Integrability in the AdS/CFT Correspondence

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References: ... , hep-th/0511082, ...
Introduction

Review methods of integrability for planar spectrum in AdS/CFT.

Outline:

- AdS/CFT, two main examples & spectrum.
- Planar limit, spin chains & integrability.
- AdS/CFT as a particle model.
- S-Matrix & asymptotic Bethe equations.
- BES equation & twist-two dimensions.
- Wilson loops & dual superconformal symmetry.
- Finite coupling, wrapping.

Why?!

- Understand all secrets of (planar) scattering amplitudes → LHC.
- Analytical non-perturbative method → quantum gravity.
- Exciting connections to condensed matter theory → superconductivity.
AdS/CFT Correspondence
Introduction to AdS/CFT

**AdS/CFT duality:** Conjectured exact duality of string theory on $\text{AdS}_{D+1} \times M^{9-D}$ background and a particular $D$-dimensional conformal field theory.

Two qualitatively different models. Why should they be equivalent?

**Symmetry groups match:** $\widetilde{\text{SO}}(D, 2)$.

**Holography:** Boundary of $\text{AdS}_{D+1}$ is conformal $\mathbb{R}^{D-1,1}$ or $\mathbb{R} \times S^{D-1}$.

**Prospects:**

- More general AdS/CFT may explain aspects of QCD strings.
- Study aspects of quantum gravity with QFT means.

**Main examples** of highly supersymmetric AdS/CFT pairs:

- Strings on $\text{AdS}_5 \times S^5$, ○ Strings on $\text{AdS}_4 \times \mathbb{C}P^3$.
- $4D \mathcal{N} = 4$ Yang–Mills theory, ○ $3D \mathcal{N} = 6$ Chern–Simons theory.

Here: Use integrability to compare spectra of both pairs of models.
Strings on $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$

$AdS_5 \times S^5$ is a coset space

$$AdS_5 \times S^5 \times 32 \ \text{fermi} = \frac{\widetilde{PSU}(2, 2|4)}{Sp(1, 1) \times Sp(2)}.$$ 

$AdS_4 \times \mathbb{CP}^3$ (with partial kappa gauge fixing) is a coset space $[Arutyunov\ Frolov][Stefanski\ 0806.4948]$

$$AdS_4 \times \mathbb{CP}^3 \times 24 \ \text{fermi} = \frac{\widetilde{OSp}(6|4, \mathbb{R})}{U(3) \times SL(2, \mathbb{C})}.$$ 

Worldsheet non-linear sigma model (Green-Schwarz superstring) $[Metsaev\ Tseytlin]$  

$$S_{\sigma} \simeq \frac{\sqrt{\lambda}}{2\pi} \int_{M^{1,1}} \left( \frac{1}{2} \text{STr} \ P \wedge *P - \frac{1}{2} \text{STr} \ Q_1 \wedge Q_2 \right).$$

Above coset spaces are $\mathbb{Z}^4$-graded symmetric spaces: $H, Q_1, P, Q_2$ are 0, 1, 2, 3-graded components of Maurer–Cartan form.
Supersymmetric CFT’s

4D $\mathcal{SU}(N) \mathcal{N} = 4$ superconformal Yang–Mills theory:

- $\mathcal{SU}(4)$ flavour symmetry, $\mathcal{SU}(2, 2)$ conformal symmetry $\rightarrow \widetilde{\mathcal{PSU}}(2, 2|4)$,
- coupling constant $g_{YM}$, theta angle $\theta$, rank $N$,
- gauge field, 6 adjoint scalars, $4 + \bar{4}$ adjoint fermions,

\[ \mathcal{L} \sim \frac{1}{g_{YM}^2} \left( \mathcal{F}^2 + (\mathcal{D}\Phi)^2 + \Psi\mathcal{D}\Psi + \Phi\Psi^2 + \Phi^4 \right). \]

3D $\mathcal{SU}(N) \times \mathcal{SU}(M) \mathcal{N} = 6$ superconformal Chern–Simons theory:

- $\mathcal{SU}(4)$ flavour symmetry, $\mathcal{Sp}(4, \mathbb{R})$ conformal symmetry $\rightarrow \widetilde{\mathcal{OSp}}(6|4, \mathbb{R})$,
- CS level $k_{CS}$, rank $N, M$,
- gauge field (non-dynamical), $4 \times \bar{M}$ and $\bar{4} \times \bar{M} \times \bar{N}$ scalars/fermions,

\[ \mathcal{L} \sim k_{CS} \left( \mathcal{A}\mathcal{F} + \mathcal{A}^3 + (\mathcal{D}\Phi)^2 + \Psi\mathcal{D}\Psi + \Phi^2\Psi^2 + \Phi^6 \right). \]
Spectrum
Spectrum of AdS/CFT

String Theory:
States: Solutions $X$ of classical equations of motion plus quantum corrections.
Energy: Charge $E_X(\lambda)$ for translation along AdS-time.

Conformal Field Theory:
States: Local operators. Gauge-inv. combinations (glueballs), e.g.

$$\mathcal{O} = \text{Tr} \Phi_1 \Phi_2 (\mathcal{D}_1 \mathcal{D}_2 \Phi_2) (\mathcal{D}_1 \Psi_2) + \ldots.$$ 

Energy: Scaling dimensions, e.g. two-point function in conformal theory

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = C |x - y|^{-2D_\mathcal{O}(\lambda)}.$$ 

AdS/CFT: String energies and CFT dimensions match, $E(\lambda) = D(\lambda)$?!
Strong/Weak Duality

Problem: Models have coupling constant: $\lambda$. Strong/weak duality.

- Perturbative regime of strings at $\lambda \to \infty$.

$$E(\lambda) = \sqrt{\lambda} E_0 + E_1 + E_2/\sqrt{\lambda} + \ldots .$$

$E_\ell$: Contribution at $\ell$ (world-sheet) loops. Limit: 1 or 2 loops.

- Perturbative conformal field theory at $\lambda \to 0$.

$$D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \ldots .$$

$D_\ell$: Contribution at $\ell$ (gauge) loops. Limit: 3 or 4 (or 5?!?) loops.

Tests impossible unless quantities are known at finite $\lambda$.
Cannot compare, not even approximately.

BMN type comparison suffers from order-of-limits.
Planar Limit

- Simplifications & surprises
- AdS/CFT integrability

String Theory:
- No string coupling $g_s = 0$.
- Strictly cylindrical worldsheet.
- No string splitting or joining.
- Worldsheet coupling $\lambda$ free.

Conformal Field Theory:
- $\mathcal{N} = 4$ SYM: $N = \infty, g_{YM} = 0$,
  finite 't Hooft coupling $\lambda = g_{YM}^2 N$.
- $\mathcal{N} = 6$ SCS: $N, M, k_{CS} = \infty$,
  finite 't Hooft coupling $\lambda = \sqrt{NM}/k_{CS}$.
- Only planar Feynman diagrams.
Spin Chains

Planar limit: only single-trace operators relevant. Translate single-trace operators to spin chain states, e.g. [Minahan Zarembo]

\[ \text{\( su(2) \) sector of \( \mathcal{N} = 4 \) SYM: } \]
\[ \mathcal{O} = \text{Tr} \phi_1 \phi_1 \phi_2 \phi_1 \phi_1 \phi_1 \phi_2 \phi_2 \]
identify \( \phi_1 \rightarrow |\uparrow\rangle, \phi_2 \rightarrow |\downarrow\rangle \)

\[ |\mathcal{O}\rangle = |\uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow\rangle \]

\[ \text{\( su(2) \times su(2) \) sector of \( \mathcal{N} = 6 \) SCS: } \]
\[ \mathcal{O} = \text{Tr} \Phi_1 \Phi_3 \Phi_2 \Phi^3 \Phi_1 \Phi^3 \Phi_2 \Phi^4 \]
identify \( \Phi_{1,2} \rightarrow |\uparrow\rangle, \Phi^{3,4} \rightarrow |\uparrow \uparrow\rangle \)

\[ |\mathcal{O}\rangle = |\uparrow \bar{\uparrow} \downarrow \bar{\uparrow} \uparrow \bar{\uparrow} \uparrow \bar{\uparrow} \downarrow \bar{\uparrow}\rangle \]


Qualitative similarity with strings:
- Many coherent spins can trace out snapshot of string embedding.
Perturbative Spin Chain Hamiltonian

Perturbative Hamiltonian (planar Feynman diagrams) in $\mathcal{N} = 4$ SYM:

$$\mathcal{H}(\lambda) = \mathcal{H}_0 + \lambda \mathcal{H}_2 + \lambda^{3/2} \mathcal{H}_3 + \lambda^2 \mathcal{H}_4 + \ldots.$$ 

Perturbative Hamiltonian in $\mathcal{N} = 6$ SCS:

$$\mathcal{H}(\lambda) = \mathcal{H}_0 + \lambda^2 \mathcal{H}_2 + \lambda^3 \mathcal{H}_3 + \lambda^4 \mathcal{H}_4 + \ldots.$$ 

Properties: • perturbatively short-ranged, • homogeneous, • dynamic.

Perturbative expansion of non-manifest symmetries $\mathcal{J} = Q, S, P, K$

$$\mathcal{J}(\lambda) = \mathcal{J}_0 + \lambda^{1/2} \mathcal{J}_1 + \ldots,$$

$$\mathcal{J}(\lambda) = \mathcal{J}_0 + \lambda \mathcal{J}_1 + \ldots.$$
Problems of Hamiltonian Approach

- Contributions at $O(\lambda^0)$: trivial; counting spins.
- Contributions at $O(\lambda^1)/O(\lambda^2)$: NN/NNN Hamiltonian.
- Contributions at higher orders hard to compute in field theory.
- Scaling dimension $D_\mathcal{O}$ as eigenvalue of the Hamiltonian $\mathcal{H}(\lambda)$.
  Diagonalisation of NN/NNN Hamiltonian hard: combinatorics/calculus.
- Complete chains: infinitely many spins $\mathcal{W} \in \{D^n\Phi, D^n\Psi, D^n\mathcal{F}\}$.
  Irreducible modules $\mathcal{V}_F$ of $\mathfrak{psu}(2,2|4)$ or $\mathcal{V}_F$ and $\mathcal{V}_{\bar{F}}$ of $\mathfrak{osp}(6|4,\mathbb{R})$.
- Hamiltonian defined modulo similarity transformations. Ambiguous!
- Part of symmetry algebra (eigenvalues label representations):
  Representation theory of $\mathfrak{psu}(2,2|4)/\mathfrak{osp}(6|4,\mathbb{R})$ useless.
- Perturbative string theory requires all orders/finite coupling.

How to obtain spectrum?
Integrability
Classical String Integrability

Integrability: Existence of family of flat connections

\[ A(x) = H + \frac{1}{2}(z^{-2} + z^2)P + \frac{1}{2}(z^{-2} - z^2)*P + z^{-1}Q_1 + zQ_2. \]

Flatness equivalent to Maurer–Cartan equations and equations of motion:

\[ 0 = dA(x) - A(x) \wedge A(x). \]

Monodromy of flat connection around worldsheet cylinder

\[ M(x) = P \exp \oint_{\gamma} A(x). \]

Eigenvalues of \( M(z) \) (as a function of \( z \))

- are independent of path \( \gamma \), base point \( \gamma(0) \) and WS diffeomorphisms,
- define spectral curve (action variables),
- fully classify classical solutions in terms of conserved charges.
NN/NNN Spin Chain Integrability

$\mathcal{N} = 4$ SYM: Complete one-loop Hamiltonian $\mathcal{H}_2$. \cite{Minahan:2003az, Minahan:2004qf}

Planar one-loop Hamiltonian completely integrable! \cite{Lipatov:1997ag, Minahan:2003az}

(To some extent in 4D gauge theory.) \cite{Lipatov:1997ag, Minahan:2003az}

NN-Hamiltonian from $\mathfrak{psu}(2, 2|4)$ R-matrix ($\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}$).

$\mathcal{N} = 6$ SCS: NNN-Hamiltonian from $\mathfrak{osp}(6|4, \mathbb{R})$ R-matrices \cite{Minahan:2003az, Bak:2007aa}

\begin{align*}
\mathcal{H}_2 & = \frac{d \log}{d u} \mathcal{R} \quad (u = 0). \\
\mathcal{H}_2 + \mathcal{H}_2 & = \frac{d \log}{d u} \mathcal{R} \quad (u = 0).
\end{align*}
Quantum Integrability in $AdS_5 \times S^5/\mathcal{N} = 4$ SYM

Higher-loop (non)compact Hamiltonian. Various string quantum corrections.

Notion of quantum integrability unclear (R-matrix?). **Indications:**

- Degenerate pairs with opposite parity.
- Local commuting charges $[\mathcal{H}, Q_r] = [Q_r, Q_s] = 0$.
- Yangian symmetry.
- Four-loop gluon scattering and integrability.
- Four-loop wrapping effects from integrability.
- Bethe equations work for quantum strings.
- Pure spinor BRST consistency of flat connection.

Useful working hypothesis. **Assumption!** Path towards proof unclear; but:

- Recursion relation for algebra in $su(1,1|2)$ sector.
- Integrability-preserving recursion relation in $su(2)$ sector.
Asymptotic Bethe Ansatz and Beyond

Procedure to obtain the spectrum of an integrable model.

- Relax periodicity: Consider model on infinite line instead of circle.
- Find particles and determine their properties.
- Derive 2-particle S-matrix. Integrability: Factorised scattering!
- Diagonalise S-matrix.
- Impose Bethe equations as periodicity conditions for eigenstates.
- Read off eigenvalues as sums over particles.
- Add finite-size corrections (S-matrix on a circle, not line). TBA.
Particle Model
**AdS\(_5 \times S^5\)** Strings: Light Cone Gauge

Obtain spectrum of \( \mathcal{H}(\lambda) \) from perturbative string theory at large \( \lambda \).

Perform light cone gauge using time from \( AdS_5 \) and great circle from \( S^5 \).

- **Vacuum**: Point-particle moving along time and great circle.
- **Excitations**: 4 coordinates on \( AdS_5 \) and 4 coordinates on \( S^5 \).
- **Fermions**: 1/2 are gauged away, 1/2 are momenta, 8 remain.

QM particle model of 8 bosonic and 8 fermionic flavours on the circle.

**Residual symmetry**

Unbroken symmetries of the vacuum state:

- \( \tilde{SO}(4, 2) \simeq \tilde{SU}(2, 2) \) reduces to \( SO(4) \times \tilde{SO}(2) \simeq SU(2) \times SU(2) \times \mathbb{R} \).
- \( \tilde{SO}(6) \simeq SU(4) \) reduces to \( SO(4) \times SO(2) \simeq SU(2) \times SU(2) \times U(1) \).
- \( \tilde{PSU}(2, 2|4) \) reduces to \( U(1) \ltimes (PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R} \).

Excitations in \((2|2) \times (2|2)\) representations of \((PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R} \).
\( \mathcal{N} = 4 \) SYM: Coordinate Space Bethe Ansatz

Gauge spectrum of \( \mathcal{H} \)? Consider spin chain states with few “excitations”.

Ferromagnetic vacuum: protected state with scalar \( \mathcal{Z} = \Phi_5 + i\Phi_6 \)

\[
|0\rangle = |\ldots \mathcal{Z} \mathcal{Z} \mathcal{Z} \ldots \rangle, \quad \delta \mathcal{H} |0\rangle = 0.
\]

Residual symmetry of \((\text{psu}(2|2) \times \text{psu}(2|2)) \ltimes \mathbb{R}\) stabilises \(\mathcal{Z}\).

One-excitation states with excitation \( \mathcal{A} \) at position \( a \), momentum \( p \)

\[
|\mathcal{A}, p\rangle = \sum_a e^{ipa} |\ldots \mathcal{Z} \ldots \mathcal{A} \ldots \mathcal{Z} \ldots \rangle, \quad \delta \mathcal{H} |\mathcal{A}, p\rangle = \delta E_A(p) |\mathcal{A}, p\rangle.
\]

\((4 + 4|4 + 4)\) flavours of excitations \( \mathcal{A} \in \{ \phi_i, D_\mu \mathcal{Z}, \psi_a, \dot{\psi}_\dot{a} \} \).

Other spin orientations in module \( \mathbb{V}_F \) are multiple coincident excitations.

Coordinate space Bethe ansatz leads to a particle model with \((2|2) \times (2|2)\) flavours transforming under \((\text{psu}(2|2) \times \text{psu}(2|2)) \ltimes \mathbb{R}\).
Residual Extended $\mathfrak{psu}(2|2)$ Algebra

Residual algebra $\mathfrak{su}(2|2)$ preserves particle number. Generators:

- $\mathcal{R}_{ab}^a$ flavour $\mathfrak{su}(2)$ rotation generator,
- $\mathcal{L}_{\alpha \beta}$ spacetime $\mathfrak{su}(2)$ rotation generator,
- $Q_{\alpha}^a$ supersymmetry generator,
- $S_{a}^b$ superboost generator,
- $\mathcal{H}$ central charge.

Algebra: $\mathcal{R}_{ab}^a, \mathcal{L}_{\alpha \beta}$ transform indices. Anticommutator of supercharges

$$\{Q_{\alpha}^a, S_{b}^\beta\} = \delta_{a}^{b} \mathcal{L}_{\alpha \beta} + \delta_{\beta}^{\alpha} \mathcal{R}_{a}^{b} + \delta_{a}^{b} \delta_{\beta}^{\alpha} \mathcal{H},$$

$$\{Q_{\alpha}^a, Q_{\beta}^b\} = \varepsilon^{\alpha \beta} \varepsilon_{ab} \mathcal{P},$$

$$\{S_{a}^{\alpha}, S_{b}^{\beta}\} = \varepsilon_{ab} \varepsilon^{\alpha \beta} \mathcal{K}.$$ 

- Additional central generators $\mathcal{P}, \mathcal{K}$ are gauge transformations.
- Exceptional threefold central extension $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3.$
Fundamental Representation of $\mathfrak{psu}(2|2) \rtimes \mathbb{R}^3$

Excitations should transform in $(2|2)$ representation of extended $\mathfrak{psu}(2|2)$. Ansatz for $(2|2)$ representation with canonical action of $R^a_b, L^\alpha_\beta$:

\[
\begin{align*}
Q^\alpha_a |\phi^b\rangle &= a \delta^b_a |\psi^\alpha\rangle, \quad \mathcal{H}|\mathcal{X}\rangle = E|\mathcal{X}\rangle, \\
Q^\alpha_a |\psi^\beta\rangle &= b \varepsilon^\alpha_\beta \varepsilon^b_a |\phi^b \mathcal{Z}^+\rangle, \quad \mathcal{P}|\mathcal{X}\rangle = P|\mathcal{X} \mathcal{Z}^+\rangle, \\
S^a_\alpha |\phi^b\rangle &= c \varepsilon^a_b \varepsilon^\alpha_\beta |\psi^\beta \mathcal{Z}^-\rangle, \quad \mathcal{K}|\mathcal{X}\rangle = K|\mathcal{X} \mathcal{Z}^-\rangle, \\
S^a_\alpha |\psi^\beta\rangle &= d \delta^\beta_\alpha |\phi^a\rangle,
\end{align*}
\]

Closure requires $ad - bc = 1$, $E = \frac{1}{2}(ad + bc)$, $P = ab$, $K = cd$. Shortening/atypicality condition for central charges: $E^2 - PK = \frac{1}{4}$. 

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Gauge Transformations and Extended $su(2|2)$

Generator $\mathcal{P}$ generates gauge transformation (similarly for $\mathcal{K}$)

\[
\mathcal{P}|\mathcal{X}\rangle \sim |[\mathcal{Z}^+, \mathcal{X}]\rangle = |\mathcal{Z}^+\mathcal{X}\rangle - |\mathcal{X}\mathcal{Z}^+\rangle = (1 - e^{ip})|\mathcal{Z}^+\mathcal{X}\rangle.
\]

$su(2|2)$ symmetry recovered for physical states (annihilated by $\mathcal{P}, \mathcal{K}$). $P, K$ are algebraically fixed functions of momentum $p$

\[
P = g\alpha(1 - e^{ip}), \quad K = g\alpha^{-1}(1 - e^{-ip}), \quad g = \frac{\sqrt{\lambda}}{4\pi}.
\]

Cyclicity condition $P = K = 0$ for physical states with zero momentum. Dispersion relation $E(p)$ algebraically determined from atypicality

\[
E = \sqrt{\frac{1}{4} + PK} = \frac{1}{2} \sqrt{1 + 16g^2 \sin^2(\frac{1}{2}p)}.
\]
**AdS$_4 \times \mathbb{CP}^3$ Strings: Light Cone Gauge**

Does this work for $AdS_4 \times \mathbb{CP}^3$ strings as well? Perform light cone gauge using time from $AdS_4$ and geodesic from $\mathbb{CP}^3$. 

- **Vacuum**: Point-particle moving along time and geodesic.
- **Excitations**: 3 coordinates on $AdS_4$ and $1 + 4$ coordinates on $\mathbb{CP}^3$.
- **Fermions**: $1/3$ are gauged away, $1/2$ are momenta, $4 + 4$ remain.

**Residual symmetry**

Unbroken symmetries of the vacuum state:

- $\widetilde{SO}(3, 2) \simeq \widetilde{Sp}(4, \mathbb{R})$ reduces to $SO(3) \times \widetilde{SO}(2) \simeq SU(2) \times \mathbb{R}$.
- $SO(6) \simeq SU(4)$ reduces to $SU(2) \times U(1)^2$.
- $\widetilde{OSp}(6|4, \mathbb{R})$ reduces to $U(1) \ltimes (PSU(2|2) \times U(1)) \ltimes \mathbb{R}$.

Excitations in $(2|2)^+ + (2|2)^- + (4|4)^0$ representations.
\( \mathcal{N} = 6 \) SCS: Coordinate Space Bethe Ansatz

What about \( \mathcal{N} = 6 \) SCS? Pick an alternating vacuum using \( \Phi_1 \) and \( \bar{\Phi}^4 \)

\[
|0\rangle = |\ldots \Phi_1 \bar{\Phi}^4 \Phi_1 \bar{\Phi}^4 \Phi_1 \bar{\Phi}^4 \ldots \rangle.
\]

Residual symmetry of \( (\text{psu}(2|2) \times \text{u}(1)) \ltimes \mathbb{R} \) stabilises \( \Phi_1 \) and \( \bar{\Phi}^4 \).

Single excitations \( (2|2)^+ \): \( \Phi_1 \rightarrow \Phi_{2,3}, \psi_\alpha \) and \( (2|2)^- \): \( \bar{\Phi}^4 \rightarrow \bar{\Phi}_{2,3}, \bar{\psi}_\alpha \).

Multiplet \( (4|4)^0 \) is double excitation. Not bound! Problem?!

Above particle \textbf{multiplet construction} applies here as well:
Gauge transformation generators \( \mathcal{P}, \mathcal{K} \) act like

\[
\mathcal{P} \mathcal{X} \sim \Phi_1 \bar{\Phi}^4 \mathcal{X} - \mathcal{X} \bar{\Phi}^4 \Phi_1, \quad \mathcal{P} \bar{\mathcal{X}} \sim \bar{\Phi}^4 \Phi_1 \bar{\mathcal{X}} - \bar{\mathcal{X}} \Phi_1 \bar{\Phi}^4.
\]

Dispersion \( E_{2|2} = \frac{1}{2} \sqrt{1 + 16h^2 \sin^2 p} \), \( E_{4|4} = \frac{1}{2} \sqrt{4 + 16h^2 \sin^2 p} \).

\( h \) is function of coupling: \( h \sim \sqrt{\lambda} \) (strings) or \( h \sim \lambda \) (CFT). Physical?!
S-Matrix
Scattering Matrix

Consider multi-particle states on non-compact worldsheet.
If particles are well-separated: plane wave partial eigenstates, e.g.

$$|\mathcal{X}_1 < \mathcal{X}_2 < \mathcal{X}_3\rangle = \sum_{k_1 \ll k_2 \ll k_3} e^{ip_1 k_1 + ip_2 k_2 + ip_3 k_3} |\ldots \mathcal{X}_1 \ldots \mathcal{X}_2 \ldots \mathcal{X}_3 \ldots \rangle$$

Construct two-particle eigenstates by fitting asymptotic regions

$$|\mathcal{X}^A \mathcal{X}^B\rangle \simeq |\mathcal{X}_{1}^A < \mathcal{X}_{2}^B\rangle + \text{UV} + S^{AB}_{CD}(p, q)|\mathcal{X}_{2}^D < \mathcal{X}_{1}^C\rangle.$$ 

S-matrix $S^{CD}_{AB}(p, q)$ encodes phase shift for permuting particles with momenta $p, q$.

Energy $\mathcal{H}|\mathcal{X}_{1}^A \mathcal{X}_{2}^B\rangle = (E(p) + E(q))|\mathcal{X}_{1}^A \mathcal{X}_{2}^B\rangle$. 

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Integrable Scattering Matrix

Scattering matrix in a generic field theory

\[ |\chi_A^1 \chi_B^2 \chi_C^3\rangle \simeq \int dp' S_{DEF}^{ABC}(p, p')|\chi_D^{1'} < \chi_E^{2'} < \chi_F^{3'}\rangle + \text{UV}. \]

Integrability: set of particle momenta \(\{p_k\}\) conserved (integrals of motion)

\[ |\chi_A^1 \chi_B^2 \chi_C^3\rangle \simeq \sum_{\pi \in S_3} (S_{\pi})^{ABC}_{DEF}(p)|\chi_{\pi(1)}^{D} < \chi_{\pi(2)}^{E} < \chi_{\pi(3)}^{F}\rangle + \text{UV}. \]

Permutation group \(S_K\) generated by pairwise permutations \(S_{k,l}\).

Need only two-particle S-matrix!

Consistency: Yang–Baxter equation

\[ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}. \]
S-Matrix Construction

Use $\mathfrak{su}(2|2)$ invariance to construct two-particle S-matrix.

From manifest $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetries

\[
\begin{align*}
S_{12}|\phi^a_1 \phi^b_2\rangle &= A_{12}|\phi^a_2 \phi^b_1\rangle + B_{12}|\phi^a_2 \phi^b_1\rangle + \frac{1}{2} C_{12}\epsilon^{ab}\epsilon^{\alpha\beta}|\psi^\alpha_2 \psi^\beta_1 Z^-\rangle, \\
S_{12}|\psi^\alpha_1 \psi^\beta_2\rangle &= D_{12}|\psi^\alpha_2 \psi^\beta_1\rangle + E_{12}|\psi^\alpha_2 \psi^\beta_1\rangle + \frac{1}{2} F_{12}\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^a_2 \phi^b_1 Z^+\rangle, \\
S_{12}|\phi^a_1 \psi^\beta_2\rangle &= G_{12}|\psi^\beta_2 \phi^a_1\rangle + H_{12}|\phi^a_2 \psi^\beta_1\rangle, \\
S_{12}|\psi^\alpha_1 \phi^b_2\rangle &= K_{12}|\psi^\alpha_2 \phi^b_1\rangle + L_{12}|\phi^b_2 \psi^\alpha_1\rangle.
\end{align*}
\]

with ten coefficient functions $A(p_1, p_2), \ldots, L(p_1, p_2)$.

- Supersymmetry fixes all functions up to one overall factor: $\sigma(p_1, p_2)$.
- Yang–Baxter equation automatically satisfied.
- Crossing relation constrains factor.
- Proposal for strong/weak interpolating phase factor.
Phase Factor

Educated guess for crossing-symmetric phase

$$\sigma(p_1, p_2) = \exp \left( i \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(\lambda) \left( q_r(p_1) q_s(p_2) - q_r(p_2) q_s(p_1) \right) \right),$$

$$c_{r,s}(\lambda) = 2 \sin \left[ \frac{1}{2} \pi (s - r) \right] (r - 1)(s - 1) \int_0^{\infty} \frac{dt}{t} \frac{J_{r-1}(t) J_{s-1}(t)}{e^{2\pi t/\sqrt{\lambda - 1}}}. $$

Strong/weak expansion agrees with expectations. E.g. plot of $c_{23}(\lambda)$
Asymptotic Bethe Equations

Diagonalisation of S-matrix (nested Bethe ansatz):
Introduce several flavours $A = 1, \ldots, R$ (rank) of Bethe excitations $u_{A,k}$ with $k = 1, \ldots, K_A$

- main excitations carry momentum & energy.
- auxiliary roots $u_{A,k}$ carry spin waves,

Equation for asymptotically $L$-periodic wave function

$$1 = e^{-i p_A(u_{A,j}) L} \prod_{B=1}^{R} \prod_{j=1}^{K_B} S_{A,B}(u_{A,k}, u_{B,j}).$$

Charge eigenvalues:

$$\exp i P = \prod_{A=1}^{R} \prod_{j=1}^{K_A} \exp i p_A(u_{A,k}) = 1, \quad E = \sum_{A=1}^{R} \sum_{j=1}^{K_A} E_A(u_{A,k}).$$
Bethe Equations for \( AdS_5 \times S^5 \) and \( \mathcal{N} = 4 \) SYM

Complete asymptotic Bethe equations

coupling constant
\[
g^2 = \frac{\lambda}{16\pi^2}
\]

relations between \( u \) and \( x^\pm \)
\[
u = x^+ + \frac{i}{2g} \quad , \quad x^+ = \frac{1}{2} \quad , \quad x^- = \frac{1}{2g}
\]
magnon momentum and energy
\[
e^\nu = \frac{x^+}{x^-} \quad , \quad e = -igx^+ + igx^- - \frac{1}{2}
\]
total energy
\[
E = L + \frac{1}{2}N + \frac{1}{2}\tilde{N} + 2\sum_{j=1}^{K} \left( \frac{iq}{x^+_j} - \frac{ig}{x^-_j} \right)
\]
local charges
\[
q_r(x^\pm) = \frac{1}{r-1} \left( \frac{i}{(x^+)^{r-1}} - \frac{i}{(x^-)^{r-1}} \right) \quad , \quad Q_r = \sum_{j=1}^{K} q_r(x^+_j)
\]

scattering factor \( \sigma \) with coefficients \( c_{r,s} \)
\[
\sigma_{12} = \exp \left( \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) \left( q_r(x^+_1)q_s(x^+_2) - q_r(x^+_2)q_s(x^+_1) \right) \right)
\]

Bethe equations
\[
1 = \prod_{j=1}^{K} \frac{x^+_j}{x^-_j}
\]
\[
1 = \prod_{j=1}^{M} \frac{u_k - u_j - ig^{-1}}{u_k - u_j + ig^{-1}} \prod_{j=1}^{N} \frac{\dot{u}_k - \dot{y}_j - 1/\dot{y}_j + ig^{-1}}{\dot{u}_k - \dot{y}_j - 1/\dot{y}_j - \frac{i}{2g}}
\]
\[
1 = \prod_{j=1}^{M} \frac{\dot{y}_k + 1/\dot{y}_k - u_j + \frac{i}{2g^{-1}}}{\dot{y}_k + 1/\dot{y}_k - u_j - \frac{i}{2g^{-1}}} \prod_{j=1}^{K} \frac{\ddot{y}_k - x^+_j}{\ddot{y}_k - x^-_j}
\]
\[
1 = \left( \frac{x^+_1}{x^-_1} \right)^L \prod_{j=1}^{K} \left( \frac{u_k - u_j + ig^{-1}}{u_k - u_j - ig^{-1}} \right) \prod_{j=1}^{N} \frac{x^+_1 - \dot{y}_j}{x^-_1 - \dot{y}_j} \prod_{j=1}^{N} \frac{x^-_1 - \dot{y}_j}{x^+_1 - \dot{y}_j}
\]
\[
1 = \prod_{j=1}^{M} \frac{\dot{y}_k + 1/\dot{y}_k - u_j + \frac{i}{2g^{-1}}}{\dot{y}_k + 1/\dot{y}_k - u_j - \frac{i}{2g^{-1}}} \prod_{j=1}^{K} \frac{\ddot{y}_k - x^+_j}{\ddot{y}_k - x^-_j}
\]
\[
1 = \prod_{j=1}^{M} \frac{\dot{y}_k + 1/\dot{y}_k - u_j + \frac{i}{2g^{-1}}}{\dot{y}_k + 1/\dot{y}_k - u_j - \frac{i}{2g^{-1}}} \prod_{j=1}^{N} \frac{x^+_1 - \dot{y}_j}{x^-_1 - \dot{y}_j} \prod_{j=1}^{N} \frac{x^-_1 - \dot{y}_j}{x^+_1 - \dot{y}_j}
\]

magic coefficients
\[
c_{r,s}(g) = 2 \sin \left( \frac{1}{2\pi} (s - r) \right) (r - 1)(s - 1) \int_0^{\infty} \frac{dt}{t} \frac{J_{r-1}(t)J_{s-1}(t)}{e^{t/2g} - 1}
\]

Asymptotic nature: Valid only up to terms \( \mathcal{O}(e^{-*L}) \) or \( \mathcal{O}(\lambda L) \).
Same as wrapping order where Hamiltonian can wrap the chain fully.

Varna 08, Niklas Beisert

NB, Dippel

NB, Staudacher

hep-th/0504190

hep-th/0511082
Bethe Equations for $AdS_4 \times \mathbb{CP}^3$ and $\mathcal{N} = 6$ SCS

Complete asymptotic Bethe equations.

coupling constant

$$h \lambda = 0 \quad h \lambda = \infty \sqrt{\lambda}$$

relations between $u$ and $x^\pm$

$$u = x^+ + \frac{1}{x^+} - \frac{i}{2h} = x^- + \frac{1}{x^-} + \frac{i}{2h}$$

magnon momentum and energy

$$e^\theta = \frac{x^+}{x^-}, \quad e = -ihx^+ + ihx^- - \frac{1}{2}$$

total energy

$$E = 2L + \frac{1}{2}N + 2 \sum_{j=1}^{K} \left( \frac{ih}{x_j} - \frac{i}{x_j} \right) + 2 \sum_{j=1}^{K} \left( \frac{ih}{x_j} - \frac{i}{x_j} \right)$$

local charges

$$q_r(x^\pm) = \frac{1}{r-1} \left( \frac{i}{(x^+)^{r-1}} - \frac{i}{(x^-)^{r-1}} \right), \quad Q_r = \sum_{j=1}^{K} q_r(x_j^\pm)$$

scattering factor $\sigma$ with coefficients $c_{r,s}$

$$\sigma_{12} = \exp \left( i \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(h) (q_r(x_1^+) q_s(x_2^+) - q_r(x_2^+) q_s(x_1^+)) \right)$$

Based on LO Bethe equations and classical spectral curve.

Bethe equations

$$1 = \prod_{j=1}^{K} \frac{x_j^+}{x_j^-} \prod_{j=1}^{K} \frac{x_j^+}{x_j^-}$$

$$1 = \left( \frac{x_k}{x_k} \right)^L \prod_{j=1}^{K} \left( \frac{u_k - u_j + ig^{-1} \sigma(u_k, u_j)}{u_k - u_j - ig^{-1} \sigma(u_k, u_j)} \right) \prod_{j=1}^{K} \sigma(u_k, u_j) \prod_{j=1}^{K} \frac{x_k^+ - y_j}{x_k^- - y_j}$$

$$1 = \left( \frac{x_k}{x_k} \right)^L \prod_{j=1}^{K} \left( \frac{u_k - u_j + ig^{-1} \sigma(u_k, u_j)}{u_k - u_j - ig^{-1} \sigma(u_k, u_j)} \right) \prod_{j=1}^{K} \sigma(u_k, u_j) \prod_{j=1}^{K} \frac{x_k^+ - y_j}{x_k^- - y_j}$$

$$1 = \prod_{j=1}^{M} \frac{y_k + 1/y_k - w_j + \frac{1}{2}g^{-1}}{y_k + 1/y_k - w_j - \frac{1}{2}g^{-1}} \prod_{j=1}^{K} \frac{y_k - x_j^+}{y_k - x_j^-} \prod_{j=1}^{K} \frac{y_k - x_j^+}{y_k - x_j^-}$$

$$1 = \prod_{j=1}^{M} \frac{y_k + 1/y_k - w_j + \frac{1}{2}g^{-1}}{y_k + 1/y_k - w_j - \frac{1}{2}g^{-1}} \prod_{j=1}^{K} \frac{y_k - x_j^+}{y_k - x_j^-} \prod_{j=1}^{K} \frac{y_k - x_j^+}{y_k - x_j^-}$$

magic coefficients

$$c_{r,s}(h) = 2 \sin \left[ \frac{\pi}{2} (s - r) \right] (r - 1)(s - 1) \int_{0}^{\infty} dt \frac{J_{r-1}(t) J_{s-1}(t)}{t e^{(2h - 1)t}}$$

Varna 08, Niklas Beisert
Finite-Twist Operators
Finite-Twist Operators

Consider deep inelastic scattering (in any 4D gauge theory)

- Operators of lowest twist $T_\mathcal{O} = D_\mathcal{O} - S_\mathcal{O}$ are dominant: $\mathcal{O} = \text{Tr} \mathcal{D}^S \Phi^T$.

- Scaling violations: Anomalous dimensions $\delta D_\mathcal{O}$ of twist-two operators. $D_\mathcal{O}$ can be expressed through generalised harmonic sums cf. [Vermaseren hep-ph/9806280]

$$ S_{\pm n}(k) = \sum_{j=1}^{k} \frac{(\pm 1)^j}{j^n}, \quad S_{\pm n,m,\ldots}(k) = \sum_{j=1}^{k} \frac{(\pm 1)^j}{j^n} S_{m,\ldots}(j). $$

Calculation of twist-2 dimension

$$ \delta D_S \sim \lambda S_1(S) + \lambda^2 (S_3(S) + \ldots) + \ldots. $$

High-spin limit: cusp dimension $D_S \sim D_{\text{cusp}} \log S$

$$ \pi^2 D_{\text{cusp}} = \frac{1}{2} \lambda - \frac{1}{96} \lambda^2 + \frac{11}{23040} \lambda^3 + \ldots. $$
Compute cusp dimension using Bethe equations. **Integral eq.:**

\[ \psi(x) = K(x, 0) - (K \ast \psi)(x), \quad A \ast B = \int_{0}^{\infty} A(y) \frac{dy \ y}{e^{2\pi y/\sqrt{\lambda}} - 1} B(y). \]

Cusp dimension: \( \pi^2 D_{\text{cusp}} = \lambda \psi(0) \). Kernel \( K = K_0 + K_1 + K_d \)

\[ K_{0,1}(x, y) = \pm \frac{x \ J_{1,0}(x) \ J_{0,1}(y) - y \ J_{0,1}(x) \ J_{1,0}(y)}{x^2 - y^2}, \quad K_d = 2K_1 \ast K_0. \]

Prediction for cusp dimension from integrability

\[ \pi^2 D_{\text{cusp}}(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{96} + \frac{11\lambda^3}{23040} - \left( \frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6} \right) \lambda^4 \pm \ldots, \]

\[ \pi E_{\text{cusp}}(\lambda) = \sqrt{\lambda} - 3 \log 2 - \frac{\beta(2)}{\sqrt{\lambda}} + \ldots, \quad \text{agreement!} \]
Gluon Scattering Amplitudes

Use gluon scattering to verify conjecture at weak coupling.
Four-gluon scattering amplitude obeys “iteration” relation

\[ A(p, \lambda) \sim A^{(0)}(p) \exp \left( 2D_{\text{cusp}}(\lambda)M^{(1)}(p) \right) . \]

Only required data: • tree level, • one loop, • cusp dimension.
Gluon scattering amplitudes constrained by unitarity.
Higher-loop supersymmetric amplitude constructible by suitable ansatz. 4-loop result in agreement with Bethe equations.

Short detour on scattering amplitudes
• Dual superconformal symmetry.
• Scattering amplitudes and light-like Wilson loops.
Detour: Dual Superconformal Symmetry

Planar amplitudes and integrals simpler than expected:

- No bubbles, no triangles, typically scalar boxes.
- Dual amplitudes and integrals are conformal.
- Similarity of momentum and position space propagators in $D = 4$.

Underlying symmetry:

- T-dual string model equivalent to original model. T-self-duality!
- Fermions require also fermionic T-duality (bosonic!).
- Dual (super)conformal symmetry = symmetry of dual model.
- Dual superconformal symmetry from string integrability.
**Detour: Light-Like Wilson Loops**

How to relate gluon scattering to a Wilson loop?

- light-like momenta $p_k^2 = 0$
- momentum conservation $\sum_k p_k = 0$
- polarisations

Set $p_k = \Delta x_k$ and match Wilson loop expectation value with amplitude.

- Follows from T-duality in string theory.
- Generic one-loop agreement.
- Four legs at two loops: Agreement (adjust renormalisation).
Finite Spins

Back: Consider operators with finite $S, T$.

Twist-2 scaling dimensions

\[
D_S = T + \frac{\lambda}{2\pi^2} S_1 - \frac{\lambda^2}{16\pi^4} (S_3 + S_{-3} - 2S_{-2,1} + 2S_1S_2 + 2S_1S_{-2}) + \frac{\lambda^3}{64\pi^6} (S_5 + \ldots) + \ldots .
\]

Asymptotic Bethe ansatz only valid to three loops.

Further related results:

- Easily diagonalisable for two-parameter family.
  Three loop results from Bethe equations.
- Generalised BES equation with soft twist-dependence.
Finite Coupling

Twist-2, spin-2 state is part of Konishi multiplet.

Four-loop Feynman diagrams

\[
D = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} - \frac{(156 - 36\zeta(3) + 90\zeta(5))\lambda^8}{4096\pi^8} + \ldots .
\]

Prediction from asymptotic Bethe ansatz correct to 3 loops.

Include Lüscher terms at four loops
Sum over bound states propagating around circle.
Precise agreement at four loops.

Agreement also at strong coupling.
Cusp for $AdS_4 \times \mathbb{CP}^3/\mathcal{N} = 6$ CSC?

What about finite-twist operators for $AdS_4 \times \mathbb{CP}^3/\mathcal{N} = 6$ CSC?

- Bethe equations almost the same.
  BES equation applies with proper definition of $g$.

$$2\pi E_{\text{cusp}}(\lambda) = \sqrt{g} - 3 \log 2 + \ldots .$$

- Semi-classical string theory result

$$\pi E_{\text{cusp}}(\lambda) = \sqrt{g} - 5 \log 2 + \ldots .$$

- Discrepancy caused by different regularisation of sum over modes.
- Apparent resolution: finite renormalisation of $g$ between schemes.
Problems in $AdS_4 \times \mathbb{CP}^3 / \mathcal{N} = 6$ CSC?

Coupling constant:

- Superficially physical because $\lambda = N/k_{CS}$ is rational.
- Apparently unphysical at strong coupling (regularisation-dependent).
- Note: function $g(\lambda)$ also unphysical in $AdS_5 \times S^5 / \mathcal{N} = 4$ SYM!

Lack of data/results and integrability assumption

- Algebraic constructions at weak coupling? Multiplets? Wrapping?
- Dressing phase the same as for $AdS_5 \times S^5 / \mathcal{N} = 4$ SYM?
- Absence of 8/16 magnons at weak coupling?! Problem?
Conclusions
Conclusions

★ Planar AdS/CFT Correspondence
- String theory & coordinate space Bethe ansatz for gauge theory: Exciting particle model.
- Residual symmetry: involves $\mathfrak{psu}(2|2)$ with central extensions.
- Unique S-matrix constructed. YBE automatically satisfied.
- Proposal for interpolating phase factor at strong and weak coupling.
- Cusp anomalous dimension computed at finite coupling.
- Full agreement with AdS/CFT!

★ Open Questions
- Promote integrability in $AdS_4 \times \mathbb{CP}^3/\mathcal{N} = 6$ CSC to solid ground.
- Find exact finite-size equations.
- Mathematical structure of integrable system.