Matrix Elements of Operators in the Thermodynamic Limit of Rational Integrable Models

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New mathematical methods in solvable models and gauge/string dualities

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- Closely related=Matsuo-Kostov determinant:

$$\langle \tilde{\boldsymbol{\theta}} | \boldsymbol{\theta} \rangle = e^{\frac{\imath L}{2} \sum_j \theta_j - \tilde{\theta}_j} \det(\mathbb{1} - K), \qquad K_{ij} = \frac{E_i}{u_i - u_j + \imath c}$$

$$E_i \equiv e^{-\imath L u_i} \frac{\prod_k (u_i - u_k + \imath c)}{\prod_{k \neq i} (u_i - u_k)}, \qquad \boldsymbol{u} \equiv \boldsymbol{\theta} \cup \tilde{\boldsymbol{\theta}}$$

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- We need to invert $1 - e^{-Ny}\mathcal{K}$, for $\mathcal{K}$.
(We have yet to discuss $\mathcal{K}$)

The Operator K

- $$K_{ij} = e^{-\imath L u_i} \frac{Q_{\boldsymbol{u}}(u_i+\imath c)}{Q'_{\boldsymbol{u}}(u_i)} \frac{1}{u_i - u_j + \imath c}$$

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- $e^{-\imath L x} \frac{Q_{\boldsymbol{u}}(x+\imath c)}{Q_{\boldsymbol{u}}(x)} \psi(x+\imath c)$ is a candidate for $\mathcal{K}\psi$.
- More precisely:

$$(\mathcal{K}\psi)(y) = \oint_{\boldsymbol{u}} \frac{e^{-\imath L x}}{y-x} \frac{Q_{\boldsymbol{u}}(x+\imath c)}{Q_{\boldsymbol{u}}(x)} \psi(x+\imath c)$$

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The $\mathcal{K}$ operator is just $1+\mathcal{K}$ with:

$$\mathcal{K} = \mathcal{P} e^{\Phi} e^{\imath c \partial}$$

with

$$e^{\Phi(x)} = e^{-\imath Lx} \frac{Q_{\boldsymbol{u}}(x+\imath c)}{Q_{\boldsymbol{u}}(x)}, \quad (\mathcal{P}f)(x) = \oint \frac{f(y)}{x-y}$$

Taking the Trace of the Resolvent

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The trace is given by:

$$\mathrm{tr}(\mathbb{1} - e^{-Ny}K)^{-1} = \sum_j \operatorname*{Res}_{\lambda \to u_j}(R_y(\lambda, u_j)) =$$

$$= \oint \oint \frac{R_y(\lambda, w)}{\lambda - w} \frac{dz}{2\pi\imath} \frac{dw}{2\pi\imath},$$

where the w integral surrounds the λ integral.

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  1. The Sutherland limit
  2. The Richardson model (equivalent to BCS)

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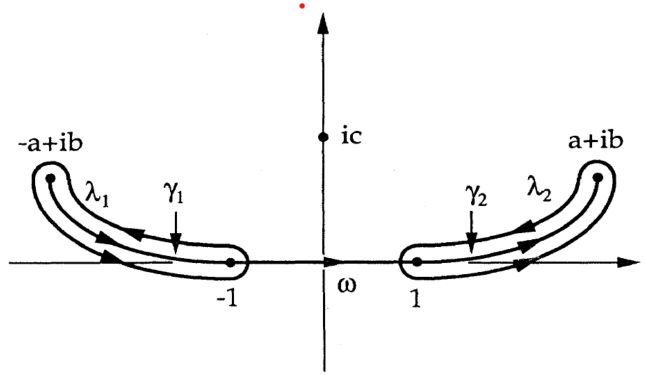

- The overlap between such states is written as:

$$\langle u|v\rangle = \oint \mathrm{Li}_2\left(e^{\imath c\Phi'(z)}\right) +$$

$$+\oint\oint \frac{\log\left(1-e^{\imath c\Phi'(z)}\right)\left(1-e^{\imath c\Phi'(z')}\right)}{(z-z')^2}$$

Technique for Sutherland case

- May be obtained by expanding tr $\log(1-\mathcal{K})$ with
$$(1-\alpha\mathcal{K})^{-1}\frac{1}{z-w}=\sum_n \alpha^n\left(\mathcal{P}e^{\Phi}e^{\imath c\partial}\right)^n\frac{1}{z-w}$$

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- Higer order terms gives rise to a Riemann Hilbert problem.

The Richardson Model

- $$H = \sum_{j,\sigma} \varepsilon_j c^\dagger_{j,\sigma} c_{j,\sigma} - \sum_{j,k} G c^\dagger_{j,\uparrow} c^\dagger_{j,\downarrow} c_{k,\uparrow} c_{k,\downarrow}$$

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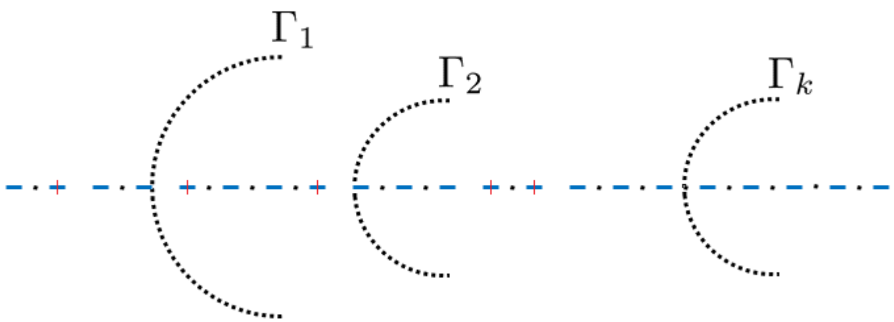
- Also a $c \to 0$ limit with macroscopic strings



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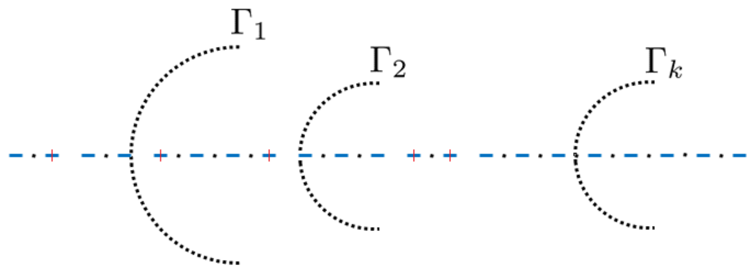
- 

- Semiclassically: $S_j^z = \frac{\wp(u_\infty - u_{\varepsilon_j}) + \wp(u_\infty + u_{\varepsilon_j}) + 2\frac{\eta}{\omega}}{2\sqrt{\prod_k \varepsilon_j - \lambda_k}}$

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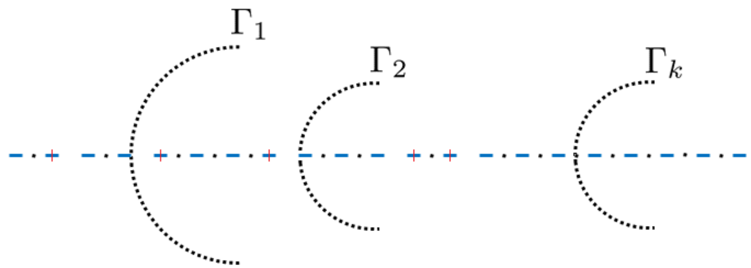
- 

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$$|\langle \text{in} - j + p | c_j | \text{in} \rangle|^2 55 = \frac{\pi^2 \sin^{-2}(u_{\varepsilon_j} - u_\infty + p\tau)}{4\omega^2 \sqrt{\prod_k \varepsilon_j - \lambda_k}}$$

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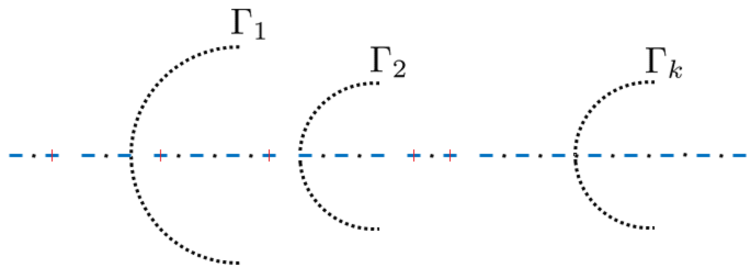
- 

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- Resolution of identity and $\sum_p$ gives agreement.

Solution through Saddle Point

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- To invert the matrix $\mathbb{1} - e^{-Ny}K$, solve:

$$R_y(z,w) - \oint \frac{e^{N(\Phi(z')-y)} R_y(z' + \imath c, w)}{z - z'} = \frac{1}{z - w}$$

with $w \in \boldsymbol{u}$ and $e^{N\Phi(x)} = e^{-\imath L x} \frac{Q_{\boldsymbol{u}}(x + \imath c)}{Q_{\boldsymbol{u}}(x)}$

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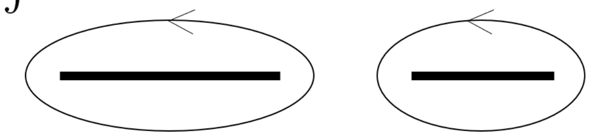
Fourier Transformed Integral Equation

The integral equation can be solved in the large N by first applying a Fourier transform.

$$\frac{1}{z-w} = R_y(z,w) - \oint \frac{e^{\Phi(z')-Ny} R_y(z'+\imath c, w)}{z-z'}$$

$$\oint e^{-N\imath Pz} R_y(z,w) = e^{NS(P,w)}$$

$$\oint e^{N(\Phi(z)-\imath Pz)} = e^{N\tilde{\varphi}(P)}$$



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For $P > 0$ this becomes $(e^{N\Phi(z)} \to e^{\imath N\tilde{\varphi}(P)})$:

$$e^{-\imath NPw} = e^{\imath NS(P,w)} - \int_0^\infty e^{\imath N[S(Q,w)+\imath cQ+\tilde{\varphi}(P-Q)]} dQ$$

Ansatz for Solving the Integral Equation

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$$e^{\imath NS(P,w)} = \begin{cases} -e^{-\imath N(\Phi(w)+(w+\imath c)P)} & P < -\frac{1}{c}\mathrm{Im}(\Phi(w)) \\ e^{-\imath NPw} & -\frac{1}{c}\mathrm{Im}(\Phi(w)) < P \end{cases}$$

Substituting the Ansatz

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Which is valid due to a saddle point justification.

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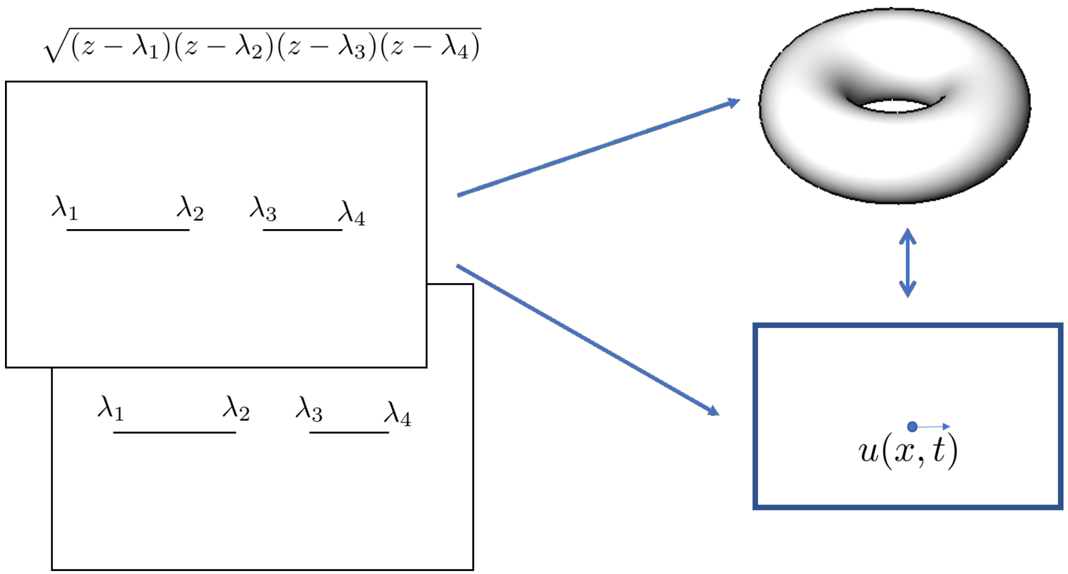
- The multi-gap spectral weight on the upper sheet of an operator from inverse scattering theory corresponds to density of Bethe roots.

The Semiclassics Field

Consider an elliptic Riemann surface



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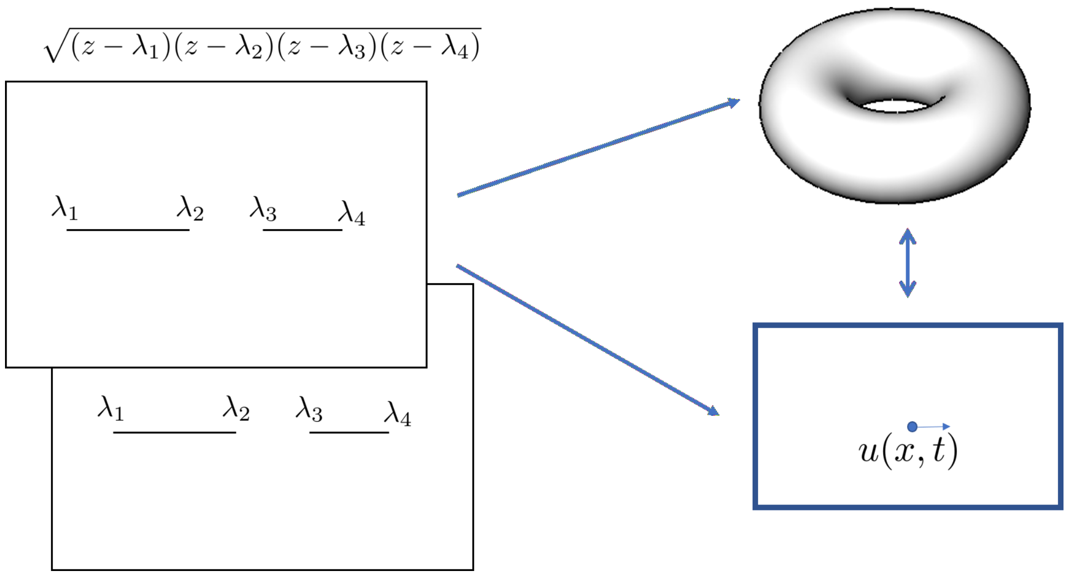

$$\psi(x,t) = e^{\frac{4\eta u_a}{\pi} u(x,t)} \frac{\sigma(u(x,t) - u_a)}{\sigma(u(x,t) + u_a)}, \qquad u(x,t) = k(x - vt)$$

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- The function $\Phi(z)$ apearing in the resolvent:

$$\Phi(z(u)) = \zeta(u+u_\infty) + \zeta(u-u_\infty) - \alpha u,$$

with $\zeta(u) = \partial \log \sigma(u)$

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$$\Phi(z(u)) = \zeta(u+u_\infty) + \zeta(u-u_\infty) - \alpha u,$$

- with $\zeta(u) = \partial \log \sigma(u)$

$$\langle \text{out}|\psi(0)|\text{in}\rangle = C \oint e^{\frac{4\eta u_a}{\pi} u(z)} \frac{\sigma(u(z)-u_a)}{\sigma(u(z)+u_a)} dz$$

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- The measure of integration is different but the object integrated over is the same

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- We approach the semiclassical limit (but do not actually take it).
- The comparison is suggestive that the approach is correct.