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Institute of Mechanics of Bulgarian Academy of Sciences, Sofia, Bulgaria

Noncommutative Geometry and Physics, old and new

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Plan:

- Physics motivations for noncommutative geometry:

- quantum mechanics
- quantum gravity
- field theories with fluxes and string theory

- Noncommutative differential geometry

- Differential geometry of graphs

Physics and Geometry...

Geometry of space and time

Galilei 1632. Galileian Relativity. Time is absolute. Space is relative (uniform motion)

Einstein 1905 (Lorentz, Poincaré). Special relativity, unification of space and time. Lenghts and time intervals are no more absolute.

Einstein 1915. General relativity Space and time depend upon matter (they are not a priori). Interaction between massive bodies is a consequence of spacetime curvature. The study of spacetime structure is the study of gravity.

Quantum physics and NC geometry

-In quantum physics phase-space becomes NC $[q, p] = i\hbar$, that is, the algebra of functions on phase space *becomes noncommutative*. Observables are then no more real functions on phase-space but self-adjoint operators in a $C*$-algebra.

Heisenberg microscope is a Gedanken experiment leading to the uncertainty principle. It is a measurement problem of an electron (a small particle) using classical optics and leads to $\Delta q \Delta p \geq \hbar/2$.

Spacetime and Quantum Mechanics I

QM (QFT) has radically changed the description and interaction of subatomic particles; spacetime structure has not been touched, it is special relativity spacetime.

Considering also the gravitational force spacetime has to become quantum as well. Indeed:

-like stars curve spacetime (light bends) similarly quantum particles curve spacetime in a “quantum way”, i.e, a quantum source should induce a quantum curvature.

Hence we face the conceptual problem of a quantum description of gravity and of spacetime. Resolution of conceptual problems of this kind historically has always been fruitful (e.g. reconciling mechanics with electromagnetism $\rightarrow$ special relativity)

Classical Mechanics$\longrightarrow$ Quantum Mechanics functions (observables) on phase space become noncommutative (phase space noncommutativity)

General Relativity $\longrightarrow$ Quantum Gravity Spacetime structure itself becomes noncommutative.

This expectation is supported by Gedanken experiments suggesting that spacetime structure is not necessarily that of a smooth manifold (a continuum of points).

Quantum spacetime effects at Planck scale $L_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm$.

Below Planck scales it is then natural to conceive a more general spacetime structure. A cell-like or lattice like structure, or a noncommutative one where (as in quantum mechanics phase-space) uncertainty relations and discretization naturally arise.

Space and time are then described by a *Noncommutative Geometry*

In this way a dynamical aspect of spacetime is encoded at a deeper kinematical level.

It is interesting to understand if on this spacetime one can consistently formulate a gravity theory. I see NC gravity as an effective theory. This theory may capture some aspects of a quantum gravity theory.

Notice also that:

- In string theory, study of string scatterings shows that generalized uncertainty principles where a minimal length occurs is natural. Also, because of T-duality, strings can be considered unable to test compactifications of spacetimes with radii smaller than the string scale.

- Loop quantum gravity predicts minimal areas and volumes.

We have reviewed conceptual motivations for the study of Quantum Spacetime and its noncommutative geometry.

Further motivation to study deformed (noncommutative) spacetimes comes from *quantum gravity phenomenology*.

Quantum gravity effects are expected to become relevant at Planck scale (10^{-33} cm), so phenomenology has to rely on indirect signatures, like:

-cosmological data near inflationary epoch (a few orders of magnitude away from regimes characterised by Planck scale energies)

-low energy but cumulative phenomena. *Example:* light travelling in a quantum spacetime could have a velocity dependent on the photons energy. Even a tiny modification of the usual dispersion relations could then be detected due to the cumulative effect of light travelling long distances.
A natural setting for this study is that of Gamma Ray Bursts (GRB) from distant galaxies [Miguejio, Smolin, Amelino-Camelia, Ellis, Mavromatos, Nanopulos].
Another possibility is that of high precision (quantum) optics experiments based on interferometry techniques e.g. [Hogan group] [Genovese group].

Spacetime and Quantum mechanics II

Noncommutativity from quantum mechanics and from fluxes

-Spacetime itself, not just phase-space, can become noncommutative also in the presence of electromagnetic fields. It is an effect of quantum theory and the presence of fluxes (magnetic field B). Example: electron in a strong magnetic field B. In this regime, due to the minimal coupling with the background gauge field $(A_x, A_y) = (0, Bx)$ associated to B, the dynamics takes place in a reduced phase space, where $q = y$, $p = Bx$ thus the electron's coordinates become noncommutative: $[x, y] = -\frac{i\hbar}{B}$.

Hence quantum theory in the presence of a magnetic field B leads to a non-commutative spacetime (not just phase space).

Let's further pursue the noncommutativity due to charges in a magnetic field. Charges source an electromagnetic field, therefore one could also consider a Maxwell theory on noncommutative spacetime.
It turns out that (under certain assumptions) there is an equivalence between a Maxwell theory (and more generally a Yang-Mills theory) on NC space and a

Maxwell theory (Yang-Mills theory) on commutative space [Connes, Douglas, Schwartz 1997] [Seiberg-Witten 1999].
This equivalence of gauge fields theories was first obtained studying low energy effective actions of open strings under open strings T-duality. In mathematics it is known as (complete or gauge) Morita equivalence [Schwarz].

Given a principal bundle $P \to M$ with structure group G we consider:
Lie(G)-valued connection 1-form A (locally $A = A^a_\mu T^a dx^\mu$ with $\{T^a\}$ is basis of Lie$G$), the curvature $F = dA - A \wedge A$, and the Yang-Mills lagrangian

$$\mathcal{L} = \frac{1}{g_{YM}} \mathrm{Tr}(F \wedge *F)$$

Yang-Mills theories can be defined on noncommutative spaces and their study has proven very fruitful:

-they provide exact low energy open string theory effective actions (in a given $\alpha' \to 0$ sector of string theory where closed strings decouple).

-they allow to realize string theory T-duality symmetry within the low energy physics of Noncommutative (Super) Yang-Mills theories.

Key example: NC d-dimensional torus T^d

NC plane $[x^i, x^j] = 2\pi i\theta^{ij} \quad \Rightarrow$ NC torus coordinates $U^i = e^{ix^i}$.

NCSYM: $U(n)$, θ^{ij}, g_{ij}, g_{SYM}, M_{ij} first Chern number $\frac{1}{2\pi}\int_{T_{ij}} TrF$.

NCSYM$'$: $U(n'), \theta'^{ij}$, g'_{ij}, g'_{SYM}, M'_{ij}.

Let $\Lambda = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in SO(d,d,\mathbb{Z})$. If

$$\theta' = (\mathcal{A}\theta + \mathcal{B})(\mathcal{C}\theta + \mathcal{D})^{-1},$$
$$\mathrm{g}'^{ij} = (\mathcal{C}\theta + \mathcal{D})^i_{\ k}\, \mathrm{g}^{kl} (\mathcal{C}\theta + \mathcal{D})^j_{\ l}\,,$$
$$g'^2_{SYM} = \sqrt{|\det(\mathcal{C}\theta + \mathcal{D})|}\; g^2_{SYM},$$
$$\begin{pmatrix} n' \\ M' \end{pmatrix} = S(\Lambda)\begin{pmatrix} n \\ M \end{pmatrix} \qquad \text{with } S(\Lambda) \text{ spinor representation}$$

then we have NCSYM $\simeq$ NCSYM$'$

[Connes, Douglas, Schwarz 1997] [Morariou, Zumino 1999]

In this context T-duality is Morita equivalence!

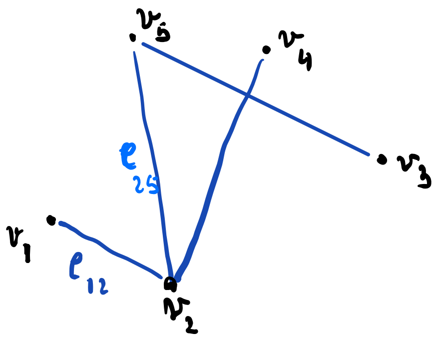
These results motivate the study of more general NC differential geometries and gauge theories.

NC instantons, NC principal bundles, ...

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

$\mathcal{V} = \{v_1, \dots, v_n\}$ vertices $|\mathcal{V}| = n$

$\mathcal{E} = \{e_{ij}\}$ edges (links) $|\mathcal{E}| = m$



Adiacency matrix $A \in M_{n \times n}$

$$A_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \text{ (exists link } e_{ij}) \\ 0 & \text{otherwise} \end{cases}$$

Incidence matrix $\nabla \in M_{n \times m}$

$$\nabla_{ev} = \begin{cases} -1 & v \xrightarrow{e} \\ 1 & \xrightarrow{e} v \\ 0 & v \text{ is not in } e \end{cases}$$

Graph Laplacian $L = \nabla^T \nabla \in M_{n \times n}$

Provide a NC diff. geometry perspective. Set $\mathcal{A} = \mathcal{C}(\mathcal{V}) = \mathbb{C}$-valued functions.

$\mathcal{A} \simeq \mathbb{C}^n$ a basis being $\delta_i \in \mathcal{A}$

$\delta_i(v_j) := \delta_{ij}$.

- $\mathcal{A}$ is C^*-algebra of functions on $\mathcal{V}$.
- Calculus on $\mathcal{A}$ defined by the edges e_{ij} = "vector fields on $\mathcal{V}$"

$\mathcal{X} :=$ quantum derivations of $\mathcal{A}$

$:= \mathbb{C}$-linear span of e_{ij}.

Then
$$A : \mathcal{A} \to \mathcal{A}$$
$$f \mapsto Af \; ; \quad (Af)(v_i) = \sum_{j \sim i} f(v_j)$$

$$\nabla f(e_{ij}) = f(v_j) - f(v_i)$$

$$W : \mathcal{A} \to \mathcal{A}$$
$$f \mapsto Wf \; ; \quad (Wf)(v_i) = \sum_{j \sim i} W_{ij} f(v_j) \quad \text{weight } W_{ij} \text{ for link } e_{ij}.$$

The NC diff. geom. methods allow:

- Space of 1-forms $\Omega^1 \simeq \mathbb{C}^m$ defined as $\mathbb{C}$-linear span of $\{\omega_{ij}\}$
  where $\langle e_{ij}, \omega_{kl}\rangle = \delta_{il}\,\delta_{jk}$

The algebra $\mathcal{A}$ is commutative, the $\mathcal{A}$-bimodule of 1-forms Ω^1 is not:

$\forall f \in \mathcal{A}$;
$$f \cdot \omega_{ij} = f(v_i)\,\omega_{ij}$$
$$\omega_{ij} \cdot f = f(v_j)\,\omega_{ij}$$

- Incidence matrix $\nabla \longleftrightarrow$ exterior derivative
$$d : \mathcal{A} \longrightarrow \Omega^1$$
$$f \longmapsto [\vartheta, f] = \vartheta f - f\vartheta$$
$$\vartheta = \sum_{ij} \omega_{ij}$$

$$df = \sum_{ij} \underbrace{\big(f(v_j) - f(v_i)\big)}_{\nabla f(e_{ij})}\,\omega_{ij}$$

Metric associated with A is $\quad (\omega_{ij}, \omega_{kl}) = A_{ij}\, \delta_{jk}\, \delta_{il}\, \delta_{e}$

then $\quad d^{+} : \Omega^{1} \longrightarrow \Omega \quad$ as adjoint to d.

Laplacian $\quad L = \nabla^{T} \nabla \; : \; \Omega \rightarrow \Omega$

$$L = d^{+} d + d\, d^{+} : \Omega \longrightarrow \Omega$$

Laplace-Beltrami

(associated to the adjacency matrix A)

Work in progress with P.A. Grassi.

Thank you