Normal frames and linear transports along paths in line bundles.

Applications to classical electrodynamics

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- The formalism is specified on line bundles and applied to a geometrical description of the classical electrodynamics.
- The inertial frames for this theory are discussed.

Contents

1. Linear transports along paths

Let (E, π, B) be a complex vector bundle with bundle (total) space E, base B, projection $\pi\colon E \to B$, and homeomorphic fibres $\pi^{-1}(x)$, $x \in B$. The base B is supposed to be a C^1 differentiable manifold. By J and $\gamma\colon J \to B$ are denoted real interval and path in B, respectively. The paths considered are generally not supposed to be continuous or differentiable unless their differentiability class is stated explicitly. If γ is a C^1 path, the vector field tangent to it is denoted by $\dot{\gamma}$.

Definition 1.1. A linear transport along paths in a bundle (E, π, B) is a map L assigning to every path γ a map L^γ, transport along γ, such that $L^\gamma\colon (s,t) \mapsto L^\gamma_{s\to t}$ where the map

$$L^\gamma_{s\to t}\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \qquad s,t \in J, \tag{1.1}$$

called transport along γ from s to t, has the properties:

$$L^\gamma_{s\to t} \circ L^\gamma_{r\to s} = L^\gamma_{r\to t}, \qquad r,s,t \in J, \tag{1.2}$$

$$L^\gamma_{s\to s} = \mathrm{id}_{\pi^{-1}(\gamma(s))}, \qquad s \in J, \tag{1.3}$$

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t} u + \mu L^\gamma_{s\to t} v, \quad \lambda,\mu \in \mathbb{C}, \tag{1.4}$$

$$u, v \in \pi^{-1}(\gamma(s)),$$

where $\circ$ denotes composition of maps and id_X is the identity map of a set X.

Let $\{e_i(s;\gamma)\}$ be a C^1 basis in $\pi^{-1}(\gamma(s))$, $s \in J$; here and henceforth the Latin indices run from 1 to $\dim \pi^{-1}(x)$, $x \in B$. We also assume the Einstein summation rule on indices repeated on different levels. So, along $\gamma\colon J \to B$ we have a set $\{e_i\}$ of bases on $\pi^{-1}(\gamma(J))$ such that the liftings $\gamma \mapsto e_i(\cdot, \gamma)$ of paths are of class C^1.

The matrix $\boldsymbol{L}(t,s;\gamma) := \left[L^i{}_j(t,s;\gamma)\right]$ (along γ at (s,t) in $\{e_i\}$) of a linear transport L along γ from s to t is defined via the expansion
$L^\gamma_{s\to t}\big(e_i(s;\gamma)\big) =: L^j{}_i(t,s;\gamma)e_j(t;\gamma)$ $s,t \in J$.

Proposition 1.1. *A non-degenerate matrix-valued function* $\boldsymbol{L}\colon (t,s;\gamma) \mapsto \boldsymbol{L}(t,s;\gamma)$ *is a matrix of some linear transport along paths* L *(in a given field* $\{e_i\}$ *of bases along* γ*) iff*

$$\boldsymbol{L}(t,s;\gamma) = \boldsymbol{F}^{-1}(t;\gamma)\boldsymbol{F}(s;\gamma) \tag{1.5}$$

where $\boldsymbol{F}\colon (t;\gamma) \mapsto \boldsymbol{F}(t;\gamma)$ *is a non-degenerate matrix-valued function.*

Proposition 1.2. *If the matrix $\boldsymbol{L}$ of a linear transport L along paths has a some representation $\boldsymbol{L}(t,s;\gamma) = {}^{\star}\boldsymbol{F}^{-1}(t;\gamma)\,{}^{\star}\boldsymbol{F}(s;\gamma)$ for some matrix-valued function ${}^{\star}\boldsymbol{F}(s;\gamma)$, then all matrix-valued functions $\boldsymbol{F}$ representing $\boldsymbol{L}$ via (1.5) are given by*

$$\boldsymbol{F}(s;\gamma) = \boldsymbol{D}^{-1}(\gamma)\,{}^{\star}\boldsymbol{F}(s;\gamma) \tag{1.6}$$

where $\boldsymbol{D}(\gamma)$ is a non-degenerate matrix depending only on γ.

Let $\{e_i(s;\gamma)\}$ be a smooth field of bases along $\gamma\colon J \to B,\ s \in J$. The explicit local action of the derivation $D\colon \gamma \mapsto D^\gamma\colon s \mapsto D^\gamma_s$, associated to L, on a C^1 lifting of paths λ is

$$D^\gamma_s \lambda = \left[\frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\lambda^j_\gamma(s)\right] e_i(s;\gamma). \qquad (1.7)$$

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Here the (2-index) coefficients $\Gamma^i{}_j$ of the linear transport L are defined by

$$\Gamma^i{}_j(s;\gamma) := \left.\frac{\partial L^i{}_j(s,t;\gamma)}{\partial t}\right|_{t=s} = -\left.\frac{\partial L^i{}_j(s,t;\gamma)}{\partial s}\right|_{t=s} \tag{1.8}$$

and, evidently, uniquely determine the derivation D generated by L.

If a matrix $\boldsymbol{F}$ determines the matrix $\boldsymbol{L}$ of a transport L according to proposition 1.1, then

$$\boldsymbol{\Gamma}(s;\gamma) := \left[\Gamma^{i}{}_{j}(s;\gamma)\right] = \left.\frac{\partial \boldsymbol{L}(s,t;\gamma)}{\partial t}\right|_{t=s} = \boldsymbol{F}^{-1}(s;\gamma)\frac{\mathrm{d}\boldsymbol{F}(s;\gamma)}{\mathrm{d}s}. \tag{1.9}$$

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A change $\{e_i\} \to \{e'_i = A^j_i e_i\}$ of the bases along a path γ with a non-degenerate C^1 matrix-valued function $A(s;\gamma) := \left[A^j_i(s;\gamma)\right]$ implies $\boldsymbol{\Gamma}(s;\gamma) = \left[\Gamma^i{}_j(s;\gamma)\right] \mapsto \boldsymbol{\Gamma}'(s;\gamma) = \left[\Gamma'^i{}_j(s;\gamma)\right]$ with

$$\boldsymbol{\Gamma}'(s;\gamma) = A^{-1}(s;\gamma)\boldsymbol{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}. \tag{1.10}$$

Reconstruction of L from $\boldsymbol{\gamma}$ ...

2. Normal frames for linear transports

Let a linear transport L along paths be given in a vector bundle (E, π, B), $U \subseteq B$ be an arbitrary subset in B, and $\gamma\colon J \to U$ be a path in U.

Definition 2.1. A frame field (of bases) in $\pi^{-1}(\gamma(J))$ is called normal along γ for L if the matrix of L in it is the identity matrix along the given path γ. A frame field (of bases) defined on U is called normal on U for L if it is normal along every path $\gamma\colon J \to U$ in U. The frame is called normal for L if $U = B$.

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Definition 2.2. A linear transport along paths (or along a path γ) is called Euclidean along some (or the given) path γ if it admits a frame normal along γ. A linear transport along paths is called Euclidean on U if it admits frame(s) normal on U. It is called Euclidean if $U = B$.

Proposition 2.1. *The following statements are equivalent in a given frame* $\{e_i\}$ *over* $U \subseteq B$*:*

(i) *The matrix of* L *is the identity matrix on* U*, i.e.* $\boldsymbol{L}(t, s; \gamma) = \mathbf{1}$ *along every path* γ *in* U*.*

(ii) *The matrix of* L *along every* $\gamma \colon J \to U$ *depends only on* γ*, i.e. it is independent of the points at which it is calculated:* $\boldsymbol{L}(t, s; \gamma) = C(\gamma)$ *where* C *is a matrix-valued function of* γ*.*

(iii) *If* E *is a* C^1 *manifold, the coefficients* $\Gamma^i{}_j(s; \gamma)$ *of* L *vanish on* U*, i.e.* $\boldsymbol{\Gamma}(s; \gamma) = 0$ *along every path* γ *in* U*.*

(iv) *The explicit local action of the derivation* D *along paths generated by* L *reduces on* U *to differentiation of the components of the liftings with respect to the* *path's parameter if the path lies entirely in* U*:* $D^{\gamma}_s\lambda = \frac{\mathrm{d}\lambda^i_{\gamma}(s)}{\mathrm{d}s}\,e_i(s;\gamma)$ *where* $\lambda = \lambda^i e_i$ *is a* C^1 *lifting of paths and* $\lambda\colon \gamma \mapsto \lambda_{\gamma}$*.*

(v) *The transport* L *leaves the vectors' components unchanged along any path in* U*, viz. we have* $L^{\gamma}_{s\to t}\big(u^i e_i(s;\gamma)\big) = u^i e_i(t;\gamma)$ *for all* $u^i \in \mathbb{C}$*.*

(vi) *The basic vector fields are* L*-transported along any path* $\gamma\colon J \to U$*:* $L^{\gamma}_{s\to t}\big(e_i(s;\gamma)\big) = e_i(t;\gamma)$*.*

Corollary 2.1. *Every linear transport along paths is Euclidean along every fixed path without self-intersections.*

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Theorem 2.1. *A linear transport along paths admits frames normal on some set (resp. along a given path) if and only if its action along every path in this set (resp. along the given path) depends only on the initial and final point of the transportation but not on the particular path connecting these points. In other words, a transport is Euclidean on* $U \subseteq B$ *iff it is path-independent on* U*.*

Proposition 2.2. *Let L be a transport along paths in a bundle (E, π, M), E and M being C^1 manifolds, and L be Euclidean on $U \subseteq M$ (resp. along a C^1 path $\gamma\colon J \to M$). Then the matrix $\mathbf{\Gamma}$ is of the form*

$$\mathbf{\Gamma}(s;\gamma) = \sum_{\mu=1}^{\dim M} \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \equiv \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \qquad (2.2)$$

in any frame $\{e_i\}$ along every (resp. the given) C^1 path $\gamma\colon J \to U$, where $\Gamma_\mu = \left[\Gamma^i{}_{j\mu}\right]_{i,j=1}^{\dim \pi^{-1}(x)}$ are some matrix-valued functions, defined on an open set V containing U (resp. $\gamma(J)$) or equal to it, and $\dot\gamma^\mu$ are the components of $\dot\gamma$ in some frame $\{E_\mu\}$ along γ in the bundle space tangent to M, $\dot\gamma = \dot\gamma^\mu E_\mu$. The functions $\Gamma^i{}_{j\mu}$ are termed 3-index coefficients of L.

Let U be an open set, e.g. $U = M$. If we change the frame $\{E_\mu\}$ in the bundle space tangent to M, $\{E_\mu\} \mapsto \{E'_\mu = B^\nu_\mu E_\nu\}$ with $B = \left[B^\nu_\mu\right]$ being non-degenerate matrix-valued function, and simultaneously the bases in the fibres $\pi^{-1}(x)$, $x \in M$, $\{e_i|_x\} \mapsto \{e'_i|_x = A^j_i(x) e_j|_x\}$, then, from (1.10) and (2.2), we see that Γ_μ transforms into Γ'_μ such that

$$\Gamma'_\mu = B^\nu_\mu A^{-1} \Gamma_\nu A + A^{-1} E'_\mu(A) = B^\nu_\mu A^{-1} \big(\Gamma_\nu A + E_\nu(A)\big) \quad (2.3)$$

where $A := \left[A^j_i\right]_{i,j=1}^{\dim \pi^{-1}(x)}$ is non-degenerate and of class C^1.

Theorem 2.2. *A C^2 linear transport L along paths is Euclidean on a neighborhood $U \subseteq M$ if and only if in every frame the matrix $\mathbf{\Gamma}$ of its coefficients has a representation (2.2) along every C^1 path γ in U in which the matrix-valued functions Γ_μ, defined on an open set containing U or equal to it, satisfy the equalities*

$$\big(R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M})\big)(x) = 0 \tag{2.4}$$

where $x \in U$ and

$$R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M}) := -\frac{\partial \Gamma_\mu}{\partial x^\nu} + \frac{\partial \Gamma_\nu}{\partial x^\mu} + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu. \tag{2.5}$$

in a coordinate frame $\{E_\mu = \frac{\partial}{\partial x^\mu}\}$ in a neighborhood of x.

Theorem 2.3. *A linear transport L along paths is Euclidean on a submanifold N of M if and only if in every frame $\{e_i\}$, in the bundle space over N, the matrix of its coefficients has a representation* (2.2) *along every C^1 path in N and, for every $p_0 \in N$ and a chart (V, x) of M such that $V \ni p_0$ and $x(p) = (x^1(p), \dots, x^{\dim N}(p), t_0^{\dim N+1}, \dots, t_0^{\dim M})$ for every $p \in N \cap V$ and constant numbers $t_0^{\dim N+1}, \dots, t_0^{\dim M}$, the equalities*

$$\big(R^N_{\alpha\beta}(-\Gamma_1, \dots, -\Gamma_{\dim N})\big)(p) = 0, \qquad \alpha, \beta = 1, \dots, \dim N \tag{2.6}$$

hold for all $p \in N \cap V$ *and*

$$R^N_{\alpha\beta}(-\Gamma_1, \ldots, -\Gamma_{\dim N}) := R_{\alpha\beta}(-\Gamma_1, \ldots, -\Gamma_{\dim M}) \\ = -\frac{\partial \Gamma_\alpha}{\partial x^\beta} - \frac{\partial \Gamma_\beta}{\partial x^\alpha} + \Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha. \quad (2.7)$$

Here $\Gamma_1, \ldots, \Gamma_{\dim N}$ *are the first* $\dim N$ *of the matrices of the 3-index coefficients of* L *in the coordinate frame* $\left\{\frac{\partial}{\partial x^\mu}\right\}$ *in the tangent bundle space over* $N \cap V$.

3. Transports and frames in line bundles

Let (E, π, M) be one-dimensional vector bundle over a C^1 manifold M; such bundles are called line bundles. Thus the (typical) fibre of (E, π, M) can be identified with $\mathbb{C}$ (resp. $\mathbb{R}$ in the real case) and then the fibre $\pi^{-1}(x)$ over $x \in M$ will be an isomorphic image of $\mathbb{C}$ (resp. $\mathbb{R}$ in the real case). Let $\gamma\colon J \to M$ be of class C^1 and L be a linear transport along paths in (E, π, M). A frame $\{e\}$ along γ consists of a single non-zero vector field $e\colon (s;\gamma) \to e(s;\gamma) \in \pi^{-1}(\gamma(s)) \backslash \{0\}$, $s \in J$, and in it the matrix of L^γ at $(t, s) \in J \times J$ is simply a number $\boldsymbol{L}(t, s; \gamma) \in \mathbb{C}$, $L^\gamma_{s \to t}(ue(s;\gamma)) = u\boldsymbol{L}(t, s;\gamma)e(t;\gamma)$ for $u \in \mathbb{C}$ and $s, t \in J$.

Te general form of $\boldsymbol{L}$ is (proposition 1.1)

$$\boldsymbol{L}(t,s;\gamma) = \frac{f(s;\gamma)}{f(t;\gamma)} \tag{3.1}$$

where $f\colon (s;\gamma) \mapsto f(s;\gamma) \in \mathbb{C}\backslash\{0\}$ is defined up to (left) multiplication with a function of γ (proposition 1.2). Respectively, due to (1.9), the matrix of the coefficient(s) of L is

$$\boldsymbol{\Gamma}(s;\gamma) = \frac{\partial \boldsymbol{L}(t,s;\gamma)}{\partial s}\Big|_{t=s} = \frac{1}{f(s;\gamma)}\frac{\mathrm{d}f(s;\gamma)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left[\ln(f(s;\gamma)\right] \tag{3.2}$$

and the transport's matrix takes the form

$$\boldsymbol{L}(t,s;\gamma) = \exp\left(-\int_s^t \boldsymbol{\Gamma}(\sigma;\gamma)\,\mathrm{d}\sigma\right). \tag{3.3}$$

A change $e(s;\gamma) \mapsto e'(s;\gamma) = a(s;\gamma)e(s;\gamma)$, with $a(s;\gamma) \in \mathbb{C}\backslash\{0\}$, of the frame $\{e\}$ implies

$$\boldsymbol{L}(t,s;\gamma) \mapsto \boldsymbol{L}'(t,s;\gamma) = \frac{a(s;\gamma)}{a(t;\gamma)}\boldsymbol{L}(t,s;\gamma) \qquad \text{(3.4a)}$$

$$\boldsymbol{\Gamma}(s;\gamma) \mapsto \boldsymbol{\Gamma}'(s;\gamma) = \boldsymbol{\Gamma}(s;\gamma) + \frac{\mathrm{d}}{\mathrm{d}s}\big[\ln(a(s;\gamma))\big]. \qquad \text{(3.4b)}$$

The explicit local action of the derivation D along paths generated by L is

$$D_s^\gamma\lambda = \Big(\frac{\mathrm{d}\lambda_\gamma(s)}{\mathrm{d}s} + \boldsymbol{\Gamma}(s;\gamma)\lambda_\gamma(s)\Big)e(s;\gamma) \qquad \text{(3.5)}$$

where $\lambda \in \mathrm{PLift}^1(E,\pi,M)$ and (1.7) was used.

Normal frames on one-dimensional vector bundles

A frame $\{e\}$ is normal for L along γ (resp. on U) iff in that frame equation (3.1) holds with

$$f(s;\gamma) = f_0(\gamma) \tag{3.6}$$

where $\gamma\colon J \to M$ (resp. $\gamma\colon J \to U$) and $f_0\colon \gamma \mapsto f_0(\gamma) \in \mathbb{C}\backslash\{0\}$ Since, in a frame normal along γ (resp. on U), it is fulfilled

$$\boldsymbol{L}(t,s;\gamma) = \mathbb{1}, \qquad \boldsymbol{\Gamma}(s;\gamma) = 0 \tag{3.7}$$

for the given path γ (resp. every path in U), in every frame $\{e' = ae\}$, we have

$$\boldsymbol{L}'(t,s;\gamma) = \frac{a(s;\gamma)}{a(t;\gamma)}, \qquad \boldsymbol{\Gamma}'(s;\gamma) = \frac{\mathrm{d}}{\mathrm{d}s}\big[\ln(a(s;\gamma)\big]. \tag{3.8}$$

In addition, for Euclidean on $U \subseteq M$ transport L, the representation

$$\mathbf{\Gamma}'(s;\gamma) = \Gamma'_\mu(\gamma(s))\dot{\gamma}'^{\,\mu}(s) \tag{3.9}$$

holds for every C^1 path $\gamma\colon J \to U$ and some $\Gamma'_\mu\colon V \to \mathbb{C}$ with V being an open set such that $V \supseteq U$ (proposition 2.2). This means (see theorem 2.1) that (3.8) holds for

$$a(s;\gamma) = a_0(\gamma(s)), \tag{3.10}$$

where $a_0\colon U \to \mathbb{C}\backslash\{0\}$, and, consequently, the equality (3.9) can be satisfied if we choose

$$\Gamma'_\mu = E_\mu(a) \tag{3.11}$$

with $a\colon V \to \mathbb{C}$, $a|_U = a_0$ and $\{E_\mu\}$ being a frame in the bundle space tangent to M which, in particular, can be a coordinate one, $E_\mu = \frac{\partial}{\partial x^\mu}$. Of course, if U is not an open set, this choice of Γ'_μ is not necessary; for example, the equality (3.9) will be preserved, if to the r.h.s. of (3.11) is added a function G'_μ such that $G'_\mu \dot{\gamma}'^{\,\mu} = 0$.

Final remark:

Final remark:

- Frames normal along injective paths always exist (corollary 2.1).
- On an arbitrary submanifold $N \subseteq M$ normal frames exist iff the functions Γ_μ satisfy the conditions (2.6) with $x \in N$ in the coordinates described in theorem 2.3.

4. The classical electromagnetic field

Now we would like to apply the above formalism to a description of the classical electromagnetic field. Before going on, we should say that the accepted natural formalism in gauge field theories, in particular in the electrodynamics, is via connections on vector bundles [1, 2]. Below we sketch an equivalent technique for an electromagnetic field.

Recall, the classical electromagnetic field is described via a real 1-form A over a 4-dimensional real manifold M (endowed with a (pseudo-)Riemannian metric g and) representing the space-time model and, usually, identified with the Minkowski space M^4 of special relativity or the (pseudo-)Riemannian space V_4 of general relativity. The electromagnetic field itself is represented by the two-form $F = \mathrm{d}A$, where "d" denotes the exterior derivative operator, with local components (in some local coordinates $\{x^\mu\}$)

$$F_{\mu\nu} = -\frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial A_\nu}{\partial x^\mu}. \tag{4.1}$$

As it is well known, the electromagnetic field, the Maxwell equations describing it, and its (minimal) interactions with other objects are invariant under a gauge transformation

$$A_\mu \mapsto A'_\mu = A_\mu + \frac{\partial \lambda}{\partial x^\mu} \tag{4.2}$$

or $A \mapsto A' = A + \mathrm{d}\lambda$, where λ is a C^2 function. As is almost evident, the electromagnetic field is invariant under simultaneous changes of the local coordinate frame, $E_\mu = \frac{\partial}{\partial x^\mu} \mapsto E'_\mu = B^\nu_\mu E_\nu$ with $B^\nu_\mu := \frac{\partial x^\nu}{\partial x'^{\,\mu}}$, and a gauge transformation (4.2):

$$A_\mu \mapsto A'_\mu = B^\nu_\mu A_\nu + E'_\mu(\lambda) = B^\nu_\mu \Big(A_\nu + \frac{\partial \lambda}{\partial x^\nu} \Big). \tag{4.3}$$

Under the transformation (4.3), the quantities (4.1) transform like components of an antisymmetric tensor,

$$F_{\mu\nu} \mapsto F'_{\mu\nu} = B^{\sigma}_{\mu} B^{\tau}_{\nu} F_{\sigma\tau} \tag{4.4}$$

due to which the 2-form F remains unchanged, $F = \mathrm{d}A = \mathrm{d}A'$. Notice, above A'_{μ} are not the components of A in $\{E'_{\mu}\}$ unless $\lambda = \mathrm{const}$ while $F'_{\mu\nu}$ are the components of F in $\left\{E'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \mathrm{d}x^{\nu}\right\}$.

Now there arises an idea for identifying the electromagnetic potentials A_{μ} with the matrices Γ_{μ} of the 3-index coefficients of some linear transport along paths in a 1-dimensional vector bundle (E, π, M). This can be done as follows.

Let M be a real 4-dimensional manifold, representing the space-time model, and (E, π, M) be a 1-dimensional real vector bundle over it. We identify the potentials A_μ of an electromagnetic field with the (local) coefficients of a linear transport L along paths in (E, π, M) whose matrix has the representation $\mathbf{\Gamma}'(s;\gamma) = \Gamma'_\mu(\gamma(s))\dot{\gamma}'^{\,\mu}(s)$ (along every path and in every pair of frames). Hence, the 3-index coefficients of L are uniquely defined and supposed to be (arbitrarily) fixed in some pair of frames.

It should be emphasized, now the (pure) gauge transformation (4.2) appears as a special case of (4.3), corresponding to a change of the frame in E and a fixed frame in $T(M)$. This means that, in the approach proposed, the gauge transformations are directly incorporated in the definition of the field potential A. This conclusion is in contrast to the situation in classical electrodynamics as in it the gauge transformations are a simple observation of 'additional' invariance of the field, which is not connected with the geometrical interpretation of the theory.

Defining the electromagnetic field by $F = \mathrm{d}A$, the equality (4.1) remains valid in a coordinate frame $\{E_\mu = \partial/\partial x^\mu\}$. Since A and F possess all of the properties they must have in electrodynamics, they represent an equivalent description of electromagnetic field. The only difference with respect to the classical description is the clear geometrical meaning of these quantities, as a consequence of which an electromagnetic field can be identified with a linear transport along paths in a one-dimensional vector bundle over the space-time. The proposed treatment of electromagnetic field is equivalent to the modern one in the bundle picture of gauge theories [1, 3], where the electromagnetic potentials are regarded as coefficients of a suitable linear connection.

In the approach proposed, the different gauge conditions, which are frequently used, find a natural interpretation as a partial fix of the class of frames in the bundle space employed.
For instance, any one of the gauges in the table 4.1 on the next slide corresponds to a class of frames for which (4.3) holds for $B_\mu^\nu = \delta_\mu^\nu$, δ_μ^ν being the Kronecker deltas, and λ subjected to a condition given in the table. (Below M is supposed to be endowed with a (pseudo-)Riemannian metric $g_{\mu\nu}$, the coordinates to be numbered as x^0, x^1, x^2, and x^3, x^0 to be the 'time' coordinate, $\partial_\mu := \partial/\partial x^\mu$, and $\partial^\mu := g^{\mu\nu}\partial_\nu$ with $[g^{\mu\nu}] := [g_{\mu\nu}]^{-1}$.)

Table 4.1: Examples of gauge conditions

Gauge	Condition on A	Condition on λ
Lorenz	$\partial^\mu A_\mu = 0$	$\partial^\mu \partial_\mu \lambda = 0$
Coulomb[a]	$\partial^k A_k = 0$	$\partial^k \partial_k \lambda = 0$
Hamilton	$A_0 = 0$	$\lambda(x) = \lambda(x^1, x^2, x^3)$
Axial	$A_3 = 0$	$\lambda(x) = \lambda(x^0, x^1, x^2)$

[a] In this row the summation over k is from 1 to 3.

Historical Remark:

The Lorenz condition and gauge are named in honor of the Danish theoretical physicist Ludwig Valentin Lorenz (1829–1891), who has first published it in 1867 [4] (see also [5, pp. 268-269, 291]); however this condition was first introduced in lectures by Bernhard G. W. Riemann in 1861 as pointed in [5, p. 291]. It should be noted that the Lorenz condition/gauge is quite often erroneously referred to as the Lorentz condition/gauge after the name of the Dutch theoretical physicist Hendrik Antoon Lorentz (1853–1928) as, e.g., in [6, p. 18] and in [7, p. 45].

5. Normal and inertial frames

Comparing (4.1) with (2.5), we get (in this section, we assume the Greek indices to run over the range 0, 1, 2, 3.)

$$F_{\mu\nu} = R_{\mu\nu}(-A_0, -A_1, -A_2, -A_3). \tag{5.1}$$

Thus, the electromagnetic field tensor F is completely responsible for the existence of frames normal for L (theorems 2.2 and 2.3). For example, if U is an open set, frames normal on $U \subseteq M$ for L exist iff $F|_U = 0$, i.e. if electromagnetic field is missing on U. Also, if N is a submanifold of M, frames normal on U for L exist iff in the special coordinates $\{x^\mu\}$, described in theorem 2.3, is valid $F_{\alpha\beta}|_U = 0$ for $\alpha, \beta = 1, \dots, \dim N$.

In the context of theorem 2.1, we can say that an electromagnetic field admits frames normal on $U \subseteq M$ iff the linear transport L corresponding to it is path-independent on U (along paths lying entirely in U). Thus, if L is path-dependent on U, the field does not admit frames normal on U. This important result is the classical analogue of a quantum effect, know as the Aharonov-Bohm effect [8, 9], whose essence is that the electromagnetic potentials directly, not only through the field tensor F, can give rise to observable physical results.

Physical meaning of the normal frames

Suppose L is Euclidean on a neighborhood $U \subseteq M$. As a consequence of (5.1) and theorem 2.2, we have $F|_U = \mathrm{d}A|_U = 0$, i.e. on U the electromagnetic field strength vanishes and hence the field is a pure gauge on U,

$$A_\mu|_U = \frac{\partial f_0}{\partial x^\mu}\Big|_U \tag{5.2}$$

for some C^1 function f_0 defined on an open set containing U or equal to it. In a frame $\{e'\}$ normal on U for L vanish the 2-index coefficients of L along any path γ in U:

$$\mathbf{\Gamma}'(s;\gamma) = A'_\mu(\gamma(s))\dot{\gamma}^\mu(s) = 0. \tag{5.3}$$

Any transformation (4.3) with $\lambda = -f_0$ transforms A_μ into A'_μ such that

$$A'_\mu|_U = 0 \tag{5.4}$$

(irrespectively of the frames $\{E_\mu\}$ and $\{E'_\mu\}$.) Hence, the one-vector frame $\{e' = \mathrm{e}^{-f_0} e\}$ in the bundle space E is normal for L on U. Therefore, in the frame $\{e'\}$, vanish the 2-index and 3-index coefficients of L, i.e. $\{e'\}$ is a strong normal frame. One can verify, all frames strong normal on a neighborhood U for L are obtainable from $\{e'\}$ by multiplying its vector e' by a function f such that $\frac{\partial f}{\partial x^\mu}\Big|_U = 0$, i.e. they are $\{b\mathrm{e}^{-f_0} e\}$ with $b \in \mathbb{R}\backslash\{0\}$ as U is a neighborhood. Thus, every frame normal on a neighborhood U for L is strong normal on U for L and vice versa.

A frame (of reference) in the bundle space in which (5.4) holds on a subset $U \subseteq M$, will be called inertial on U for the electromagnetic field considered. In other words, the frames inertial on U for a given electromagnetic field are the ones in which its potentials vanish on U. Thus, every frame inertial on U is strong normal on it and vice versa.

So, in a frame inertial on $U \subseteq M$ for an electromagnetic field it is not only a pure gauge, but in such a frame its potentials vanish on U.

Relying on known results [10, 11, 12], we can assert the existence of frames inertial at a single point and/or along paths without self-intersections for every electromagnetic field, while on submanifolds of dimension not less than two such frames exist only as an exception if (and only if) some additional conditions are satisfied, i.e. for some particular types of electromagnetic fields.

Links with the (strong) equivalence principle

In [13] was demonstrated that the (ordinary strong) equivalence principle is a provable theorem and the inertial frames in a gravity theory based on a linear connection (or other derivation) are the frames normal for it. Such frames are called inertial for the gravitational field under consideration.

Let there be given a physical system consisting of pure or, possibly, interacting gravitational and electromagnetic fields which are described via, respectively, a linear connection ∇ in the tangent bundle $(T(M), \pi_T, M)$ (or the tensor algebra) over the space-time M and a linear transport along paths in a 1-dimensional vector bundle (E, π_E, M) over M.

- The frames inertial for an electromagnetic field, if any, in the bundle space E are completely independent of any frame in the bundle space $T(M)$ tangent to M.
- The frames inertial for the gravity field, i.e. the ones normal for ∇, if any, are frames in $T(M)$ and have nothing in common with the frames in E, in particular with the frames normal for L, if any.

- The frames inertial for an electromagnetic field, if any, in the bundle space E are completely independent of any frame in the bundle space $T(M)$ tangent to M.

- The frames inertial for the gravity field, i.e. the ones normal for ∇, if any, are frames in $T(M)$ and have nothing in common with the frames in E, in particular with the frames normal for L, if any.

So, if a frame $\{E_\mu\}$ in $T(M)$ is inertial on $U \subseteq M$ for the gravity field and a frame $\{e\}$ in E inertial on U for the electromagnetic field, the frame $\{e \times E_\mu\} = \{(e, E_\mu)\}$ in the bundle space of $(E \times T(M), \pi_E \times \pi_T, M \times M)$ over $M \times M$ can be called simply inertial on U (for the system of gravity and electromagnetic fields).

Thus, in an inertial frame, if any, the potentials of both, gravity and electromagnetic, fields vanish. We can assert the existence of inertial frames at every single space-time point and/or along every path without self-intersections in it. On submanifolds of dimension higher than one, inertial frames exist only for some exceptional configurations of the fields which can be described on the base of the results in the cited works.

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