsf{P}^{\gamma_1}. \tag{8.27}$$

(iv) *If $\gamma_{r,x}\colon \{r\} = [r,r] \to \{x\}$ for some given $r \in \mathbb{R}$ and $x \in M$, then*

$$\mathsf{P}^{\gamma_{r,x}} = \mathsf{id}_{\pi^{-1}(x)}. \tag{8.28}$$

(v) *If $\bar{\gamma}_p\colon s \in [\sigma,\tau] \to \bar{\gamma}(s) := \mathsf{P}^{\gamma|[\sigma,s]}(p)$ is the lifting of γ through $p \in \pi^{-1}(\gamma(\sigma))$ defined by P and the C^1 paths $\gamma_i\colon [\sigma_i,\tau_i] \to M$ are such that $\gamma_1(\sigma_1) = \gamma_2(\sigma_2)$ and $\dot{\gamma}_1(\sigma_1) = \dot{\gamma}_2(\sigma_2)$, then (a) the lifted path*

$$\bar{\gamma}_p \text{ is of class } C^1 \tag{8.29}$$

for any C^1 path γ and (b) the lifted paths $\bar{\gamma}_{1;p}$ and $\bar{\gamma}_{2;p}$ have equal tangent vectors at p,

$$\dot{\bar{\gamma}}_{1;p}(\sigma_1) = \dot{\bar{\gamma}}_{2;p}(\sigma_2). \tag{8.30}$$

(vi) *If $\gamma_i\colon [\sigma_i, \tau_i] \to M$, $i = 1, 2$, are two C^1 paths and $\gamma_1(\sigma_1) = \gamma_2(\sigma_2)$, then for every $a_1, a_2 \in \mathbb{K}$ there exists a C^1 path $\gamma_3\colon [\sigma_3, \tau_3] \to M$ such that $\gamma_3(\sigma_3) = \gamma_1(\sigma_1)$ $(= \gamma_2(\sigma_2))$ and the vector tangent to the lifted path $\bar{\gamma}_{3;p}\colon t_3 \in [\sigma_3, \tau_3] \to \bar{\gamma}_{3;p}(t_3) := \mathsf{P}^{\gamma_3|[\sigma_3.t_3]}(p)$, with $p \in \pi^{-1}(\gamma_3(\sigma_3))$, at σ_3 is*

$$\dot{\bar{\gamma}}_{3;p}(\sigma_3) = a_1\dot{\bar{\gamma}}_{1;p}(\sigma_2) + a_2\dot{\bar{\gamma}}_{2;p}(\sigma_2), \tag{8.31}$$

where $\bar{\gamma}_{i;p}\colon t_i \in [\sigma_i, \tau_i] \to \bar{\gamma}_{i;p}(t_i) := \mathsf{P}^{\gamma_i|[\sigma_i.t_i]}(p)$.

Remark 8.6. As a result of (8.25), some properties of the parallel transports generated by connections, like (8.26) and (8.27), are sufficient to be formulated/proved only for canonical paths $[0, 1] \to M$.

Proof. The proofs of (8.25)–(8.31) can be found in a number of works, for example in [3, 4, 6, 11, 30, 32, 33, 149, 150, 155]. Alternatively, the reader can prove them by applying the definitions given in this book. (See also subsections IV.14.1 and IV.14.2 in a case of a vector bundle.) □

Definition 8.3 (cf. definition IV. 14.11 on page 310)**.** A mapping (8.24) satisfying the equalities (8.25)–(8.31) will be called *(axiomatically defined) parallel transport.*

Proposition 8.2. *Let P be the parallel transport assigned to a C^m, with $m \in \mathbb{N} \cup \{0\}$, connection on a* smooth, *of class C^{m+1}, bundle (E, π, M). Then P is smooth, of class C^m, in a sense that, if $\gamma\colon [\sigma, \tau] \to M$ is a C^1 path, then P^γ is in the set of C^m diffeomorphisms between the fibres $\pi^{-1}(\gamma(\sigma))$ and $\pi^{-1}(\gamma(\tau))$,*

$$\mathsf{P}\colon \gamma \mapsto \mathsf{P}^\gamma \in \mathrm{Diff}^m\big(\pi^{-1}(\gamma(\sigma)), \pi^{-1}(\gamma(\tau))\big) \qquad \gamma\colon [\sigma, \tau] \to M. \tag{8.32}$$

Proof. See [16, 106, 107]. □

The axiomatic approach to parallel transport was developed mainly on the ground on the properties (8.25)–(8.32) of the parallel transports assigned to connections. However, this topic is out of the range of the present monograph and the reader is referred to the literature cited at the beginning of the present section.

Now we shall present a modified version of [115, p. 13, theorem 3.1].

Theorem 8.2. *Let I be a transport along paths in bundle (E, π, M) and $\gamma\colon [\sigma, \tau] \to M$. If I satisfies the conditions (*8.8*) from theorem* 8.1, [3] *then the mapping*

$$\mathsf{I}\colon \gamma \mapsto \mathsf{I}^\gamma := I^\gamma_{\sigma\to\tau}\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma, \tau] \to M \tag{8.33}$$

*is a parallel transport, i.e. it possess the properties (*8.25*)–(*8.31*), with I for P. Besides, if I is smooth in a sense that*

$$I^\beta_{s\to t} \in \mathrm{Diff}^m\big(\pi^{-1}(\beta(s)), \pi^{-1}(\beta(t))\big) \qquad \beta\colon J \to M \quad s, t \in J \tag{8.34}$$

[3] Recall, according to exercise 8.2, the equations (8.8) imply (8.4) and (8.5).

for some $m \in \mathbb{N} \cup \{0\}$*, then the mapping* (8.33) *satisfies* (8.32)*, with* I *for* P.

Conversely, suppose the mapping

$$\mathsf{P}\colon \gamma \mapsto \mathsf{P}^{\gamma}\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma,\tau] \to M \tag{8.35}$$

is a parallel transport, i.e. it satisfies (8.25)*–*(8.31)*, and define the mapping*

$$P\colon \beta \mapsto P^{\beta}\colon (s,t) \mapsto P^{\beta}_{s\to t} = \mathsf{P}^{(\beta|[\sigma,\tau])\circ\chi_t^{[\sigma,\tau]}} \circ \Big(\mathsf{P}^{(\beta|[\sigma,\tau])\circ\chi_s^{[\sigma,\tau]}}\Big)^{-1} \qquad \beta\colon J \to M, \tag{8.36}$$

where $s,t \in J$*,* $\sigma,\tau \in J$ *are such that* $\sigma \le \tau$ *and* $[\sigma,\tau] \ni s,t$*,* [4] *and* $\chi_s^{[\sigma,\tau]}\colon [\sigma,\tau] \to [\sigma,s]$ *are for* $s > \sigma$ *arbitrary orientation preserving* C^1 *diffeomorphisms (depending on* β *via the interval* $[\sigma,\tau]$*). Then the mapping* (8.36) *is a transport along paths in* (E,π,M)*, which transport satisfies the conditions* (8.8)*, with* P *for* I*. Besides, under the same assumptions, the condition* (8.32) *for* P *implies* (8.34)*, with* P *for* I*, where* P *is given by* (8.36)*.*

Remark 8.7. Instead by (8.36), the transport P along paths generated by a parallel transport P can be defined equivalently as follows. For a path $\gamma\colon [\sigma,\tau] \to M$ and $s,t \in [\sigma,\tau]$, we put (see (8.25))

$$P\colon \gamma \mapsto P^{\gamma}\colon (s,t) \mapsto P^{\gamma}_{s\to t} = \mathsf{P}^{\gamma\circ\chi_t^{[\sigma,\tau]}} \circ \Big(\mathsf{P}^{\gamma\circ\chi_s^{[\sigma,\tau]}}\Big)^{-1} = \mathsf{P}^{\gamma|[\sigma,t]} \circ \big(\mathsf{P}^{\gamma|[\sigma,s]}\big)^{-1}$$
$$\gamma\colon [\sigma,\tau] \to M \tag{8.37a}$$

Now, for an arbitrary path $\beta\colon J \to M$, with J being closed or open at one or both its ends, we set

$$P\colon \beta \mapsto P^{\beta}\colon (s,t) \mapsto P^{\beta}_{s\to t} = \begin{cases} P^{\beta|[s,t]}_{s\to t} & \text{for } s \le t \\ \big(P^{\beta|[t,s]}_{t\to s}\big)^{-1} & \text{for } s \ge t \end{cases} \qquad \beta\colon J \to M \quad s,t \in J. \tag{8.37b}$$

It can easily be verified that (8.37) are tantamount to

$$P\colon \beta \mapsto P^{\beta}\colon (s,t) \mapsto P^{\beta}_{s\to t} = \begin{cases} \mathsf{P}^{\beta|[s,t]} & \text{for } s \le t \\ \big(\mathsf{P}^{\beta|[t,s]}\big)^{-1} & \text{for } s \ge t \end{cases} \qquad \beta\colon J \to M \quad s,t \in J. \tag{8.38}$$

Proof. Suppose I satisfies (8.8). To begin with, we notice that (8.8) imply (8.4) and (8.5) by virtue of exercise 8.2. Equation (8.25) follows from (8.33), and (8.5). Equation (8.26) is a consequence of (8.33), (8.5) and (8.6) for $\gamma\colon [0,1] \to M$ and $\tau_{\text{-}}(t) = 1 - t$ for $t \in [0,1]$:

$$\mathsf{I}^{\gamma_{\text{-}}} = I^{\gamma\circ\tau_{\text{-}}}_{0\to 1} = I^{\gamma}_{\tau_{\text{-}}(0)\to\tau_{\text{-}}(1)} = I^{\gamma}_{1\to 0} = (I^{\gamma}_{0\to 1})^{-1} = (\mathsf{I}^{\gamma})^{-1}.$$

[4] In particular, one can set $\sigma = \min(s,t)$ and $\tau = \max(s,t)$ or, if J is a closed interval, define σ and τ as the end points of J, i.e. $J = [\sigma,\tau]$.

To prove (8.27), we set $\gamma_{1,2}\colon [0,1]\to M$, $\tau_1(t)=2t$ for $t\in[0,1/2]$ and $\tau_2(t)=2t-1$ for $t\in[1/2,1]$ and, applying equations (8.33), (8.4), (8.5), (8.6) and (8.2), we find:

$$\begin{aligned}\mathsf{I}^{\gamma_1\gamma_2} &= I^{\gamma_1\gamma_2}_{0\to1} = I^{\gamma_1\gamma_2}_{1/2\to1}\circ I^{\gamma_1\gamma_2}_{0\to1/2} = I^{\gamma_1\gamma_2|[1/2,1]}_{1/2\to1}\circ I^{\gamma_1\gamma_2|[0,1/2]}_{0\to1/2}\\ &= I^{\gamma_2\circ\tau_2}_{1/2\to1}\circ I^{\gamma_1\circ\tau_1}_{0\to1/2} = I^{\gamma_2}_{\tau_2(1/2)\to\tau_2(1)}\circ I^{\gamma_1}_{\tau_1(0)\to\tau_1(1/2)} = I^{\gamma_2}_{0\to1}\circ I^{\gamma_1}_{0\to1} = \mathsf{I}^{\gamma_2}\circ\mathsf{I}^{\gamma_1}.\end{aligned}$$

Next, equation (8.28) is a direct corollary from (8.33) and (8.3). At last, the properties (v) and (vi) are trivial consequences from (8.33) and respectively (8.8a)-(8.8b) and (8.8c). So, I is a parallel transport, which, evidently, satisfies (8.32) with I for P, if (8.34) is valid.

Conversely, suppose P is a parallel transport and P is defined via (8.36), or, equivalently (see remark 8.7), via (8.38). The verification of (8.1)–(8.3) can be done by applying (8.35) and (8.25)–(8.28). Therefore P is a parallel transport along paths. Similarly, the conditions (8.8) follow from the properties (v) and (vi) of the parallel transport P. At last, the validity of (8.28), with P for I, is a consequence from (8.32) and (8.38). □

Definition 8.4. A transport I along paths which has the properties (8.8) will be called *parallel* transport along paths.

Theorem 8.2 simply says that there is a bijective correspondence between the parallel transports along paths and the parallel transports.

Definition 8.5. If I is a parallel transport along paths, then we say that the parallel transport (8.33) is *generated by* (*defined by, assigned to*) I. Respectively, if P is a parallel transport, then we say that the (parallel) transport along paths (8.36) is *generated by* (*defined by, assigned to*) P.

Proposition 8.3. *If I is a parallel transport along paths and I is the assigned to it parallel transport, then the parallel transport along paths defined by I coincides with I. Conversely, if P is a parallel transport and P is the parallel transport along paths generated by P, then the parallel transport defined by P coincides with P.*

Proof. The assertions are consequences from definition 8.5, theorem 8.2 and remark 8.7. □

Let us now return to the connection Δ^I generated by a transport I along paths, introduced in theorem 8.1.

Proposition 8.4. *If $\gamma\colon J\to M$ is a C^1 injective path and $p\in\pi^{-1}(\gamma(s_0))$ for some $s_0\in J$, then there is a unique Δ^I-horizontal lift of γ (relative to Δ^I) in E through p and it is exactly the path $\bar\gamma_{s_0,p}$ defined by (8.7).*

Proof. Simply apply the definitions (8.7) and (8.9) and use the properties (8.1) and (8.2) of the transports along paths ($t_0 \in J$):

$$\begin{aligned}\dot{\bar{\gamma}}_{s_0,p}(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \big(I^\gamma_{s_0\to t}(p)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \big(I^\gamma_{t_0\to t}(I^\gamma_{s_0\to t_0}(p))\big) \\ = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \big(I^\gamma_{t_0\to t}(\bar{\gamma}_{s_0,p}(t_0))\big) = \dot{\bar{\gamma}}_{t_0,\bar{\gamma}_{s_0,p}(t_0)}(t_0) \in \Delta^I_{\bar{\gamma}_{s_0,p}(t_0)}.\end{aligned}$$

□

Remark 8.8. If γ is not injective and $\gamma(s_0) = \gamma(t_0)$ for some $s_0, t_0 \in J$ such that $s_0 \neq t_0$, then the paths $\bar{\gamma}_{s_0,p}, \bar{\gamma}_{t_0,p} \colon J \to E$ need not to coincide as $\bar{\gamma}_{s_0,p} = I_{t_0\to s_0} \circ \bar{\gamma}_{t_0,p}$, due to (8.7) and (8.2). Therefore the Δ^I-horizontal lift of a non-injective path through a point in E lying in a fibre over a self-intersection point of the path, if any, may not be unique. Similar is the situation for an arbitrary connection. (This range of problems is connected with the so-called holonomy groups.)

Proposition 8.5. *The parallel transport* I *generated by the connection* Δ^I*, defined by a transport along paths* I*, is such that*

$$\mathsf{I}^\gamma = I^\gamma_{\sigma\to\tau} \qquad \text{for } \gamma\colon [\sigma,\tau] \to M. \tag{8.39}$$

Proof. According to definition 3.2 and (8.7), we have:

$$\begin{aligned}\mathsf{I}\colon \gamma \mapsto \mathsf{I}^\gamma \colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma,\tau] \to M \\ \mathsf{I}^\gamma \colon p \mapsto \mathsf{I}^\gamma(p) = \bar{\gamma}_{\sigma,p}(\tau) = I^\gamma_{\sigma\to\tau}(p) \qquad p \in \pi^{-1}(\gamma(\sigma)).\end{aligned}$$

□

Corollary 8.1. *The parallel transport* I *generated by the connection* Δ^I*, assigned to a transport* I *along paths, is a parallel transport, i.e. it satisfies (*8.25*)–(*8.31*) with* I *for* P*.*

Proof. This result is a particular case of proposition 8.1. An alternative proof can be carried out by using (8.1)–(8.8), (8.39), and the definitions of inverse path and product of paths. The assertion is also a consequence of (8.39) and theorem 8.2. □

8.3. Connections and parallel transports along paths

Until this point, we have studied how a transport along paths generates a connection (theorem 8.1) and parallel transport (proposition 8.5). Besides, theorem 8.2 establishes a bijective correspondence between particular class of transports along paths and mappings having (some of) the main properties of the parallel transports generated by connections. Below we shall pay attention, in a sense, to the opposite links, starting from a connection on a bundle.

Proposition 8.6. *Let* P *be the parallel transport assigned to a connection* Δ^h *on a bundle* (E, π, M)*. The mapping*

$$P\colon \gamma \mapsto P^\gamma\colon (s,t) \mapsto P^\gamma_{s\to t} \qquad \gamma\colon J \to M \tag{8.40a}$$

defined by

$$P^\gamma_{s\to t} = \begin{cases} \mathsf{P}^{\gamma|[s,t]} & \text{for } s \le t \\ \left(\mathsf{P}^{\gamma|[t,s]}\right)^{-1} & \text{for } s \ge t \end{cases} \tag{8.40b}$$

is a transport along paths in (E, π, M)*. Moreover,* P *is parallel transport along paths, i.e. it satisfies the equations (*8.8*) from theorem* 8.1 *with* P *for* I*.*

Proof. One should check the conditions (8.1)–(8.3), and (8.8) with P for I. The relations (8.1)–(8.3) follow directly from definition 3.2 of a parallel transport generated by a connection. The rest conditions are consequences of (8.40) and a simple, but tedious, application of the properties (8.25)–(8.31) of the parallel transports. Alternatively, this proposition is a consequence of the second part of theorem 8.2 and remark 8.7. □

Remark 8.9. Applying (8.25)–(8.27), the reader can verify that

$$P^\gamma_{s\to t} = F^{-1}(t;\gamma) \circ F(s;\gamma) \tag{8.41}$$

with

$$F(r;\gamma) = \begin{cases} \mathsf{P}^{\gamma|[r,w]} & \text{for } r \le w \\ \left(\mathsf{P}^{\gamma|[w,r]}\right)^{-1} & \text{for } r \ge w \end{cases} \qquad r = s,t \tag{8.42}$$

for any (arbitrarily) fixed $w \in J$. This result is a special case of the general structure of the transports along paths [114, theorem 3.1].

Definition 8.6. The parallel transport along paths, defined by a connection Δ^h on a bundle through proposition 8.6, will be called parallel transport along paths *assigned to (defined by, generated by)* the connection Δ^h.

Corollary 8.2. *Let* Δ^I *be the connection generated by a parallel transport* I *along paths according to theorem* 8.1*. If* I *is the parallel transport assigned to* Δ^I*, then the transport along paths assigned to* I *(or* Δ^I*), as described in proposition* 8.6*, coincides with the initial transport* I *along paths.*

Proof. Substitute (8.39) into (8.40), with I for P and I for P. □

Corollary 8.3. *Let* P *be the transport along paths assigned to a connection* Δ^h *(via its parallel transport* P*) according to proposition* 8.6*. The connection* Δ^P *generated by* P*, as described in theorem* 8.1*, coincides with the initial connection* Δ^h*,* $\Delta^P = \Delta^h$*.*

Proof. On one hand, if $p \in E$, the space Δ_p^P consists of the vectors tangent at s_0 to the paths $\bar{\gamma}_{s_0,p}\colon t \mapsto P_{s_0\to t}^{\gamma}(p)$, with $\gamma\colon J \to M$ of class C^1, $s_0 \in J$, and $\pi(p) = \gamma(s_0)$, due to theorem 8.1. On another hand, Δ_p^h consists of the vectors tangent at s_0 to the paths $\tilde{\gamma}_{s_0,p}\colon t \mapsto \mathsf{P}^{\gamma|[s_0,t]}(p)$, by virtue of definition 3.2. Equation (8.40b) says that both types of paths coincide, $\tilde{\gamma}_{s_0,p} = \bar{\gamma}_{s_0,p}$, so that their tangent vectors at $t = s_0$ are identical and, consequently, Δ_p^P and Δ_p^h are equal as sets, $\Delta_p^P = \Delta_p^h$, for all $p \in E$. □

8.4. Recapitulation

Roughly speaking, the above series of results says that a connection Δ^h is equivalent to a mapping P (the assigned to it parallel transport) satisfying (8.24)–(8.31) or to a mapping P (the assigned to it parallel transport along paths) satisfying (8.1)–(8.3) and (8.8) (with P for I). Besides, the smoothness of Δ^h is equivalent to the one of P or P. Let us summarize these results as follows.

Theorem 8.3. *Given a connection Δ^h on a bundle (E, π, M), there exists a unique parallel transport I along paths in (E, π, M) which generates Δ^h via (8.9), i.e. $\Delta^I = \Delta^h$. Besides, the parallel transport P defined by Δ^h is given by (8.33), i.e. $\mathsf{P} = \mathsf{I}$.*

Proof. See theorem 8.1, proposition 8.5 and theorem 8.2. □

Theorem 8.4. *Given a parallel transport I along paths in a bundle (E, π, M), then there exists a unique connection Δ^h on (E, π, M) such that the parallel transport P along paths assigned to Δ^h coincides with I, $P = I$. Besides, the connection Δ^I generated by I is identical with Δ^h, $\Delta^I = \Delta^h$.*

Proof. Apply theorem 8.1 and corollaries 8.2 and 8.3. □

Theorem 8.5. *Given a parallel transport along paths in a bundle, there is a unique (axiomatically defined) parallel transport generating it. Conversely, given a parallel transport, there is a unique parallel transport along paths generating it.*

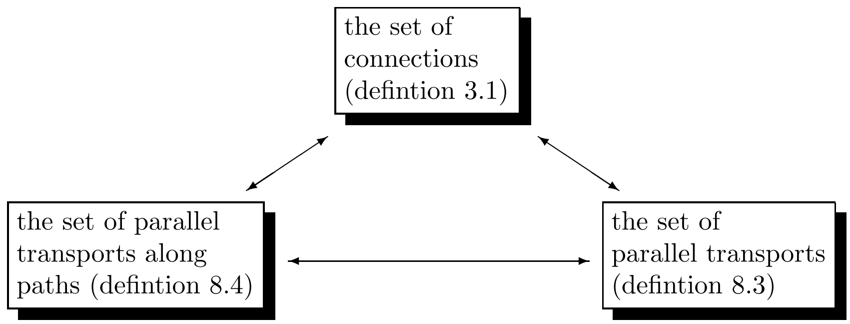
Proof. This statement is a reformulation of theorem 8.2. □

Theorem 8.6. *Given a parallel transport P, there exists a unique connection Δ^h generating it. Besides, the parallel transport assigned to Δ^h coincides with P.*

Proof. See theorems 8.5 and 8.4 and definitions 3.2 and 8.5. □

Theorem 8.7. *Given a connection Δ^h, there is a unique parallel transport P such that the defined by it parallel transport P along paths generates Δ^h, $\Delta^P = \Delta^h$. Besides, P coincides with the parallel transport assigned to Δ^h.*

Figure 8.1: Mappings between the sets of parallel transports, connections and parallel transports along paths in differentiable bundles



Proof. Apply theorems 8.3 and 8.5. □

The above results can be summarized in the commutative diagram shown on figure 8.1, the mappings in which are described via theorems 8.3–8.7. Besides, if one of these objects is smooth, so are the other ones corresponding to it via the bijections constructed in the present section.

We end with the main moral of the above theorems. In a case of a differentiable bundle, the concepts "connection," "(axiomatically defined) parallel transport" and "parallel transport along path" are equivalent in a sense that there are bijective mappings between the sets of these objects. Besides, if one of these objects is smooth, so are the other ones corresponding to it via the bijections constructed in the present section.

8.5. The case of topological bundles

Now we want to say a few words concerning bundles which are *not* differentiable. The definition 3.1 of a connection is strongly related to the concept of the tangent bundle $T(E)$ of the bundle space E and, consequently, it is senseless in a case when E is a manifold of class C^0 of a topological space. As we have demonstrated above (see theorems 8.3 and 8.4), a connection can equivalently be described via a suitable parallel transport which, by definition 8.3, is a mapping possessing the properties (8.25)–(8.31). One observes at first sight that the properties (8.25)–(8.28) do not depend on the differentiable structure of the bundle and, on the contrary, the properties (8.29)–(8.31) are senseless if E is not of class C^1 or higher. This separation of the properties of the parallel transports assigned to connections to ones that depend and do not depend on the differentialble structure of the bundles points to a natural way for generalizing the concept of a parallel transport (and hence of a connection) on bundles with class of smoothness not higher than C^0.

Definition 8.7. *A parallel transport* P in a *topological* bundle (E, π, B) is a mapping (8.24) (with B for M) which has the properties (8.25)–(8.28). Such a mapping will also be called *topological parallel transport* if there is a risk to mix it with the one defined via definition 8.3 in a case of a differentiable bundle.

Proposition 8.7. *A parallel transport in a differentiable bundle of class C^1 is a topological parallel transport.*

Proof. Compare definitions 8.3 and 8.7. □

Relying on our experience with connections and parallel transports in differentiable bundles, *the concepts "connection" and "(topological) transport" in topological bundles should be identified* and, hence considered as equivalent. As the following result shows, the set of (topological) parallel transports is in a bijective correspondence with certain subset of transports along paths in topological bundles.

Theorem 8.8. *Let I be a transport along paths in an arbitrary topological bundle (E, π, B) and $\gamma\colon [\sigma, \tau] \to B$. If I satisfies the conditions (8.4) and (8.5), then the mapping*

$$\mathsf{I}\colon \gamma \mapsto \mathsf{I}^\gamma := I^\gamma_{\sigma\to\tau}\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma, \tau] \to M \tag{8.43}$$

is a parallel transport, i.e. it possess the properties (8.25)–(8.28), with I for P.

Conversely, suppose the mapping

$$\mathsf{P}\colon \gamma \mapsto \mathsf{P}^\gamma\colon \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)) \qquad \gamma\colon [\sigma, \tau] \to M \tag{8.44}$$

is a parallel transport, i.e. satisfies (8.25)–(8.28), and define the mapping

$$P\colon \beta \mapsto P^\beta\colon (s,t) \mapsto P^\beta_{s\to t} = \begin{cases} \mathsf{P}^{\beta|[s,t]} & \textit{for } s \le t \\ \left(\mathsf{P}^{\beta|[t,s]}\right)^{-1} & \textit{for } s \ge t \end{cases} \qquad \beta\colon J \to B \quad s,t \in J. \tag{8.45}$$

Then the mapping (8.45) is a transport along paths in (E, π, B), which transport satisfies the conditions (8.4) and (8.5), with P for I.

Proof. To prove the first part of the theorem, we define $\tau_-\colon t \mapsto 1-t$, $\tau_1\colon t \mapsto 2t$ and $\tau_2\colon t \mapsto 2t-1$ for $t \in [0,1]$. Applying (8.2)–(8.5) and using the notation of proposition 8.1, we get:

$$\begin{aligned}
\mathsf{I}^{\gamma\circ\chi} &= I^{\gamma\circ\chi}_{\sigma\to\tau} = I^{\gamma}_{\chi(\sigma)\to\chi(\tau)} = \mathsf{I}^\gamma \\
\mathsf{I}^{\gamma_-} &= I^{\gamma\circ\tau_-}_{0\to1} = I^{\gamma}_{\tau_-(0)\to\tau_-(1)} = I^\gamma_{1\to0} = (I^\gamma_{0\to1})^{-1} = (\mathsf{I}^\gamma)^{-1} \\
\mathsf{I}^{\gamma_1\gamma_2} &= I^{\gamma_1\gamma_2}_{0\to1} = I^{(\gamma_1\gamma_2)|[1/2,1]}_{1/2\to1} \circ I^{(\gamma_1\gamma_2)|[0,1/2]}_{0\to1/2} = I^{\gamma_2\circ\tau_2}_{1/2\to1} \circ I^{\gamma_1\circ\tau_1}_{0\to1/2} = I^{\gamma_2}_{0\to1} \circ I^{\gamma_1}_{0\to1} = \mathsf{I}^{\gamma_2} \circ \mathsf{I}^{\gamma_1} \\
\mathsf{I}^{\gamma_{r,x}} &= I^{\gamma_{r,x}}_{r\to r} = \mathsf{id}_{\pi^{-1}(x)}.
\end{aligned}$$

To prove the second part, we first note that (8.3) and (8.5) follow from (8.28) and (8.25), respectively, due to (8.45). As a result of (8.45), we have, e.g. for $s \le t$,

$$P^{\gamma|J'}_{s\to t} = \mathsf{P}^{(\gamma|J')|[s,t]} = \mathsf{P}^{\gamma|[s,t]} = P^{\gamma}_{s\to t}$$

by virtue of $[s,t] \subseteq J'$. At last, we shall prove (8.2) for $r \le s \le t$; the other cases can be proved similarly. Let us fix some orientation-preserving bijections $\tau' \colon [0,1] \to [r,s]$ and $\tau'' \colon [0,1] \to [s,t]$. Then:

$$P^{\beta}_{s\to t} \circ P^{\beta}_{r\to s} = \mathsf{P}^{\beta|[s,t]} \circ \mathsf{P}^{\beta|[r,s]} = \mathsf{P}^{\beta\circ\tau''} \circ \mathsf{P}^{\beta\circ\tau'} = \mathsf{P}^{(\beta\circ\tau')(\beta\circ\tau'')}$$
$$= \mathsf{P}^{\beta\circ\tau} = \mathsf{P}^{\beta|[r,t]} = P^{\beta}_{r\to t}$$

where $\tau \colon a \mapsto \tau'(2a)$ for $a \in [0,1/2]$ and $\tau \colon a \mapsto \tau''(2a-1)$ for $a \in [1/2,1]$, so that $\tau(0) = \tau'(0) = r$ and $\tau(1) = \tau''(1) = t$. □

Remark 8.10. In fact, this proof is contained implicitly in the proof of theorem 8.2.

Definition 8.8. In a topological bundle, a transport I along paths which has the properties (8.4) and (8.5) will be called *parallel* transport along paths. If I is a parallel transport along paths, then we say that the parallel transport (8.43) is *generated by* (*defined by, assigned to*) I. Respectively, if P is a parallel transport, then we say that the (parallel) transport along paths (8.45) is *generated by (defined by, assigned to)* P.

Theorem 8.8 simply says that there is a bijective correspondence between the set of parallel transports along paths and the one of parallel transports.

Theorem 8.8 and definition 8.8 are analogues of respectively Theorem 8.2 and definition 8.4 in a case of a topological bundle and contain them as special cases when differentiable bundles are considered. The last assertion is confirmed by the result of exercise 8.2 that equations (8.4) and (8.5) are consequences from the equations (8.8). Thus, in terms of transports along paths, equations (8.4) and (8.5) single out the parallel transports which are independent of the differentiable structure of the bundles in which they act. The same result is expressed by the properties (8.25)–(8.28) of the parallel transports assigned to connections while (8.29)–(8.31) are sensible only in differentiable bundles and single out from the topological parallel transports the ones defined by connections.

We can summarize the above considerations in the commutative diagram presented on figure 8.2 on the next page the mappings in which are described via theorem 8.8.

As a conclusion, we can say that in topological bundles the concept of connection does not survive and there is a bijection between the sets of "parallel transports" and "parallel transports along paths" and, in that sense, these concepts are equivalent.

Figure 8.2: Mappings between the sets of (topological) parallel transports and parallel transports along paths in topological bundles

$$\boxed{\begin{array}{l}\text{the set of parallel}\\ \text{transports along}\\ \text{paths (defintion 8.8)}\end{array}} \longleftrightarrow \boxed{\begin{array}{l}\text{the set of (topological)}\\ \text{parallel transports}\\ \text{(defintion 8.7)}\end{array}}$$

8.6. The case of vector bundles

The purpose of this section is to be presented some links between transports and connections in vector bundles in a case these objects are linear, i.e. when they are compatible with the vector structure of the bundle.

Corollary 8.4. *In a case of a vector bundle:*
(a) A parallel transport I along paths is linear if and only if the assigned to it parallel transport I is is linear;
(b) A parallel transport P is linear if and only if the generated by it parallel transport P along paths is linear.

Proof. The results follow from theorem 8.2 and proposition 8.3. □

Corollary 8.5. *In a case of a vector bundle:*
(a) The linear connections are the only connections that can be generated by linear parallel transports along paths;
(b) The linear parallel transports along paths are the only transports along paths that can be generated by linear connections (via the assigned to them parallel transports).

Proof. The assertions are consequences from definition 4.1, corollary 8.4 and proposition 8.6. □

Theorem 8.9. *Let Δ^h be a* linear *connection on a* vector *bundle (E, π, M), E and M being C^1 manifolds. Let $\{u^I\} = \{x^\mu \circ \pi, u^a\}$ be vector bundle coordinates on E generated by a frame $\{e_a\}$ in E, $\{X_I\}$ be the frame adapted to $\{u^I\}$ and $\Gamma^a_{b\mu}$ be the 3-index coefficients of Δ^h in $\{X_I\}$ (see theorem* 4.1*). If P is the parallel transport along paths assigned to Δ^h, then the coefficients of P in $\{e_a\}$ along a C^1 path $\gamma\colon J \to M$ are ($s \in J$)*

$$\Gamma^a{}_b(s;\gamma) = \Gamma^a_{b\mu}(\gamma(s))\dot{\gamma}^\mu(s). \tag{8.46}$$

Conversely, if I is a C^1 linear *transport along paths in (E, π, M) and in $\{e_a\}$ its coefficients have a representation* (8.46) *along every C^1 path γ for some functions $\Gamma^a_{b\mu}$, then there exists a unique linear connection Δ^h on (E, π, M) such* *that the parallel transport P along paths generated by Δ^h coincides with I, $P = I$; besides, $\Gamma^a_{b\mu}$ are exactly the 3-index coefficients of Δ^h.*

Proof. According to corollaries 8.2 and 8.3 and proposition 8.4, the Δ^h-horizontal lift of γ through $p \in \pi^{-1}(\gamma(s_0))$, $s_0 \in J$ is given by $\bar\gamma_{s_0,p}(s) = P^\gamma_{s_0\to s}(p)$ and therefore

$$\dot{\bar\gamma}^a_{s_0,p}(s) = -\Gamma^a{}_b(s;\gamma)\bar\gamma^b_{s_0,p}(s)$$

by virtue of (8.15). On another hand, the parallel transport equation (3.39′a) implies

$$\dot{\bar\gamma}^a_{s_0,p}(s) = \Gamma^a_\mu(\bar\gamma_{s_0,p}(s))\dot{\bar\gamma}^\mu_{s_0,p}(s),$$

where $\Gamma^a{}_\mu$ are the 2-index coefficients of Δ^h. Inserting here (4.13) and comparing the result with the previous displayed equation, we obtain (8.46), due to $\pi\circ\bar\gamma_{s_0,p} = \gamma$ and (8.11).

Conversely, suppose (8.46) holds. Then I satisfies the conditions (a)–(c) from theorem 8.1 (see examples 8.1a–8.1c), so (8.9) defines a connection Δ^h. Then the parallel transport P along paths generated by Δ^h coincides with I, $P = I$, by virtue of corollary 8.2. As a result of proposition 8.4, (8.46) and (8.15), the Δ^h horizontal lift of γ is such that

$$\dot{\bar\gamma}_{s_0,p}(s) = -\Gamma^a_{b\mu}(\gamma(s))\dot\gamma^\mu(s)\bar\gamma^b_{s_0,p}(s).$$

Comparing this result with the parallel transport equation (3.39′a), we get (see also (8.11))

$$\Gamma^a_\mu(\bar\gamma_{s_0,p}(s)) = -\Gamma^a_{b\mu}(\gamma(s))\bar\gamma^b_{s_0,p}(s)$$

which entails $\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot u^b$ due to $\pi\circ\bar\gamma_{s_0,p} = \gamma$ and the arbitrariness of γ and $p \in \pi^{-1}(\gamma(s_0))$. At last, theorem 4.1 on page 350 says that the connection Δ^h is linear. $\square$

Theorem 8.9 simply means that a linear transport along paths is a parallel transport along paths (generated by a linear connection) if and only if equation (8.46) holds in some (and hence in any) frame $\{e_a\}$. In this sense, a linear transport along paths is equivalent to a linear connection iff equation (8.46) is valid in some (and hence in any) frame $\{e_a\}$. We can also say that theorem 8.9 describes the subset of all linear transports along paths that can be generated by linear connections, viz. these are the transports with coefficeints given by equation (8.46).

Exercise 8.3. Derive corollary 8.5 from theorem 8.9.

Corollary 8.6. *Let L be a linear transport along paths in a vector bundle (E,π,M), E and M being C^1 manifolds. Then L admits frames normal on $U \subseteq M$ (resp. along a C^1 path $\gamma\colon J\to M$), if along every C^1 path $\gamma\colon J\to M$ (resp. along the given path γ) it coincides with some parallel linear transport along paths (generated by some linear connection) in (E,π,M).*

Proof. Apply theorem 8.9 and proposition IV.6.1. □

In fact, the last result is an invariant reformulation of proposition IV.6.1. Note, along a fixed/given path γ, the result is trivial as a representation like (8.46) always exists along a fixed/given path. Besides, according to corollary IV.5.1, frames normal along injective paths always exist.

Example 8.2. As an application of theorem 8.9, we shall prove that the evolution transport arising in bundle quantum mechanics [127] (see (IV.16.3)) is generally *not a parallel* linear transport along paths. [5] To show this, we note that the coefficients of the evolution transport are (see [128, eq. (2.22)] or (IV.16.4))

$$\Gamma^a{}_b(t;\gamma) = -\frac{1}{\mathrm{i}\hbar}(\boldsymbol{H}^m_\gamma(t))^a_b \tag{8.47}$$

where i is the imaginary unit, $\hbar$ is the Planck's constant (devided by 2π) and $\boldsymbol{H}^m_\gamma(t)$ is the matrix-bundle Hamiltonian. One can choose a special frame which is independent of γ and is such that [128, remark 2.1] $\boldsymbol{H}^m_\gamma(t) = \boldsymbol{\mathcal{H}}(t)$, where $\boldsymbol{\mathcal{H}}(t)$ is the matrix of the usual Hilbert space Hamiltonian. So, in this frame, we have

$$\Gamma^a{}_b(t;\gamma) = -\frac{1}{\mathrm{i}\hbar}\mathcal{H}^a_b(t) \tag{8.48}$$

with $\mathcal{H}^a_b(t)$ being the matrix elements of the system Hamiltonian which may depend on the (time) parameter t but does not depend on the path γ. Therefore equation (8.46) is generally not valid [6] and consequently the evolution transport cannot be generated by a linear connection in the general case. In particular, this conclusion is valid for all time-independent Hamiltonians, $\frac{\partial \mathcal{H}(t)}{\partial t} = 0$.

[5] The below-presented proof is not quite rigorous as theorem 8.9 is proved in the finite-dimensional case while the evolution transport acts in generally infinite dimensional Hilbert bundle.

[6] The same result can be prooved by using the general expression (IV.16.5) for the matrix-bundle Hamiltonian instead of the one in special frame mantioned above.