Spectral Data for Real Regular KP-2 solutions on Rational Degenerations of M-curves and tropical M-curves

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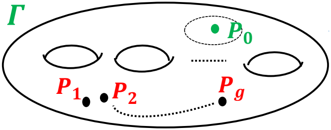
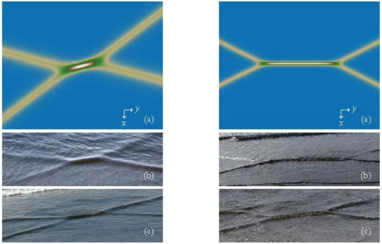
KP-2 equation and algebraic geometry

Wikipedia





M.A. Ablowitz and D.E. Baldwin Phys. Rev E , v. 86 (2012)

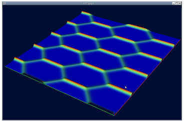
$$\text{KP}-2 \text{ equation} \quad [\text{KP}-1970]: \ (-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

is the first member of the most relevant $2+1$ integrable hierarchy [ZS-1974].

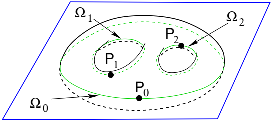
Problem: investigate relations of spectral problems for real-regular finite-gap solutions and real regular multi-line soliton solutions to solve problems in real algebraic geometry and tropical geometry

Tropicalization of (real) regular KP-2 finite–gap solutions

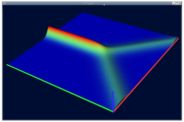
Real regular finite–gap KP-2 solutions $\leftrightarrow$ non-special divisors on regular M curves



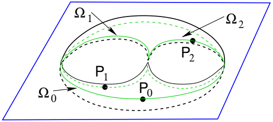
$\leftrightarrow$ [DN-1988]



tropicalization (solitonic limit) $\downarrow$



$\leftarrow$



KP-2 solitons $\leftarrow$ spectral data on reducible rational curves

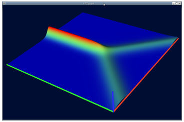
[Nak-2018],[Nak-2019]: Tropicalization in the Sato Grassmannian in special cases

[ACSS-21], [AFMS-23], [FM-24]: Computational approach to tropicalization of algebraic curves using KP theory

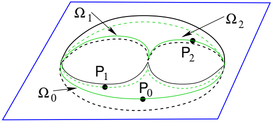
[Ich-23] 1-dimensional families of M–curves degenerate to rational curves and corresponding regular finite-gap solutions degenerate to soliton solutions

Spectral problems for real regular KP-2 finite–gap solutions [AG-2018...]

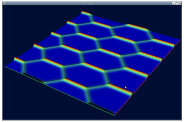
Real regular multiline KP-2 solitons $\rightarrow$ spectral data on reducible spectral curve fulfilling DN theorem [AG-2018...]



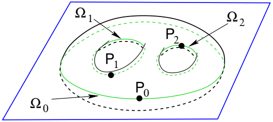
$\rightarrow$ [AG-2018...]



[AG-2018...] $\downarrow$



$\leftrightarrow$ [DN-1988]



Real regular finite–gap KP-2 solutions $\leftrightarrow$ spectral data on M–curves fulfilling DN theorem

Real regular soliton data are points in $Gr^{\mathrm{TNN}}(k,n)$ encoded by planar bicolored (plabic) networks in the disk.

[AG-2018a,AG-2018b,AG-2019,AG-2022c]: Use combinatorial structure of $Gr^{\mathrm{TNN}}(k,n)$ to solve a degenerate spectral problem for real regular multiline KP solitons on tropical M–curves: we prove that plabic graphs are dual to the topological model of such tropical curve and consistently assign spectral data (divisor) solving a system of relations on the graph.

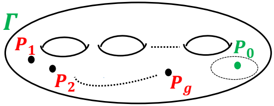
Intermezzo: two relevant families of KP solutions

• KP-2 regular finite gap solutions (Krichever 1976, Dubrovin-Natanzon 1988)

• KP-2 real regular multiline soliton solutions (Matveev 1979, Freeman and Nimmo 1983, Malanyuk 1991, Chakravarthy-Kodama 2009, Kodama-Williams 2013,2014,...)

Algebraic geometric data:

(Γ, P_0, ζ) $\zeta^{-1}(P_0)=0$



Families of regular quasi–periodic solutions $\boxed{u(\vec{x})}$ on (Γ, P_0), Γ non–singular genus g algebraic curve with marked point P_0, are parametrized by non special divisors $\boxed{\mathcal{D} = (P_1, \ldots, P_g)}$. Here $\vec{x} = (x, y, t)$.

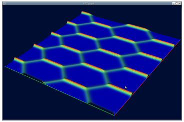
There exists a unique normalized KP wave–function $\boxed{\Psi(P, \vec{x})}$, meromorphic on $\Gamma \backslash \{P_0\}$, with poles in $\mathcal{D}$ and asymptotics at P_0 $(\zeta^{-1}(P_0) = 0)$:

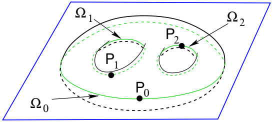
$$\Psi(\zeta, \vec{x}) = \left(1 - \frac{w_1(\vec{x})}{\zeta} + O(\zeta^{-2})\right) e^{\zeta x + \zeta^2 y + \zeta^3 t + \cdots} \quad (\zeta \to \infty).$$

$$u(\vec{x}) = 2\partial_x^2 \log \Theta(xU^{(1)} + yU^{(2)} + tU^{(3)}) + c_1$$

[DN-1988]: Smooth, real (quasi–)periodic KP-2 solutions $u(x, y, t)$ correspond to real and regular divisors on smooth M–curves:

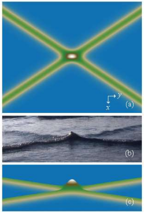
- Γ possesses an antiholomorphic involution which fixes the maximum number $g + 1$ of ovals, $\Omega_0, \ldots, \Omega_g$;
- $P_0 \in \Omega_0$ (infinite oval) and the divisor points $P_j \in \Omega_j$, $j = 1, ..., g$ (finite ovals).

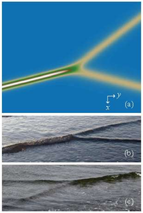




Question: how to effectively construct such solutions? How to identify M–curves and spectral data fulfilling DN theorem?

Idea: associate degenerate spectral problems on reducible M–curves to real–regular KP-2 multi-line solitons solutions, check DN theorem for the degenerate problem and open gaps





Left: $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{7} \end{bmatrix}$ $k_1 := \frac{-7}{12}, k_2 := \frac{-1}{12}, k_3 := \frac{1}{12}, k_4 := \frac{7}{12}$

Right: $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}, k_1 := 0, k_2 := \frac{1}{2}, k_3 := 1$

[Mat-1979], [FN-1983], [Mal-1991] $A \in Mat_{\mathbb{R}}(k,n)$, $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$:

$f^{(i)}(x,y,t) = \sum_{j=1}^{n} A^i_j \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t), \quad i \in [k]$

$$\tau(x,y,t) = \mathrm{Wr}_x(f^{(1)},\dots,f^{(k)}) = \sum_{1\le j_1<\cdots<j_k\le n} \Delta_{[j_1,\dots,j_k]}(A) E_{j_1,\dots,j_k}(x,y,t)$$

KP-2 soliton solution: $u(x,y,t) = 2\partial_x^2 \log(\tau(x,y,t))$

- same $u(x,y,t)$ if recombine rows of $A \implies [A] \in Gr(k,n) = GL_{\mathbb{R}}(k)\backslash Mat_{\mathbb{R}}(k,n)$
- u is bounded for real (x,y,t) $\iff$ $[A] \in Gr^{\mathrm{TNN}}(k,n) = GL^+_{\mathbb{R}}(k)\backslash Mat^{\mathrm{TNN}}_{\mathbb{R}}(k,n)$ [KW-2013])

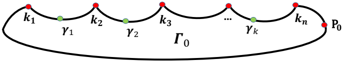
Direct spectral problem for KP-2 solitons [Mal-1991]

Soliton data: $(\mathcal{K},[A])$ $\mapsto$ Sato algebraic geometric data: $(\Gamma_0, P_0, \zeta; \mathcal{D}_S^{(k)})$

Γ_0 copy of $\mathbb{CP}^1$, ζ such that $\zeta^{-1}(P_0)=0$ and $\zeta(\kappa_i)=\kappa_i$.

$\mathfrak{D} = \partial_x^k - \mathfrak{w}_1(\mathbf{t})\partial_x^{k-1} - \ldots - \mathfrak{w}_k(\mathbf{t}) = W\partial_x^k$, $\qquad \mathfrak{D} f_i(\vec{x}) \equiv 0$

W = Dressing operator in Sato Grassmannian for $(\mathcal{K},[A])$!



Sato divisor $\mathcal{D}_{\mathrm{S},\Gamma_0} = \{\gamma_j : \gamma_j^k - \mathfrak{w}_1(\vec{x}_0)\gamma_j^{k-1} - \cdots - \mathfrak{w}_{k-1}(\vec{x}_0)\gamma_j - \mathfrak{w}_k(\vec{x}_0) = 0\}$

$\gamma_j \in [\kappa_1, \kappa_n]$, $j \in [k]$ and for a.a. $\vec{x}_0$ γ_j are distinct.

Incompleteness of Sato algebraic–geometric data:

$$k = \deg(\mathcal{D}_{\mathrm{S},\Gamma_0}) < \dim(Gr^{\mathrm{TNN}}(k,n)) = k(n-k)$$

Conclusion: it is not possible to reconstruct the soliton solution from the degree k divisor $\mathcal{D}_{\mathrm{S},\Gamma_0}$!!!!

The degenerate spectral problem associated to KP-2 solitons [AG2018,...]

Recap:

- KP-2 multi-line soliton solutions are represented by points $[A] \in Gr^{\mathrm{TNN}}(k,n)$

Direct spectral problem [Mal-1991]: $(\Gamma_0 = CP^1, P_0)$ and divisor $P_1, \ldots, P_k \in [\kappa_1, \kappa_n]$

[AG-2018, AG-2019, AG-2022c]: Complete Sato divisor using Krichever approach to degenerate finite–gap solutions !!!!

Step 1) Construct a rational reducible M–curve Γ such that Γ_0 is a component.

Question: how to choose the curve? We had the idea to use the planar graphs classifying the soliton data in $Gr^{\mathrm{TNN}}(k,n)$ to get the topological model of the curve

Step 2) Extend KP-2 wave function from Γ_0 to Γ so that DN thm holds true (need to control the value of the wave-function at nodes and intersection of rational components)

We had the idea to use systems of relations on the graph since the edges represent nodes and intersection points of the rational components

$\diamond$ Total non–negativity $\Longrightarrow$ reality and regularity DN conditions: one divisor point in each finite oval

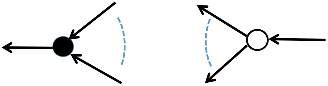
◇ [Lusz-1990s] generalizes the classical notion of total positivity in GL_n to reductive Lie groups and generalized partial flag varietes; cell decomposition of $(G/P)_{\geq 0}$ (Rietsch, Ph.D. thesis).

◇ [Pos-2006] characterizes the cell decomposition of $Gr^{\text{TNN}}(k,n)$ combinatorially and using graph theory:

A positroid cell $\mathcal{S}_{\mathcal{M}}^{\text{TNN}}$ in $Gr^{\text{TNN}}(k,n)$ is the equivalence class of the totally non–negative $k \times n$ matrices sharing the same matroid (=the same list of positive maximal minors, all other maximal minors are zero). $\mathcal{S}_{\mathcal{M}}^{\text{TNN}}$ is represented by a Young diagram filled with the Le–rule.

$\mathcal{S}_{\mathcal{M}}^{\text{TNN}}$ is represented by an equivalence class of perfectly orientable planar bicolored graphs in the disk (real positive weights on edges of the graph) :

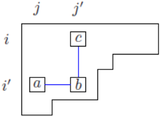
- n univalent vertices on the boundary of the disk and k of them are sources in each perfect orientation;
- At each internal black vertex, exactly one edge oriented outward;
- At each internal white vertex, exactly one edge oriented inward.



Representation of $\mathcal{S}_{\mathcal{M}}^{\text{TNN}}$ via Le–diagrams [Pos-2006]

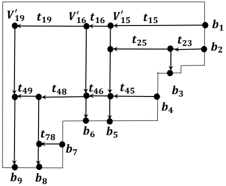
Postnikov constructs a bijection between $\mathcal{S}_{\mathcal{M}}^{\text{TNN}} \subset Gr^{\text{TNN}}(k,n)$ and { Le–diagrams } in $k \times n$ boxes.

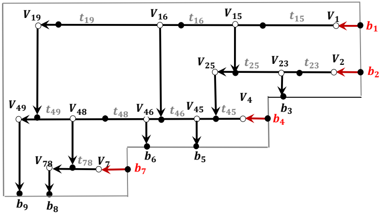
A Le–diagram is a filling of Young diagram with 0's and 1's s.t. for any 3 boxes (i',j), (i,j'), (i',j'), with $i < i'$, $j < j'$, $a, c = 1 \implies b = 1$:



Le diagram (tableau) $\Longleftrightarrow$ perfectly oriented bipartite Le–graph (network) in the disk:

t_{19}	0	t_{16}	t_{15}	0	1
0	0	0	t_{25}	t_{23}	2
t_{49}	t_{48}	t_{46}	t_{45}	4	
0	t_{78}	7			



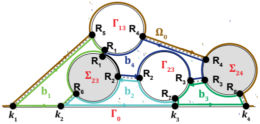


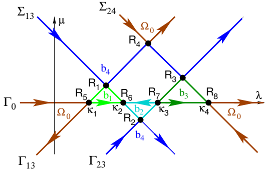
[Pos-2006]: Classification of planar networks in the disk representing the same point $[A] \in \mathcal{S}_{\mathcal{M}}^{\text{TNN}}$ using moves and reductions

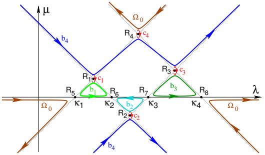
- Take soliton data in $\mathcal{S}_{\mathcal{M}}^{\mathrm{TNN}}$ and choose a graph in the disk $\mathcal{G}$ representing $\mathcal{S}_{\mathcal{M}}^{\mathrm{TNN}}$ in Postnikov classification.
- $\mathcal{G}$ is dual to the reducible rational curve Γ:

$\mathcal{G}$	Γ
Boundary of disk	Sato component Γ_0
Boundary vertex b_l	Marked point κ_l on Γ_0
Internal black vertex V'_s	Copy of $\mathbb{CP}^1$ denoted Σ_s
Internal white vertex V_l	Copy of $\mathbb{CP}^1$ denoted Γ_l
Edge e	Double point
Face f	Oval

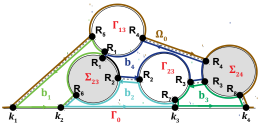
- Perturb Γ to Γ_ϵ opening gaps so that Γ_ϵ is an M–curve of genus $g = F - 1$, where F is the number of faces of the graph ($g \geq \dim \mathcal{S}_{\mathcal{M}}^{\mathrm{TNN}}$, [AG-2019]: have = for the Le–graph)

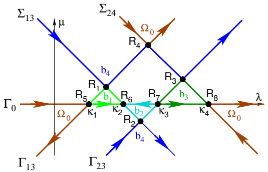


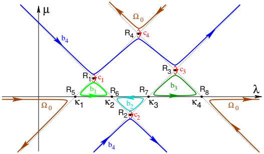




Soliton lattices of KP-2 and desingularization of spectral curves in $Gr^{TP}(2,4)$ [AG-2018b]





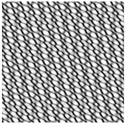
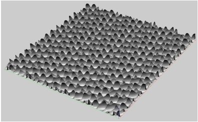
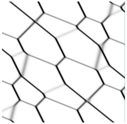
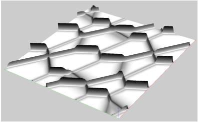


$$0 = P_0(\lambda, \mu) = \mu \cdot (\mu - (\lambda - \kappa_1)) \cdot (\mu + (\lambda - \kappa_2)) \cdot (\mu - (\lambda - \kappa_3)) \cdot (\mu + (\lambda - \kappa_4)).$$

Genus 4 M–curve after desingularization:

$$\Gamma(\varepsilon) \ : \qquad P(\lambda, \mu) = P_0(\lambda, \mu) + \varepsilon(\beta^2 - \mu^2) = 0, \qquad 0 < \varepsilon \ll 1,$$

$\beta = \frac{\kappa_4 - \kappa_1}{4} + \frac{1}{4} \max\{\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \kappa_4 - \kappa_3\}$,
$\kappa_1 = -1.5, \quad \kappa_2 = -0.75, \quad \kappa_3 = 0.5, \quad \kappa_4 = 2.$

Level plots for KP-2 finite gap solutions: $\epsilon = 10^{-2}$ [left], $\epsilon = 10^{-18}$ [right].
Horizontal axis is $-60 \le x \le 60$, vertical axis is $0 \le y \le 120$, $t = 0$.
White (black) = lowest (highest) value of u.

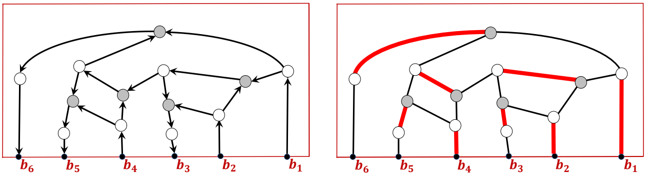
• Dimer models were introduced in [Kas-1961] and [TF-1961] to describe crystal surfaces at equilibrium like partially dissolved salt crystals.

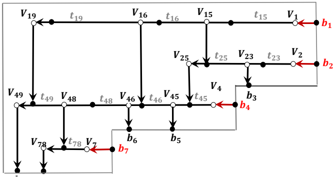
[PSW-2009]: Dimer configuration on $\mathcal{G} = (\mathcal{V} = \mathcal{B} \cup \mathcal{W}, \mathcal{E})$ is a collection M of edges of $\mathcal{G}$ that contains exactly once internal vertices, and at most once the n boundary vertices.

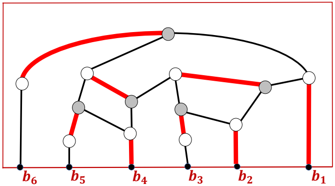
$$k = \partial M = \{i \in [n] : \text{ black boundary vertex } b_i \in M\} \cup \{i \in [n] : \text{ white boundary vertex } b_i \notin M\}.$$

Perfect orientations $\Longleftrightarrow$ dimer configurations

Example: $Gr^{TP}(3,6)$:







[Pos-2006]: $[A] \in Gr^{\text{TNN}}(k,n)$:

$$A^r_j = (-1)^{\sigma_{i_r j}} \sum_{P: b_{i_r} \mapsto b_j} (-1)^{\text{Wind}(P)} wt(P)$$

$\sigma_{i_r j} = \#\{$ sources between i_r and $j\}$;

$$wt(P) = \prod_{e \in P} t_e$$

[Lam-2016]: Weight of dimer state M:

$$wt(M) = \prod_{e \in M} t_e$$

The partition function $Z(G,t;\partial M)$ relative to $\partial M = I$ is the I-th Plücker coordinate of $[A] \in Gr^{\text{TNN}}(k,n)$:

$$Z(G, wt; \partial M) = \sum_{M \,:\, \partial M = I} wt(M) = D_I(A)$$

[AG-2022a, AG-2022b]: $[A]$ can be computed solving a geometric system of relations
[AG-2022c]: these relations provide the value of the KP-2 wavefunction at the double points of Γ (Step 2)
[A-2021]: the system of relations is associated to a Kasteleyn sign matrix in the case of bipartite graphs

Classical Kasteleyn theorem: count dimer configurations in a planar bipartite graph as the determinant of a $|\mathcal{W}| \times |\mathcal{B}|$ square matrix K whose entries K_b^w are ± 1 if there is an edge joining $b \in \mathcal{B}$ and $w \in \mathcal{W}$, and 0 otherwise.

[Speyer 2016]: $\mathcal{G} = (\mathcal{B} \cup \mathcal{W}, \mathcal{E})$ planar bipartite graph in the disk representing $\mathcal{S}_{\mathcal{M}}^{\mathrm{TNN}} \in Gr^{\mathrm{TNN}}(k,n)$ with black boundary vertices: $|\mathcal{W}| = N + k$, $|\mathcal{B}| = N + n$.

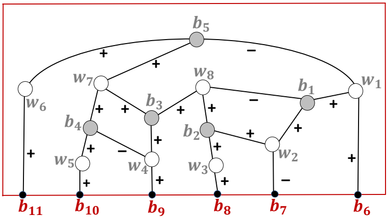
Label black vertices s.t. boundary vertices are labeled clockwise in increasing order $b_{N+1}, \dots, b_{N+n}$. Then there exists $\sigma : \mathcal{E} \mapsto \{\pm 1\}$ such that, given the Kasteleyn matrix K^{wt}:

$$(K^{wt})_b^w = \begin{cases} \sigma_{bw} t_{bw}, & \text{if } (b,w) \text{ is an edge ;} \\ 0, & \text{otherwise,} \end{cases}$$

then $|\det(K^{wt})_I|$ are the Plücker coordinates of $[A]$ in Postnikov parametrization of $Gr^{\mathrm{TNN}}(k,n)$, and

$$K^{wt} \mapsto \begin{array}{c} \\ N \\ k \end{array} \overset{\begin{array}{cc} N & n \end{array}}{\left(\begin{array}{c|c} \mathrm{Id}_N & * \\ \hline 0 & A \end{array} \right)}$$

The proof of Speyer is topological. In our construction we provide an explicit characterization of such signatures



$\mathcal{G} = (\mathcal{V} = \mathcal{B} \cup \mathcal{W}, \mathcal{E})$ with boundary vertices of equal color. A function $\sigma : \mathcal{E} \mapsto \{\pm 1\}$ is a Kasteleyn signature in the sense of Speyer if and only if, for any finite face Ω, the total signature of the face fulfills

$$\prod_{e \in \partial\Omega} \sigma(e) = (-1)^{\frac{|\Omega|}{2}+1},$$

where $|\Omega|$ = number of edges bounding the face Ω,

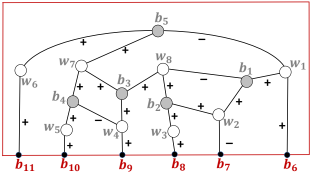
[A-2021] There is a unique equivalence class of Kasteleyn signatures on $\mathcal{G}$ and it coincides with the geometric signature constructed in [AG-2022a, AG-2022b] in the case of bipartite graphs

- K^{wt} Kasteleyn matrix $\qquad V$ a vector space

Kasteleyn system of relations $(v = \{v_b : b \in \mathcal{B}\}, R_w)$:

▷ v_b is an element in V assigned to the black vertex $b \in \mathcal{B}$;

▷ At white vertex $w \in \mathcal{W}$: $\quad 0 = R_w(v) \equiv \sum_{b\in\mathcal{B}} (K^{wt})^w_b \, v_b \equiv \sum_{b\in\mathcal{B}} \sigma_{bw}\, t_{bw}\, v_b.$

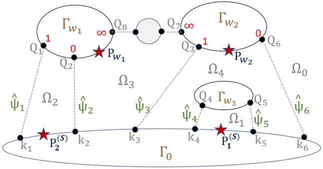


$$K^{wt} = \begin{array}{c} \\ N \\ k \end{array} \begin{array}{c} \begin{array}{cc} N & n \end{array} \\ \left(\begin{array}{c|c} \mathrm{Id}_N & * \\ \hline 0 & A \end{array}\right) \end{array}$$

◇ KP soliton wave function on Γ_0: $\quad 0 \equiv \mathfrak{D} f_i(\vec{x}) \equiv \sum_{j=1}^n A^i_j\, \psi(\kappa_j, x, y, t)$

If assign at boundary vertex b_j: $v_{b_j} = \psi(\kappa_j, x, y, t)$, then the system $R_w(v) = 0$ is solvable and gives $\psi(\kappa, x, y, t)$ at the intersection/nodal points of the reducible curve dual to the graph!

[AG-2022a, AG-2022b]: explicit solution to the linear system at internal vertices (generalization of Talaska's formula)



• Generalized Talaska formula gives the value of the KP wave function at the double points:

$$\psi(Q,\vec{x}) = \sum_{j=1}^{n} (E_e)_j\, \psi(\kappa_j,\vec{x})$$

• Kasteleyn relations at white trivalent vertices rule the position of the KP divisor in the ovals:

$$\gamma_w \equiv \zeta(P_w) = \frac{K^{\sigma,wt}_{w,b_1}\,\psi(Q_1,\vec{x}_0)}{K^{\sigma,wt}_{w,b_1}\,\psi(Q_1,\vec{x}_0) + K^{\sigma,wt}_{w,b_2}\psi(Q_2,\vec{x}_0)}$$

• Kasteleyn face signature implies one divisor point in each finite oval

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