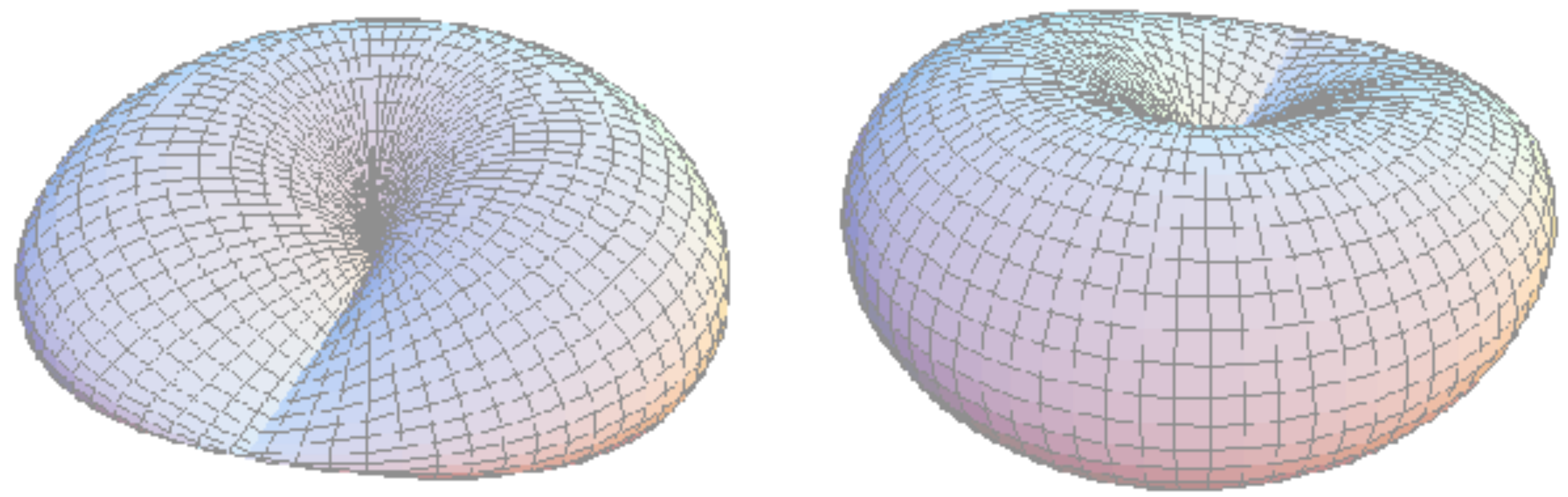


Unoriented QFTs

João Caetano

CERN

New Mathematical Methods in Solvable Models and Gauge/String Dualities

Varna 2022

Famously, unoriented QFTs appear in 2D as the worldsheet theory of unoriented strings

Unoriented string theory is relevant in many situations.

For example, massless unoriented open strings correspond to SO(N) or Sp(N) gauge bosons

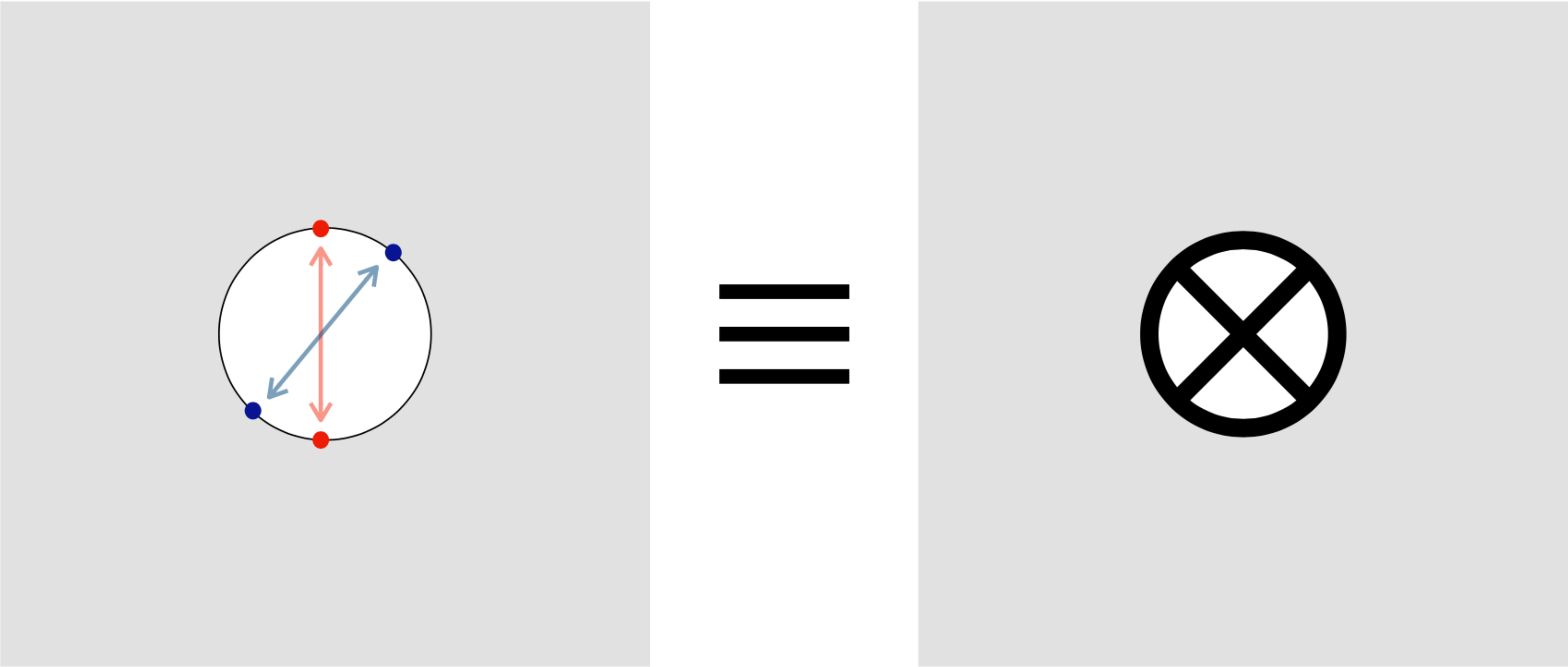
Other (related) examples are orientifolds in string theory which gauge worldsheet parity and give rise to unoriented strings.

These are relevant in string compactifications and (attempts of) constructing de-Sitter solutions in string theory.

Let’s then first talk about unoriented QFTs in 2D

In 2D, the way to construct unorientable manifolds where we will place the QFT is by adding crosscaps

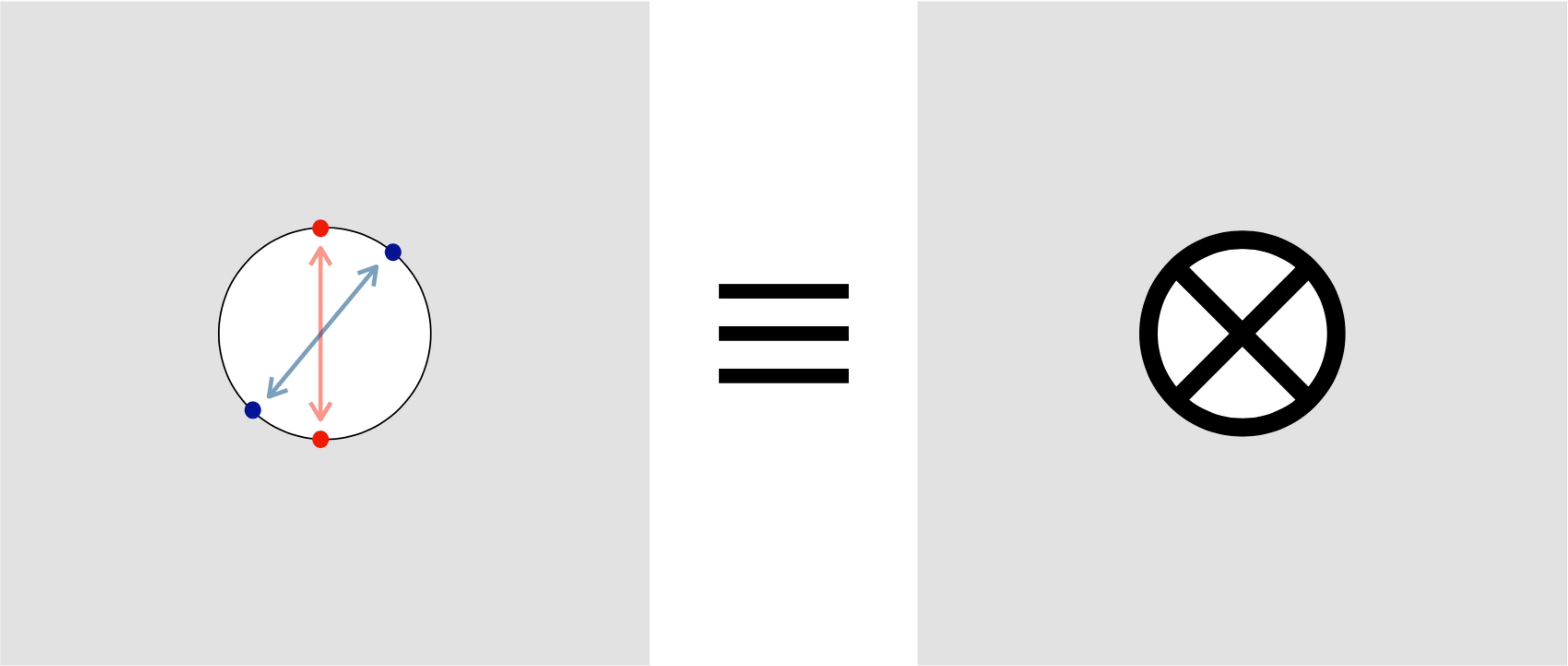
Crosscaps



The state created by this procedure is the **crosscap state**

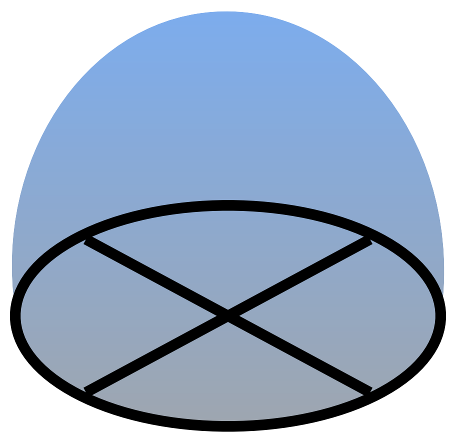
Crosscaps

$$z \sim -1/\bar{z}$$



The state created by this procedure is the **crosscap state**

- Insert one crosscap state on S^2: $\mathbb{RP}^2$



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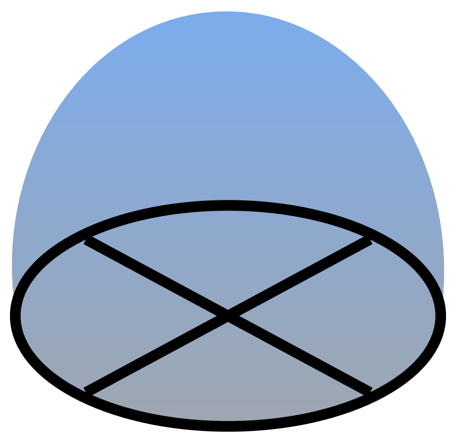

- Insert two crosscap states on S^2: Klein bottle

Non-orientable manifolds

Crosscap states have some analogies with boundary states

Boundary states in 2D are usually studied in 2 ways:

- Fixed points of RG and CFT techniques

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THE BOUNDARY AND CROSSCAP STATES IN CONFORMAL FIELD THEORIES

NOBUYUKI ISHIBASHI

Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan

Received 20 June 1988

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BOUNDARY CONDITIONS, FUSION RULES AND THE VERLINDE FORMULA

John L. CARDY

Department of Physics, University of California, Santa Barbara, CA 93106, USA

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Boundary operators in conformal field theory are considered as arising from the juxtaposition of different types of boundary conditions. From this point of view, the operator content of the theory in an annulus may be related to the fusion rules. By considering the partition function in such a geometry, we give a simple derivation of the Verlinde formula.

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Systematic construction of conformal boundary conditions

- Use integrable models (∞ conserved charges)

- Use integrable models (∞ conserved charges)

For special boundaries, called integrable boundaries, one can follow their RG flow

BOUNDARY *S* MATRIX AND BOUNDARY STATE IN TWO-DIMENSIONAL INTEGRABLE QUANTUM FIELD THEORY

SUBIR GHOSHAL* and ALEXANDER ZAMOLODCHIKOV†‡

Department of Physics and Astronomy, Rutgers University,
PO Box 849, Piscataway, NJ 08855-0849, USA

Received 29 November 1993

We study integrals of motion and factorizable S matrices in two-dimensional integrable field theory with boundary. We propose the "boundary cross-unitarity equation," which is the boundary analog of the crossing-symmetry condition of the "bulk" S matrix. We derive the boundary S matrices for the Ising field theory with boundary magnetic field and for the boundary sine–Gordon model.

For crosscap states, they have been studied in CFTs

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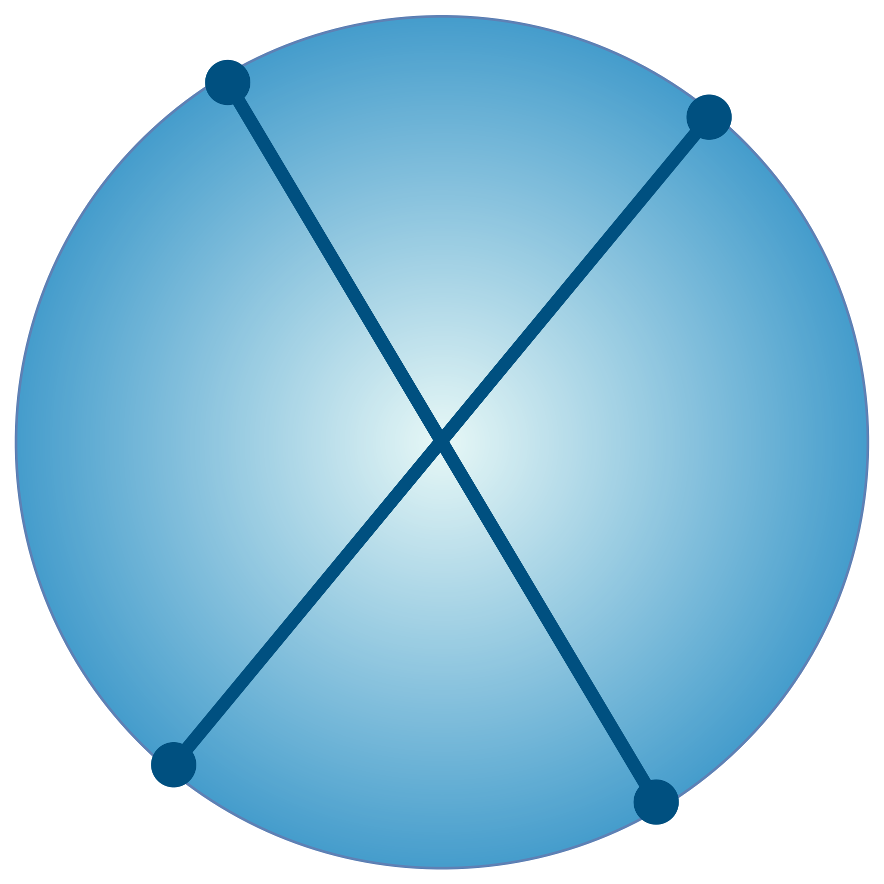
But never studied in integrable models (as far as I am aware…)

4 dimensions

The analog of the crosscap in 4D is $\mathbb{RP}^4$ (instead of $\mathbb{RP}^2$)

$$\mathbb{RP}^4 = S^4 \big/ \{X^\mu \sim -X^\mu\}$$



- Simplest unorientable 4-manifold
- Locally conformally flat, but **not** globally

Outline and outcomes

- A two-dimensional integrable model in flat space remains integrable on a crosscap

$$\Leftrightarrow$$

crosscap states preserve integrability

- $\mathcal{N} = 4$ Supersymmetric Yang-Mills on $\mathbb{RP}^4$ *with gauged charge conjugation* is integrable in the planar limit

- $\mathcal{N} = 4$ Supersymmetric Yang-Mills on $\mathbb{RP}^4$ *without gauged charge conjugation* is not integrable, but one can study it as well by holography + localization + bootstrap

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[JC, Rastelli' 22]

Crosscaps
in
2 dimensions

Crosscap overlaps $\langle \mathscr{C} \mid \Psi \rangle$

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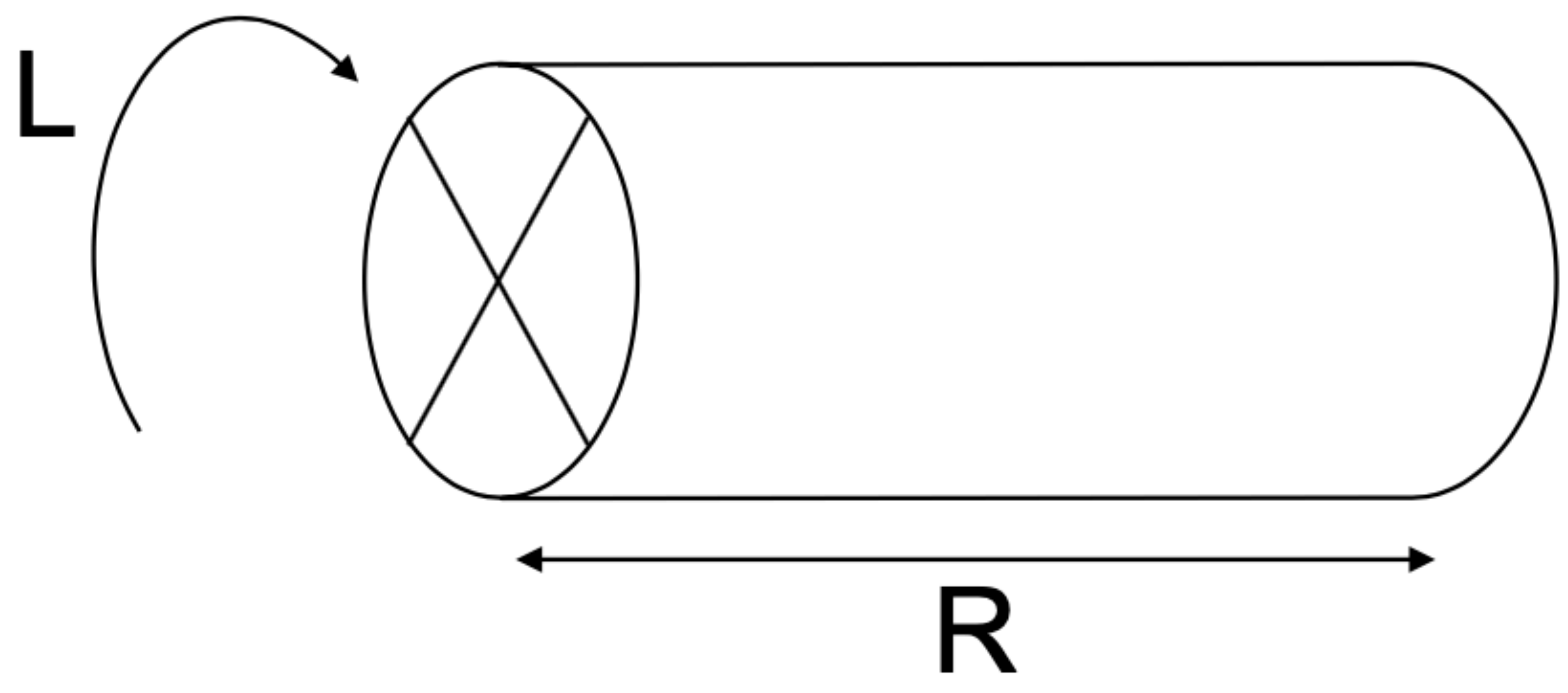
- Klein bottle partition function in two channels

Crosscap overlaps $\langle \mathscr{C} \,|\, \Psi \rangle$

- Klein bottle partition function in two channels

Tree channel (closed string)



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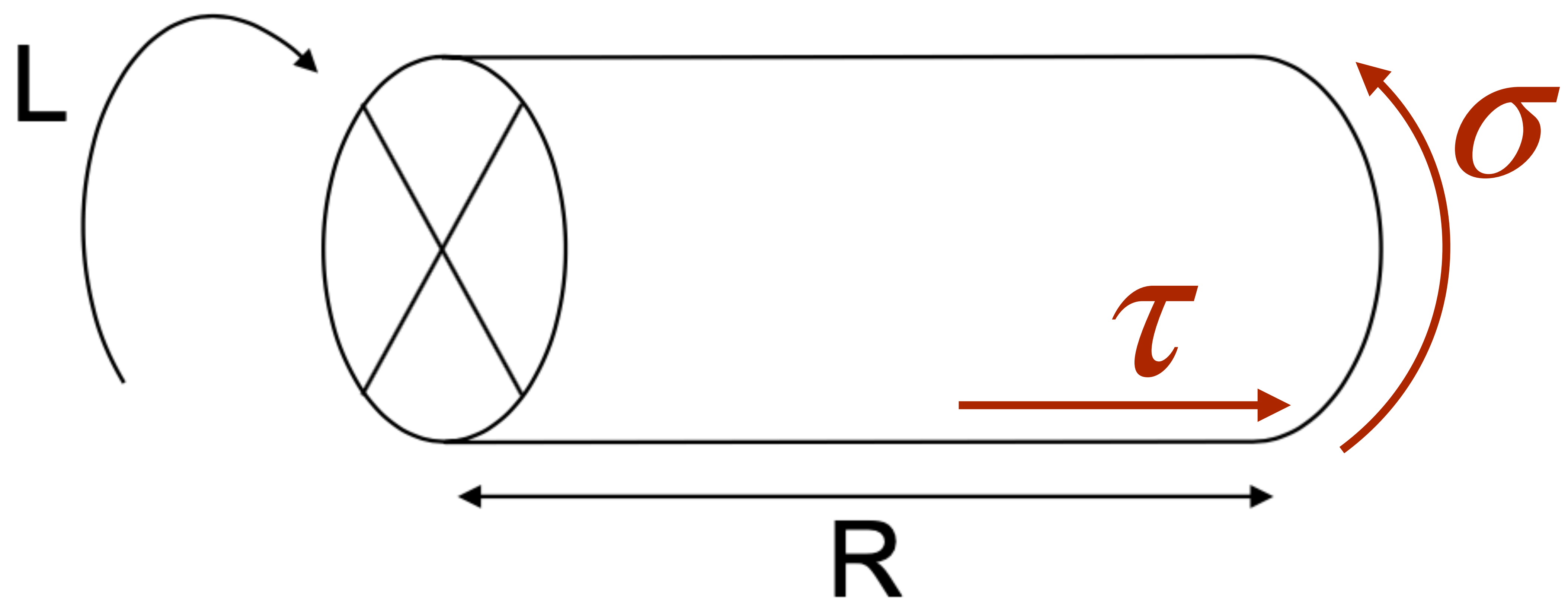

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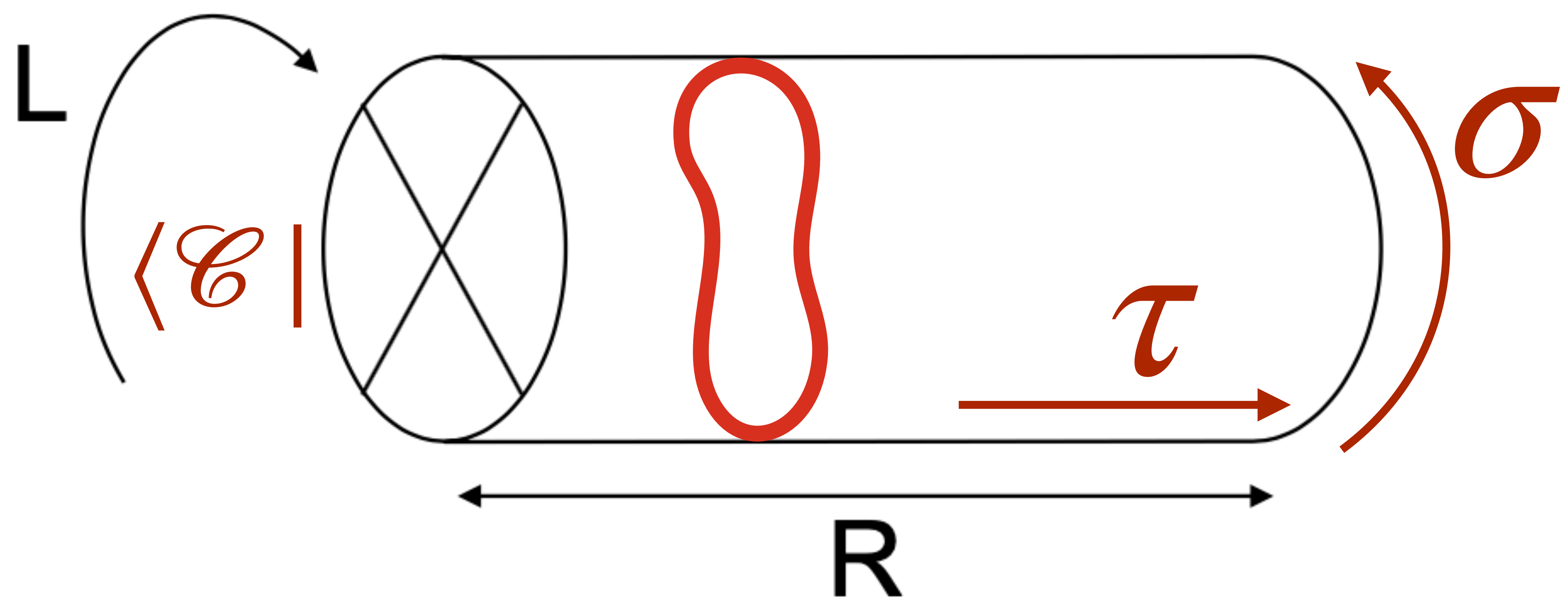

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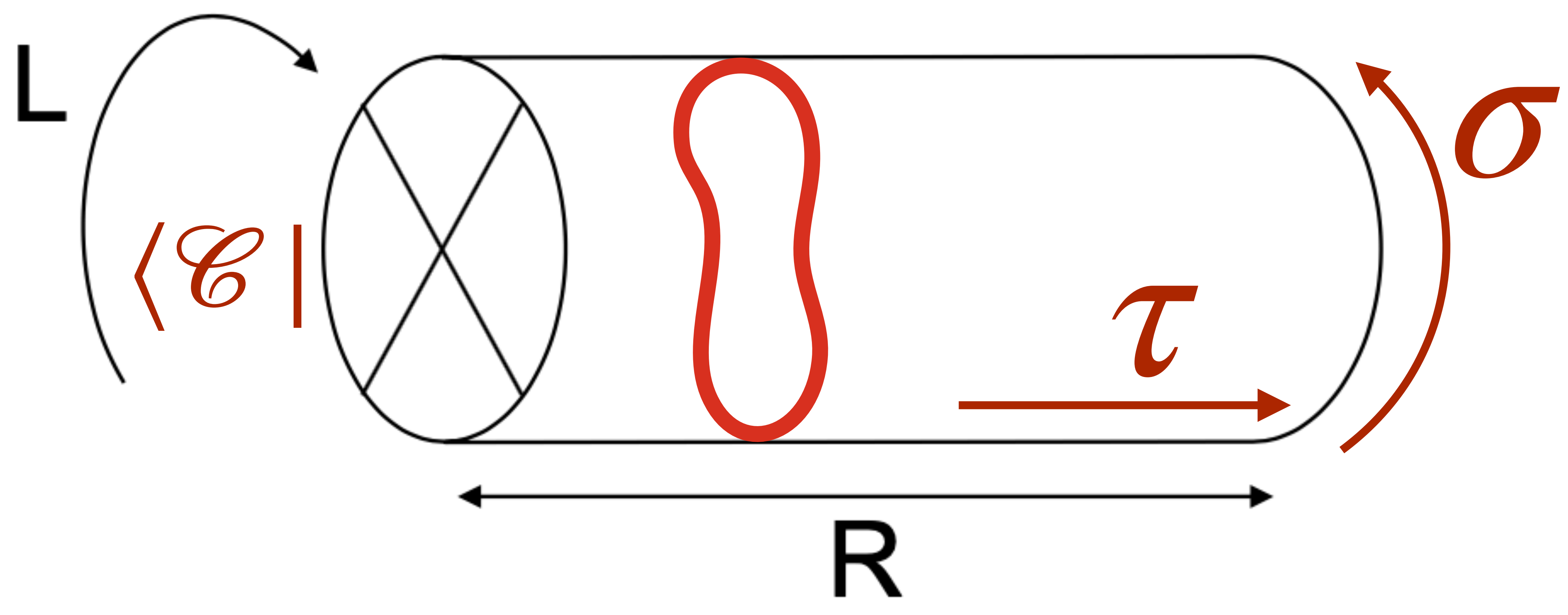

$$Z_{\mathbb{K}}(R,L) = \sum_{\psi_L} e^{-E_{\psi_L} R} \left| \langle \mathscr{C} | \psi_L \rangle \right|^2 \overset{R\to\infty}{=} e^{-E_{\Omega_L} R} \left| \langle \mathscr{C} | \Omega_L \rangle \right|^2 + \cdots$$

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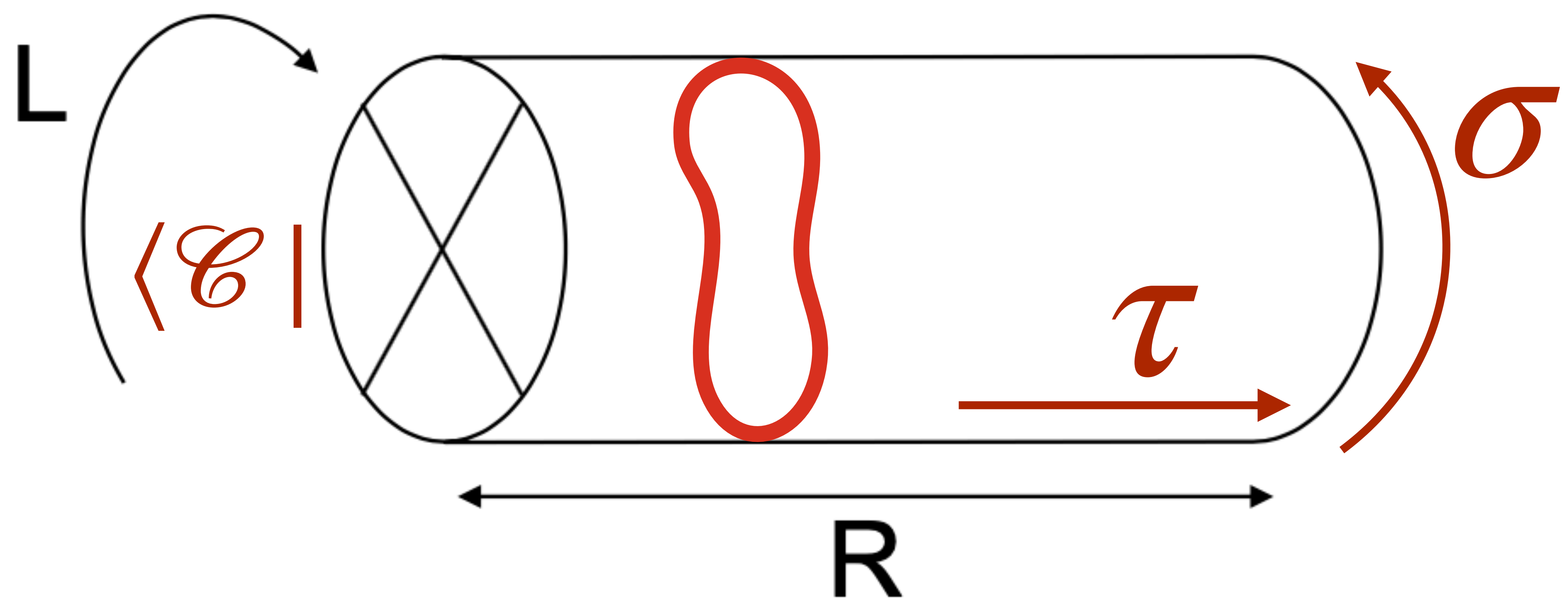

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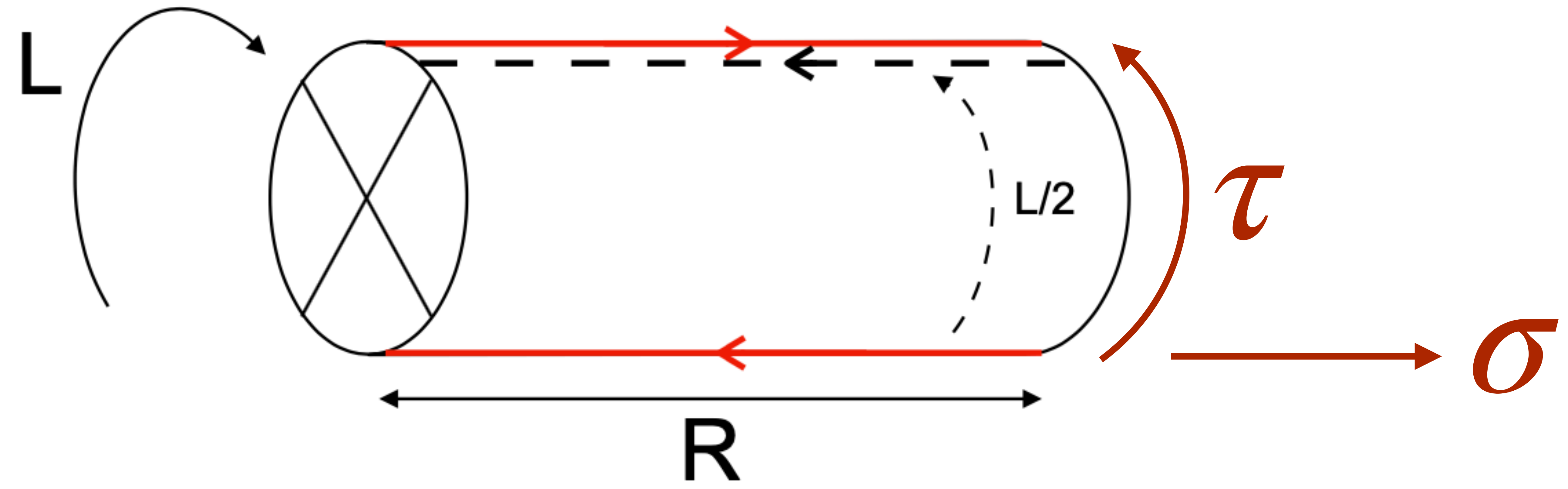
Ground state

Loop channel (open string)



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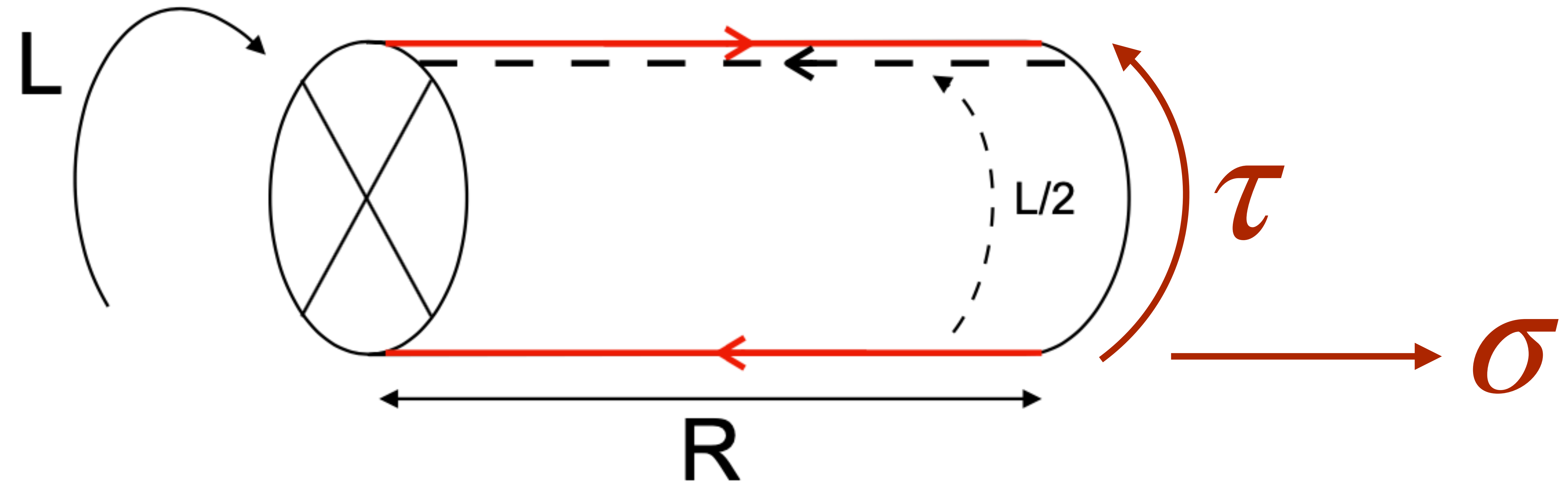
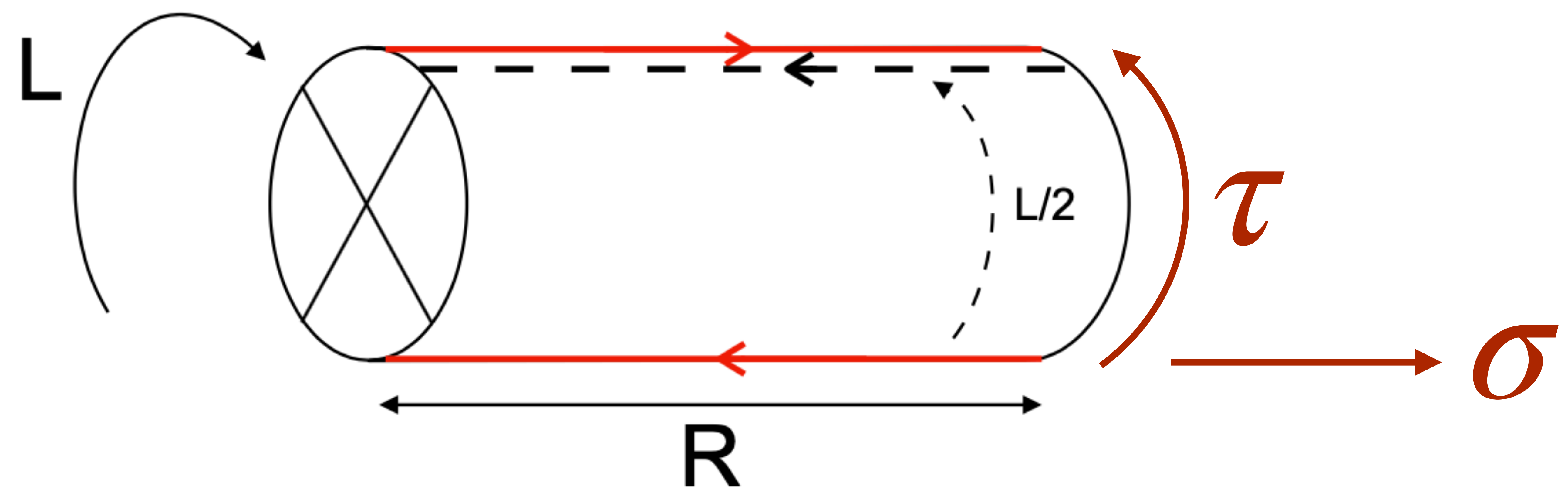


$$Z_{\mathbb{K}}(R,L) = \mathrm{Tr}_{2R}\left[\Pi\, e^{-HL/2}\right] = \sum_{\psi_{2R}} e^{-E_{\psi_{2R}} L/2} \langle \psi_{2R} | \Pi | \psi_{2R} \rangle$$

Parity operator

Loop channel (open string)



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Parity operator

$$= \sum_{\psi_{2R}} \epsilon_{\psi_{2R}} e^{-E_{\psi_{2R}}L/2}$$

Parity eigenstates

Parity eigenvalues ± 1

Loop channel (open string) = Tree channel (closed string)

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$$\lim_{R\to\infty} Z_{\mathbb{K}}(R,L) = \lim_{R\to\infty} \left[\sum_{\psi_{2R}} \epsilon_{\psi_{2R}} e^{-E_{\psi_{2R}} L/2} \right] \simeq e^{-E_{\Omega_L} R} \left| \langle \mathscr{C} \,|\, \Omega_L \rangle \right|^2$$

$\langle \mathscr{C} \,|\, \Omega_L \rangle$ controls the density of states weighted by the parity ϵ_ψ

Loop channel (open string) = Tree channel (closed string)

$$F_{\mathbb{K}} \equiv -\lim_{R\to\infty} \log Z_{\mathbb{K}}(R, L)$$

Parity-weighted **free energy**

Loop channel (open string) = Tree channel (closed string)

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***Parity-weighted* free energy**

$$= RE_{\Omega_L} - \log\left[\,|\langle \mathscr{C} \,|\, \Omega_L\rangle|^2\,\right] + O(1/R)$$

$$\lim_{R\to\infty} Z_{\mathbb{K}}(R, L) \simeq e^{-E_{\Omega_L}R}\left|\langle \mathscr{C} \,|\, \Omega_L\rangle\right|^2$$

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extensive piece

$\mathcal{O}(1)$ piece

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extensive piece

$\mathcal{O}(1)$ piece

- Same structure as the thermal free energy of a system with boundaries
- In that case, $\mathcal{O}(1)$ piece defines the **boundary entropy** or **g-function**

Crosscap overlap in Integrable models

$$\boxed{\lim_{R\to\infty} \mathrm{Tr}_{2R}\left[\Pi\, e^{-\hat{H}L/2}\right]} \simeq e^{-E_\Omega R} \left|\langle \mathscr{C} \,|\, \Omega_L \rangle\right|^2$$

Large volume partition function

(in integrable models)

$$\longleftrightarrow$$

Thermodynamic Bethe Ansatz + $\mathcal{O}(1)$ fluctuation

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- Energy eigenstates for $R \to \infty \Leftrightarrow M$ excitations labelled by $|\{p_j\}\rangle$

$$1 = e^{2ip_jR} \prod_{k\neq j} S(p_j, p_k)$$

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- For Bethe states with standard normalization: $\Pi\,|\{p_j\}\rangle = \mathbf{1}\,|\{-p_j\}\rangle$

Crosscap overlap in Integrable models

$$\mathrm{Tr}_{2R}\left[\Pi\, e^{-\hat{H}L/2}\right] = \boxed{\sum_{\{p_j\}=\{-p_j\}} e^{-\frac{L}{2}\sum_j E(p_j)}} \simeq e^{-E_\Omega R}\left|\langle \mathscr{C} \,|\, \Omega_L \rangle\right|^2$$

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Standard thermal sum

with the parity invariant constraint $\{p_j\} = \{-p_j\}$

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Standard thermal sum

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Apply **standard TBA** techniques to compute the **saddle point** and its **fluctuations**

Crosscap overlap in Integrable models

Result: **"Simplest"** g-function

$$\left|\langle \mathscr{C} | \Omega_L \rangle\right| = \sqrt{\left(1 + \sqrt{\frac{Y(0)}{1 + Y(0)}}\right) \frac{\det\left[1 - \hat{G}_-\right]}{\det\left[1 - \hat{G}_+\right]}}$$

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Y-function $$0 = LE(u) + \log Y(u) - \log(1+Y) \star \mathscr{K}_+(u)$$

Dispersion relation

$$\mathscr{K}_\pm(u,v) = \frac{1}{i}\partial_u \left[\log S(u,v) \pm \log S(u,-v)\right]$$

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Fredholm determinants: $\hat{G}_\pm \cdot f(u) = \int_0^\infty \frac{dv}{2\pi}\frac{\mathscr{K}_\pm(u,v)}{1+1/Y(v)}f(v)$

- Can be generalized for any **excited state** $|\langle\mathscr{C}|\Psi_L\rangle|$ using analytic continuation of this formula, similar to Dorey-Tateo trick.

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$$\hat{G}^{\bullet}_{\pm}\cdot f(u) = \sum_k \frac{i\mathscr{K}_{\pm}(u,u_k)}{\partial_u \log Y(\tilde{u}_k)} f(\tilde{u}_k) + \int_0^{\infty}\frac{dv}{2\pi}\frac{\mathscr{K}_{\pm}(u,v)}{1+1/Y(v)}f(v)$$

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- Asymptotic limit

$$|\langle \mathscr{C} | \Psi_L \rangle| \overset{L\to\infty}{=} \sqrt{\frac{\det G_+}{\det G_-}}$$

$$\left(G_{\pm}\right)_{1 \le i,j \le \frac{M}{2}} = \left[L\partial_u p(u_i) + \sum_{k=1}^{\frac{M}{2}} \mathscr{K}_{+}(u_i, u_k)\right] \delta_{ij} - \mathscr{K}_{\pm}(u_i, u_j)$$

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Crosscap states in spin chains

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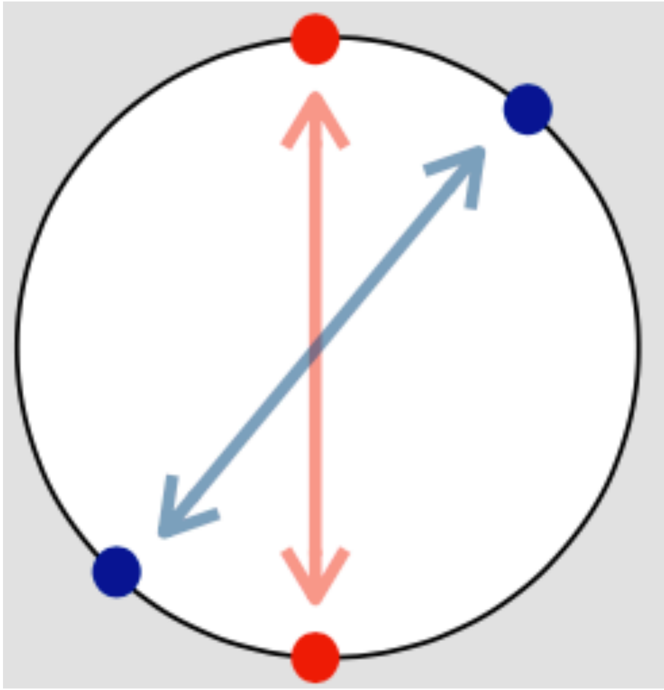
- XXX SU(2) spin chain $H_{\mathrm{SU(2)}} \propto \sum_j \overrightarrow{S}_j \overrightarrow{S}_{j+1}$

Crosscap states in spin chains



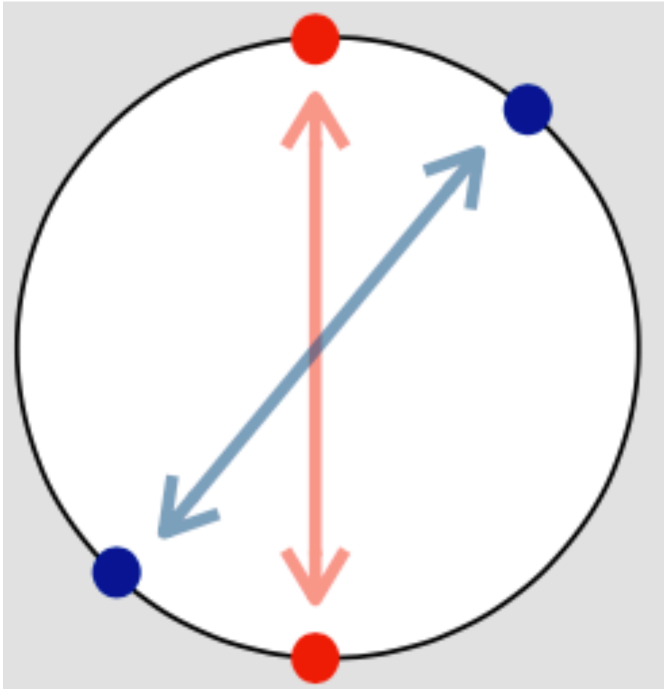
- XXX SU(2) spin chain $\quad H_{\mathrm{SU(2)}} \propto \sum_j \vec{S}_j \vec{S}_{j+1}$

- Mimic the definition in field theory: identify states on antipodal sites of the chain:

$$|c\rangle\rangle_j \equiv |\uparrow\rangle_j \otimes |\uparrow\rangle_{j+\frac{L}{2}} + |\downarrow\rangle_j \otimes |\downarrow\rangle_{j+\frac{L}{2}}$$

site j

Crosscap states in spin chains



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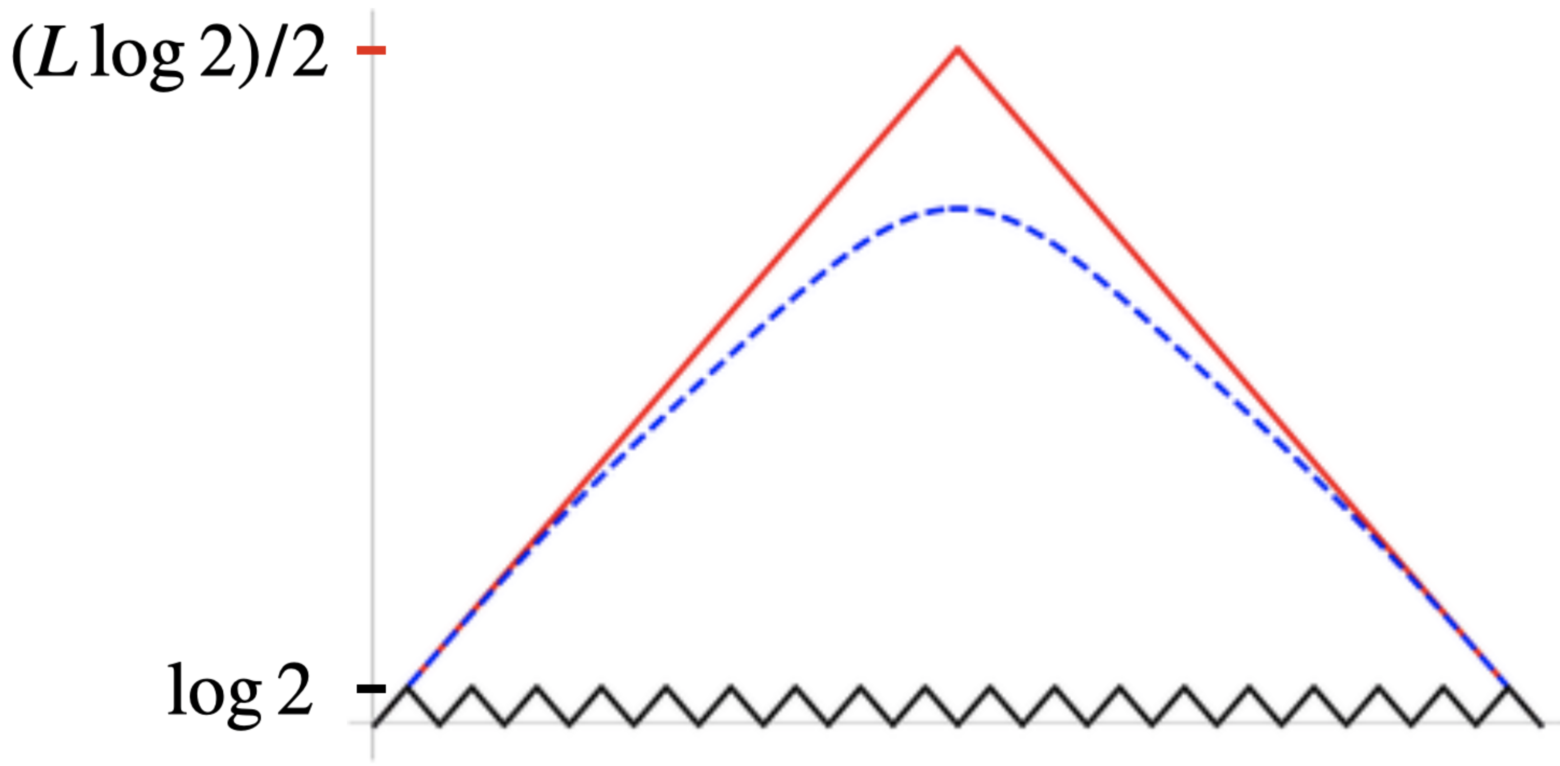
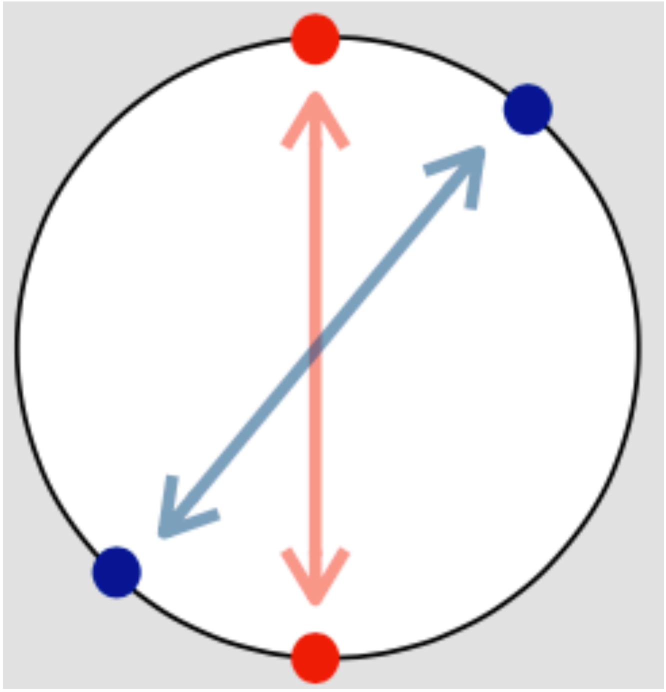


$$|\, \mathscr{C} \rangle \equiv \prod_{j=1}^{\frac{L}{2}} \Big(|\, c \rangle\!\rangle_j \Big)^{\otimes}$$

Long-range entangled
(As opposed to the short-range entangled in spin chain boundary state)

$$|\, b \rangle\!\rangle_j \sim \# |\uparrow\,\rangle_j \otimes |\uparrow\,\rangle_{j+1} + \# |\downarrow\,\rangle_j \otimes |\uparrow\,\rangle_{j+1} + \# |\uparrow\,\rangle_j \otimes |\downarrow\,\rangle_{j+1}$$

Crosscap states in spin chains



- XXX SU(2) spin chain $$H_{\mathrm{SU(2)}} \propto \sum_j \vec{S}_j \vec{S}_{j+1}$$

- Mimic the definition in field theory: identify states on antipodal sites of the chain:

$$|c\rangle\!\rangle_j \equiv |\uparrow\rangle_j \otimes |\uparrow\rangle_{j+\frac{L}{2}} + |\downarrow\rangle_j \otimes |\downarrow\rangle_{j+\frac{L}{2}}$$

site j

$$|\mathscr{C}\rangle \equiv \prod_{j=1}^{\frac{L}{2}} \left(|c\rangle\!\rangle_j \right)^{\otimes}$$

Long-range entangled
(As opposed to the short-range entangled in spin chain boundary state)

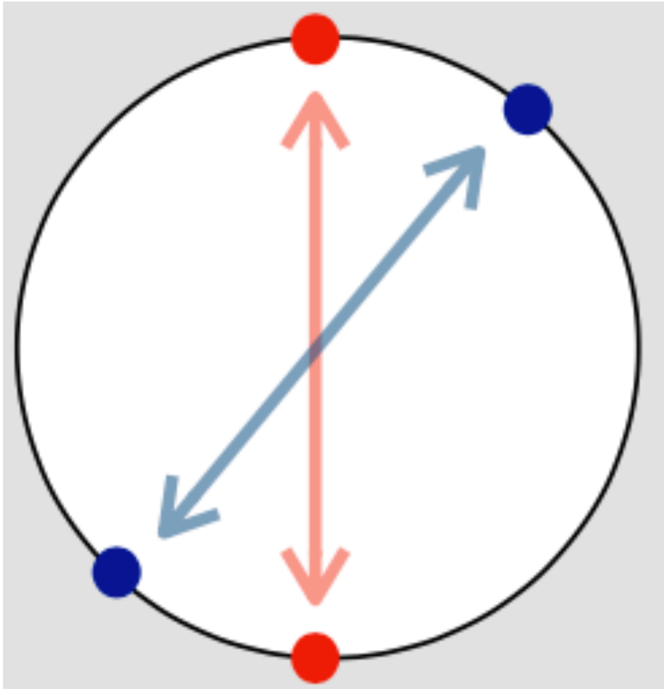
- One can show: $$\big(T(u) - T(-u)\big)\,|\mathscr{C}\rangle = 0 \Leftrightarrow Q_{2n+1}|\mathscr{C}\rangle = 0$$

(∞ many conserved charges)

[Ghoshal, Zamolodchikov]

[Piroli, Pozsgay, Vernier]

Crosscap states in spin chains



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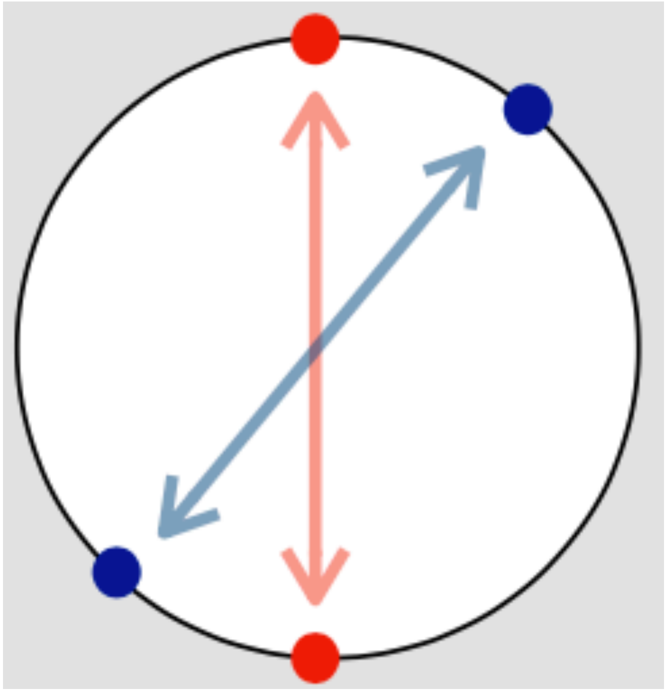

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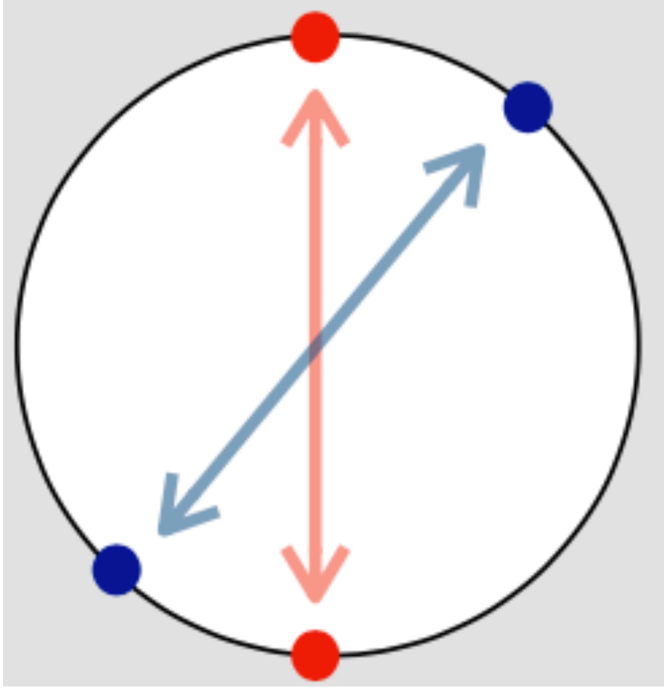

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Bethe state

$$\frac{\langle \mathscr{C} | \mathbf{u} \rangle}{\sqrt{\langle \mathbf{u} | \mathbf{u} \rangle}} = \sqrt{\frac{\det G_+}{\det G_-}}$$

Proven recently:

[Gombor'22]
[Ekman'22]

Gaudin type matrix: $(G_\pm)_{1\le i,j\le \frac{M}{2}} = \left[L\partial_u p(u_i) + \sum_{k=1}^{\frac{M}{2}} \mathscr{K}_+(u_i,u_k) \right] \delta_{ij} - \mathscr{K}_\pm(u_i,u_j)$

Crosscap states in spin chains

$$\frac{\langle \mathscr{C} \,|\, \mathbf{u} \rangle}{\sqrt{\langle \mathbf{u} \,|\, \mathbf{u} \rangle}} = \sqrt{\frac{\det G_+}{\det G_-}}$$

Boundary overlap: $$\frac{\langle \mathscr{B} \,|\, \mathbf{u} \rangle}{\sqrt{\langle \mathbf{u} \,|\, \mathbf{u} \rangle}} = (\text{non-universal factor}) \times \sqrt{\frac{\det G_+}{\det G_-}}$$

N=4 SYM on $\mathbb{RP}^4$
&
worldsheet crosscap

N=4 SYM on $\mathbb{RP}^4$

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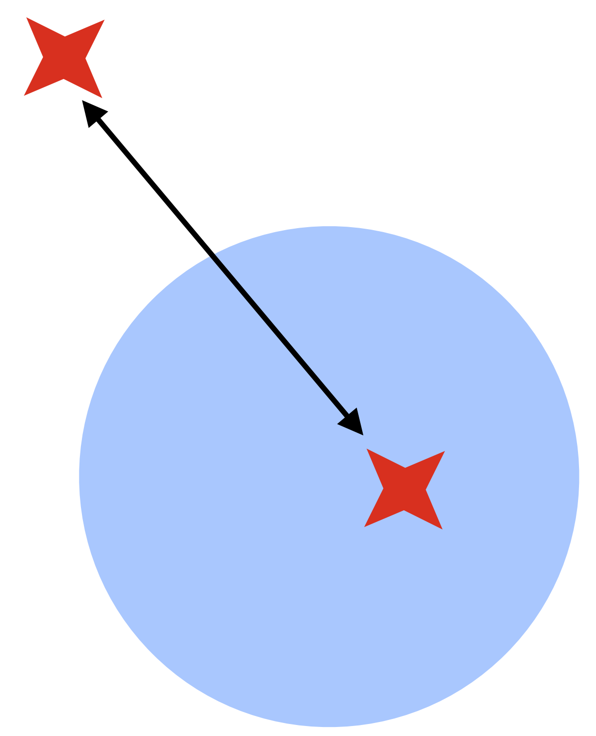
- CFT$_d$ on $\mathbb{RP}^d$:

$$\mathfrak{so}(d+1,1) \to \mathfrak{so}(d+1) : \quad K_\mu - P_\mu \,, \;\; M_{\mu\nu} \quad \cancel{D}$$



$$x^\mu \leftrightarrow -\frac{x^\mu}{|x|^2}$$

N=4 SYM on $\mathbb{RP}^4$

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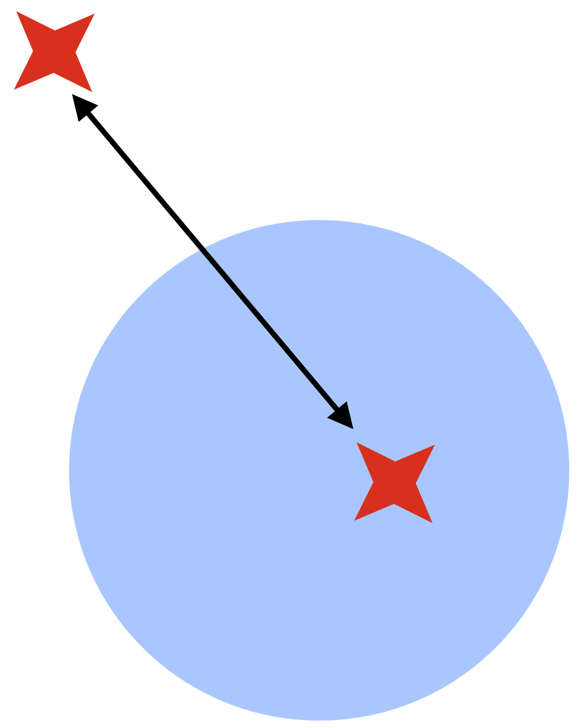

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- New CFT data: $\langle \mathcal{O} \rangle$

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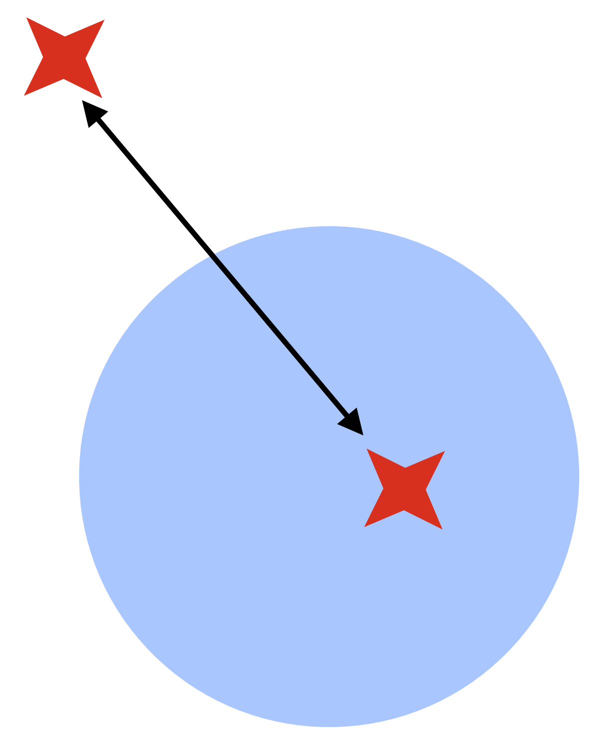

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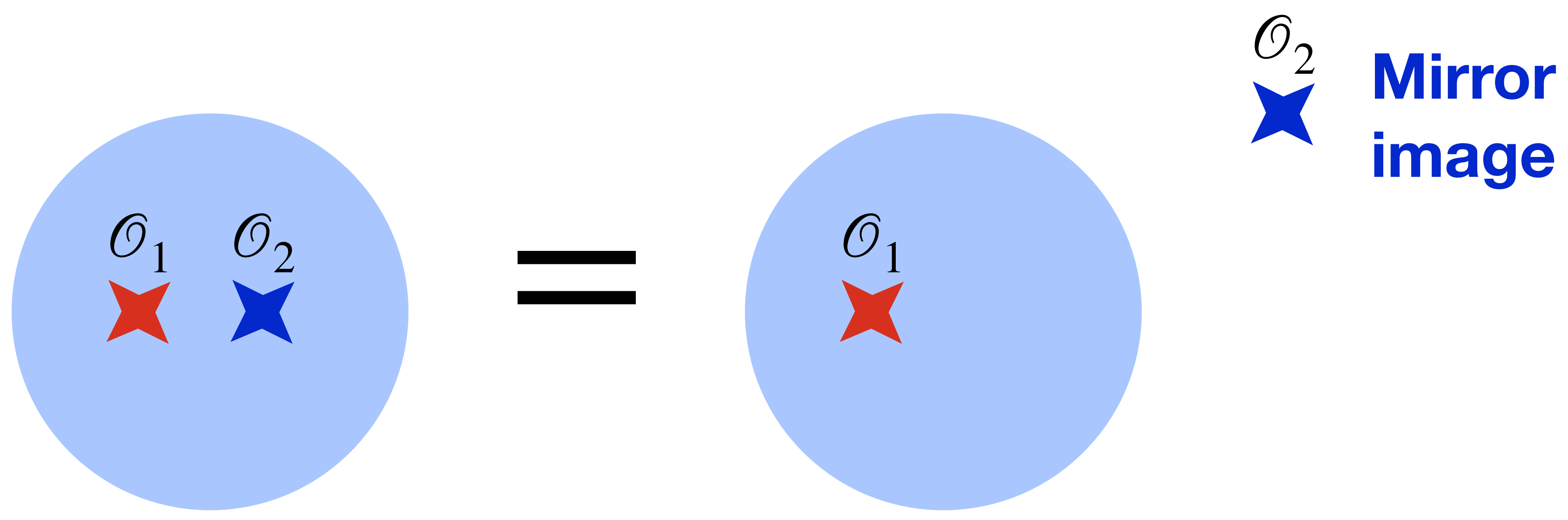
- New CFT data: $\langle \mathcal{O} \rangle$

- Local OPE data (OPE coeffs and dimensions) remains the same

- New bootstrap condition:



N=4 SYM on $\mathbb{RP}^4$

- Antipodal map on susy generators: $Q \leftrightarrow S$
- Preserves half supersymmetry "$Q + S$": **1/2-BPS setup**
- Field identification on antipodal points is fixed by spacetime symmetries:

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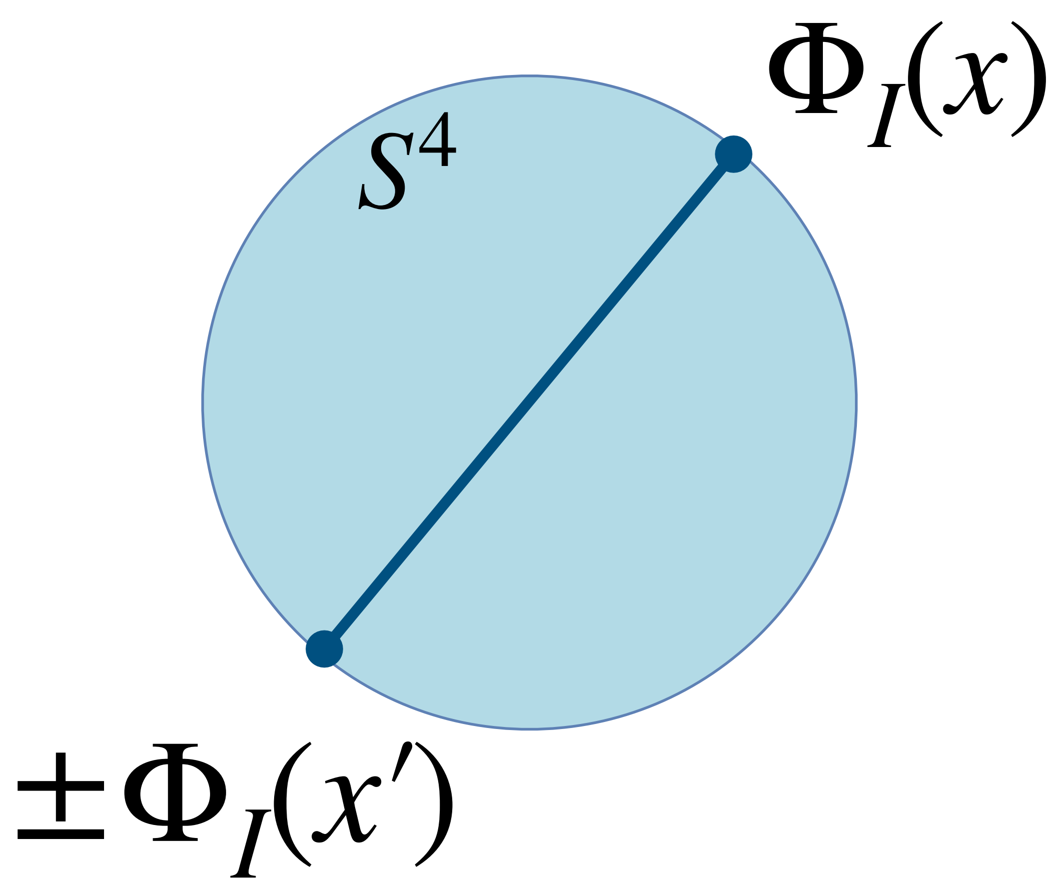

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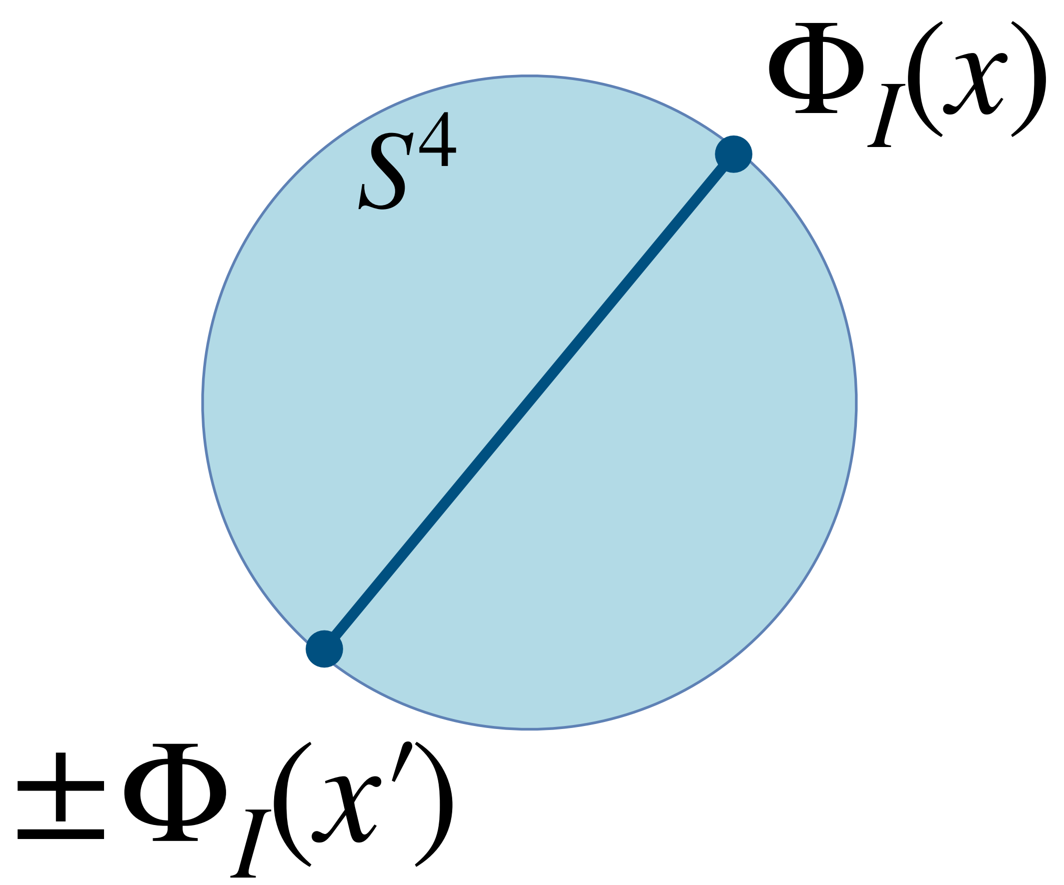

$$\Phi_{1,2,3} \leftrightarrow \Phi_{1,2,3} \quad \Phi_{4,5,6} \leftrightarrow -\Phi_{4,5,6} \qquad \mathrm{R-symmetry:} \quad SO(6) \rightarrow SO(3) \times SO(3)$$

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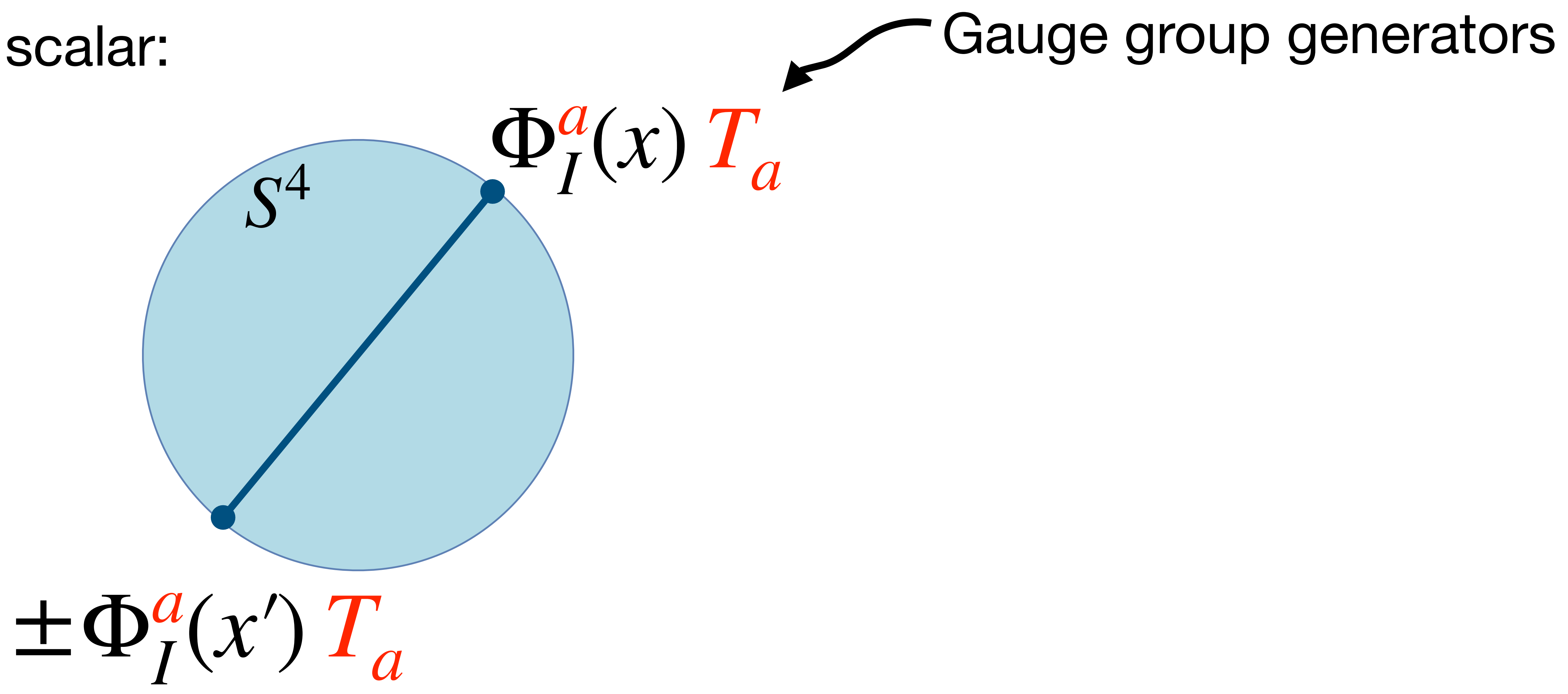

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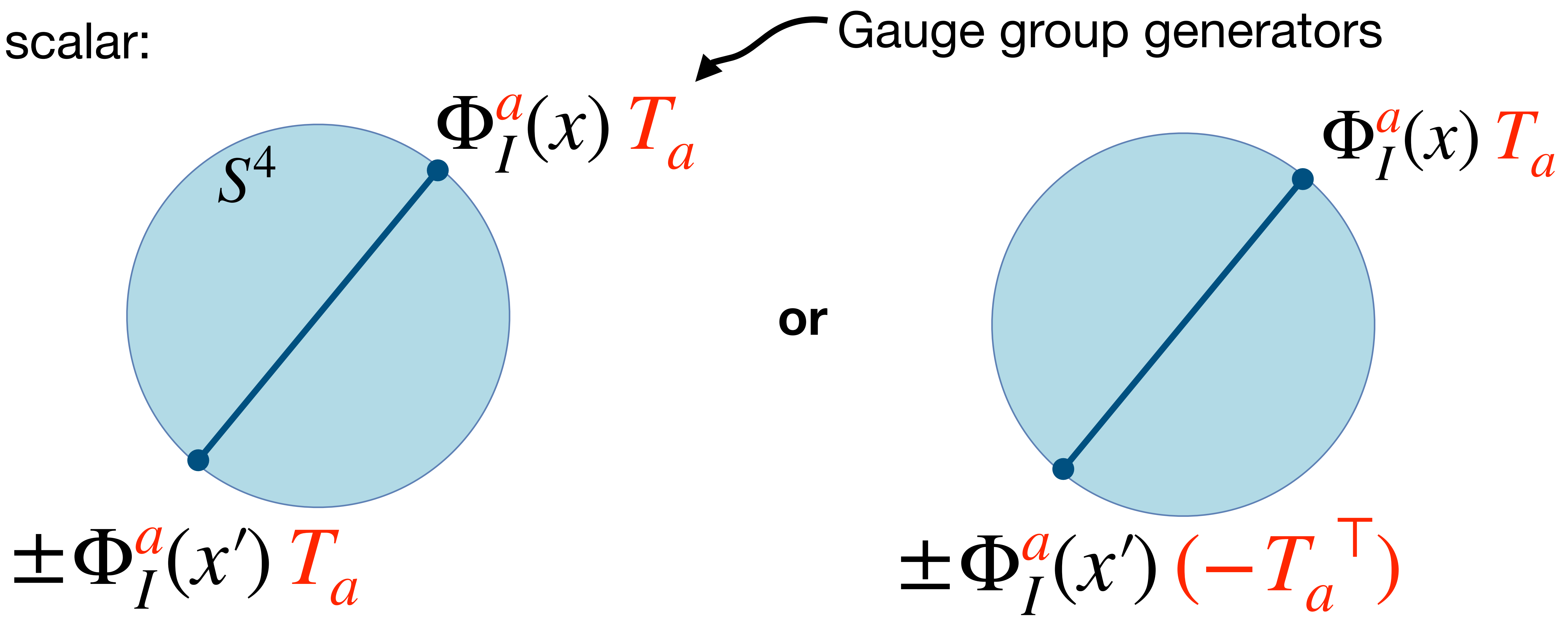

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Gauge **charge conjugation**
(Outer automorphism of SU(N))

New observables, $\langle \mathscr{O} \rangle \neq 0$

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Consider a single trace $\mathcal{O} \sim$ Tr$[\chi_1 \ldots \chi_L]$ at tree level

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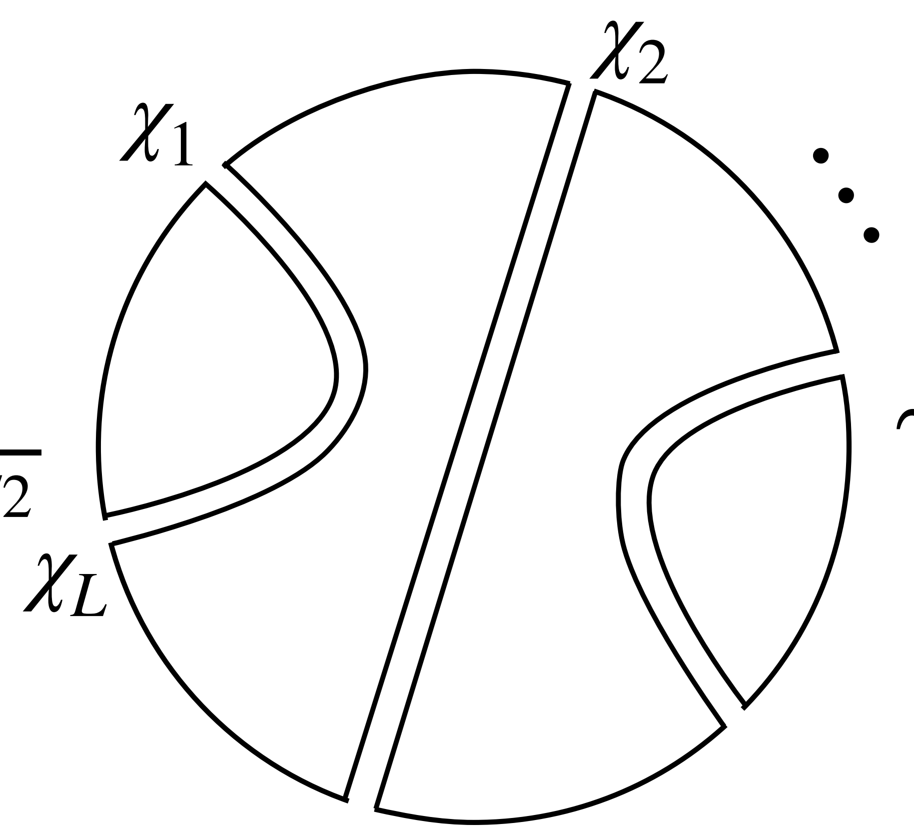
$$\langle \mathcal{O} \rangle \sim \frac{1}{N^{L/2}}$$



$$\sim N$$

$$\langle \Phi(x)^a_b \Phi(y)^c_d \rangle_{\mathbb{RP}^4} = \frac{\|}{(x-y)^2} \pm \frac{\|}{(1+xy)^2}$$

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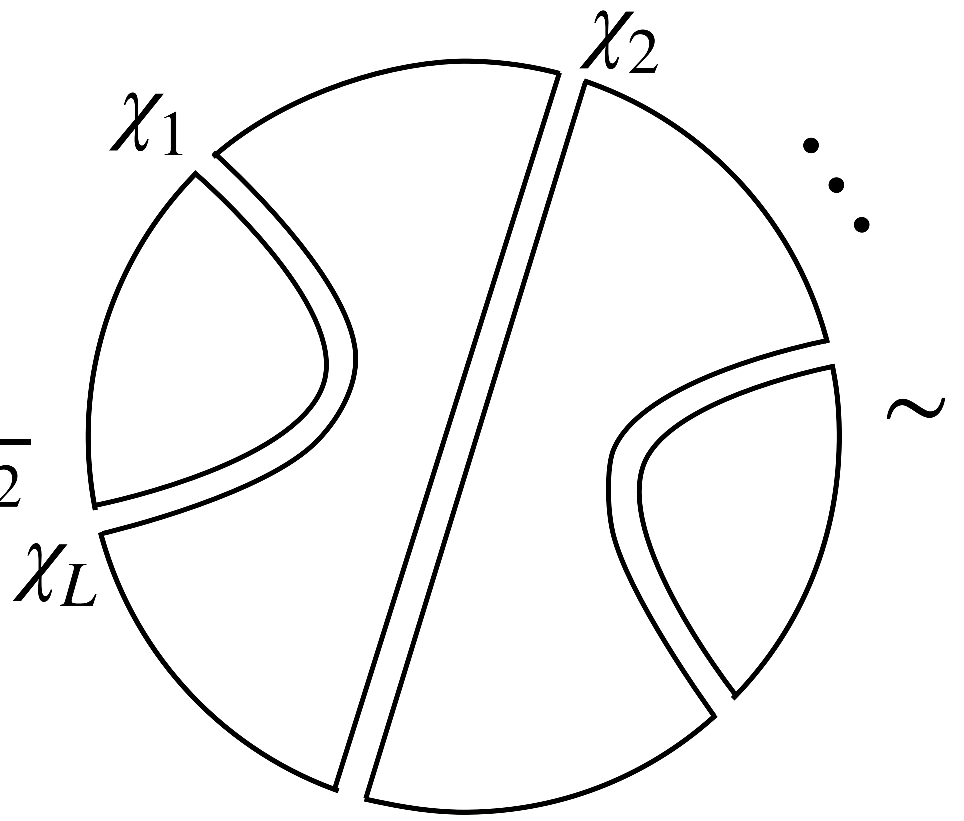

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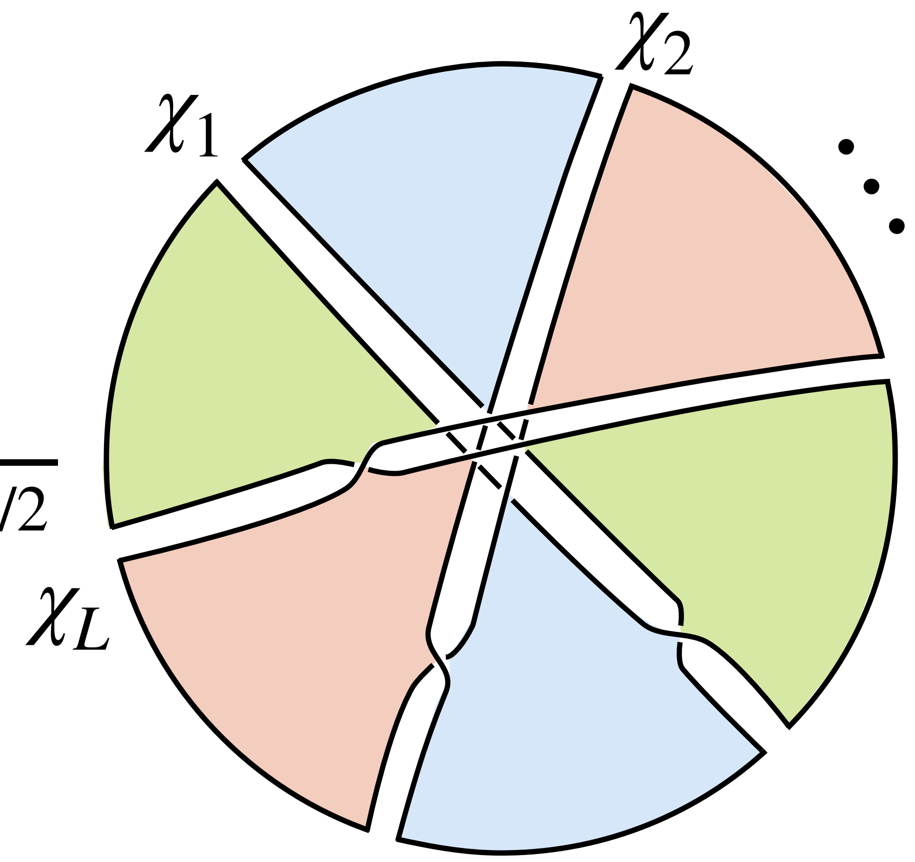

$\sim N^0$

(Identification of antipodal points on the spin chain)

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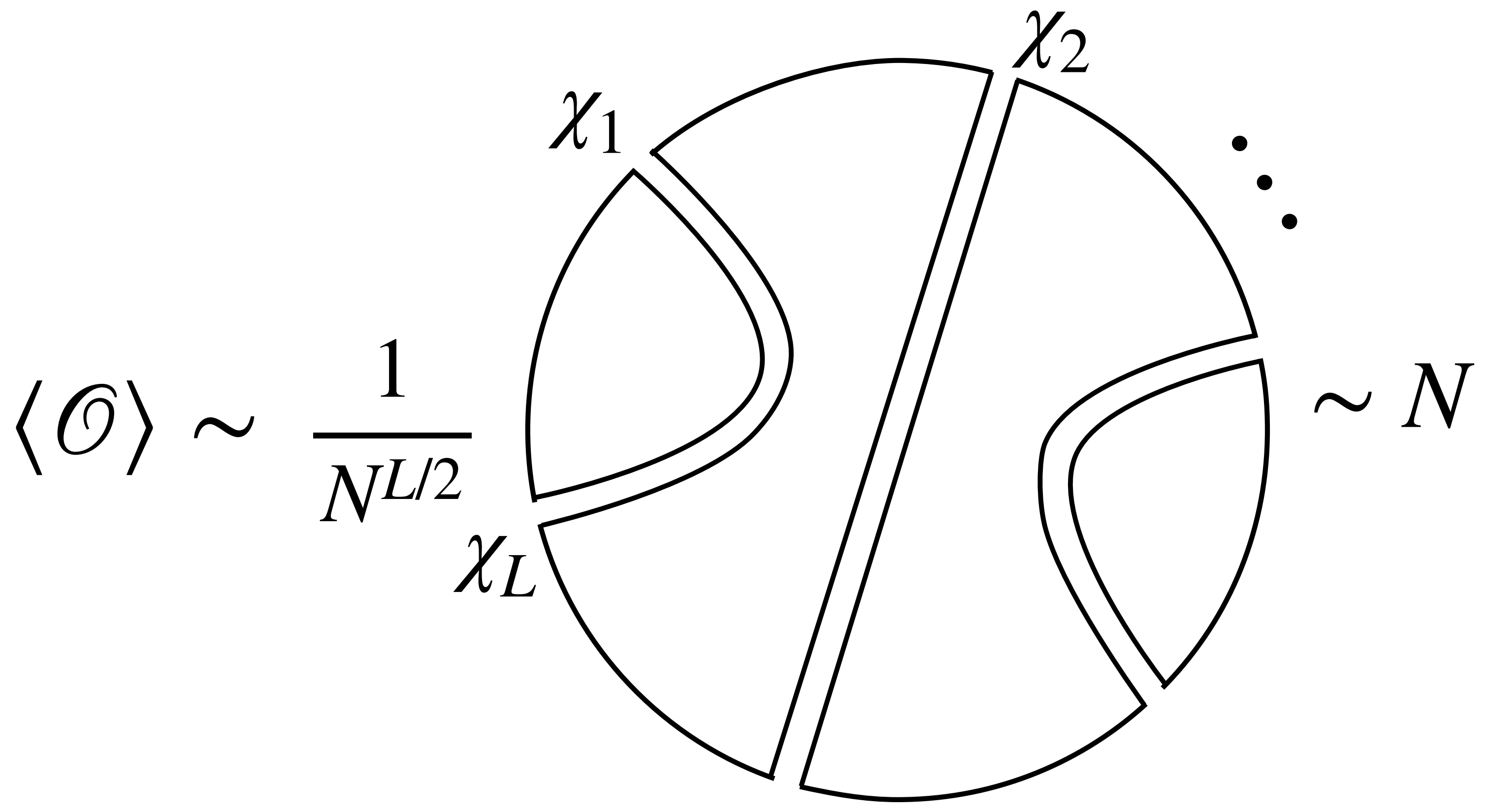

- New analytic classical (Euclidean) background asymptotic to $AdS_5 \times S^5$
- Explicit analytic (Euclidean) solution from a 10D uplift of 5D $\mathcal{N} = 8$ gauged supergravity

[JC, Rastelli' 22]

- Not integrable

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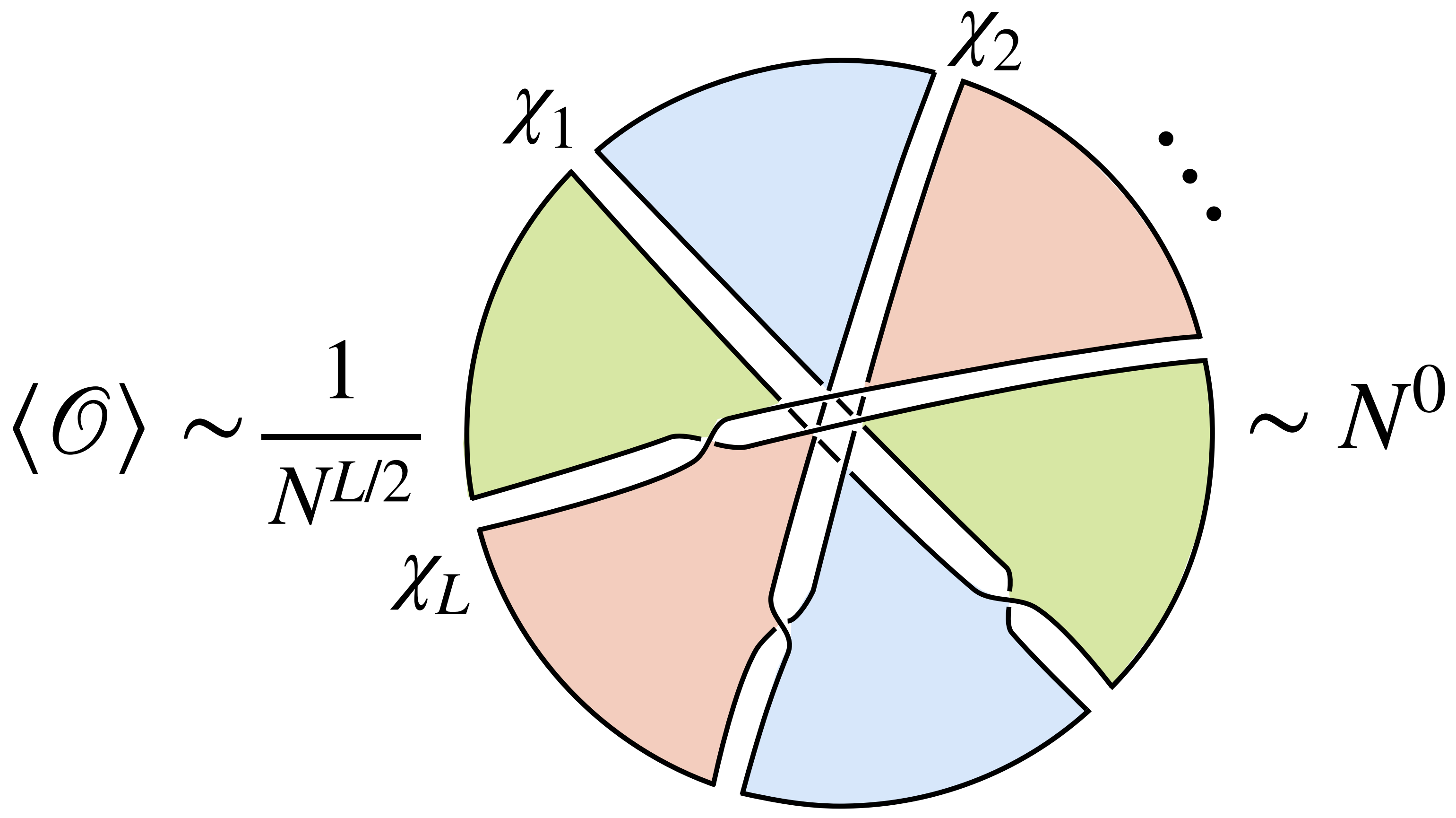

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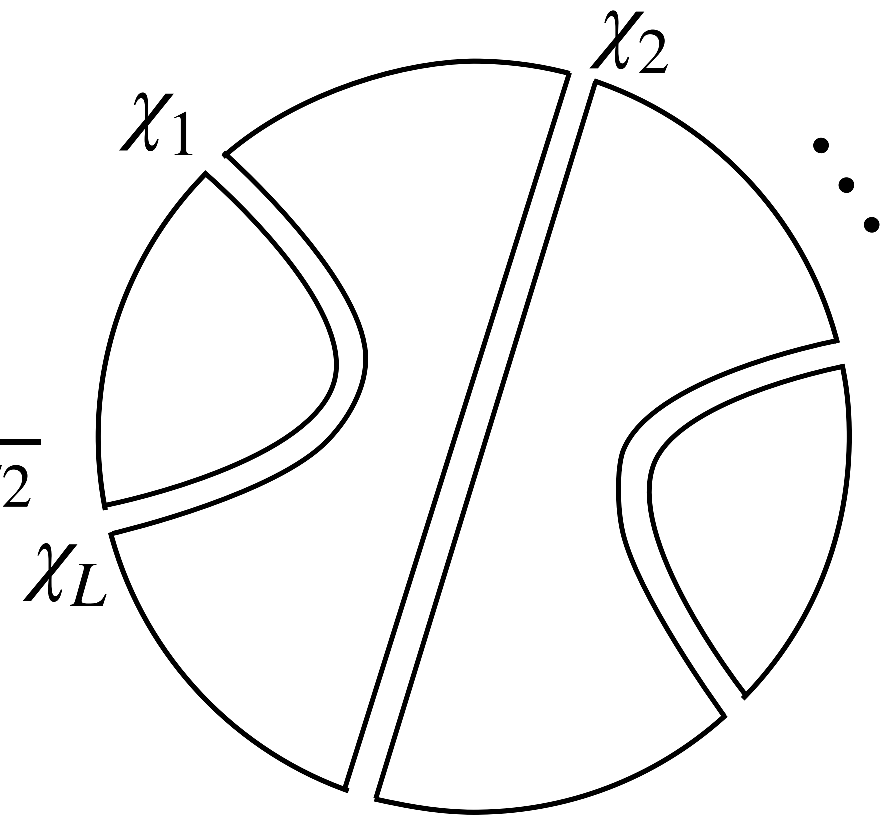

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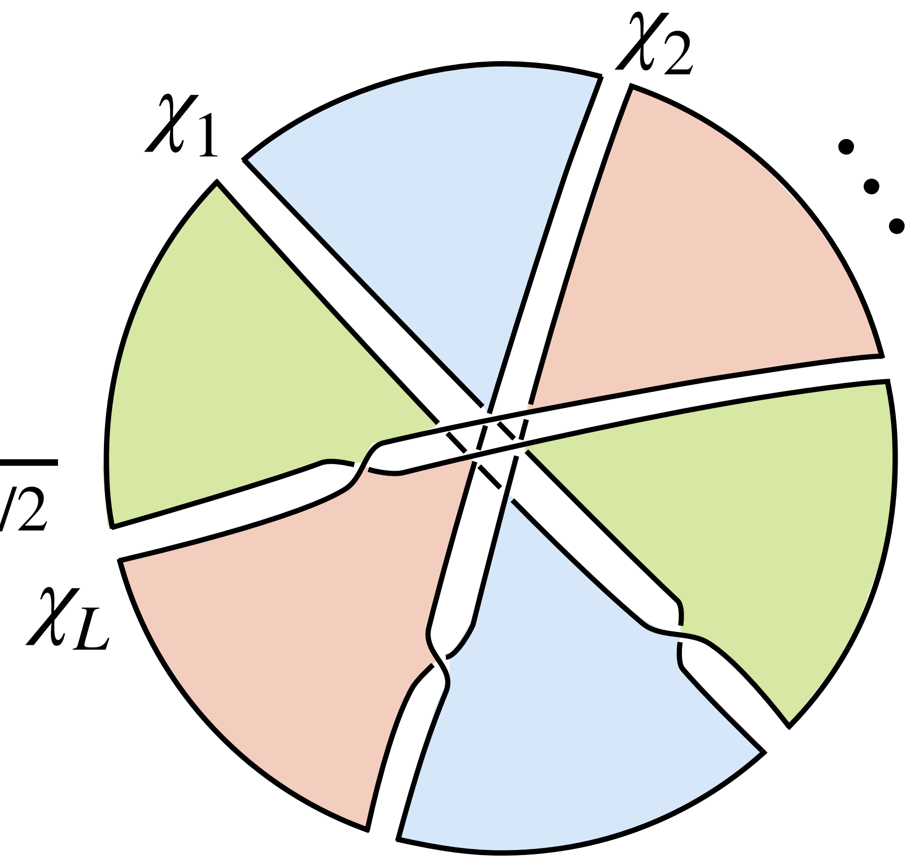

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(Identification of antipodal points on the spin chain)

- Background still $AdS_5 \times S^5$ with a probe orientifold O1 plane extended inside the S^5
- Integrable setup

New observables, $\langle \mathcal{O} \rangle \neq 0$

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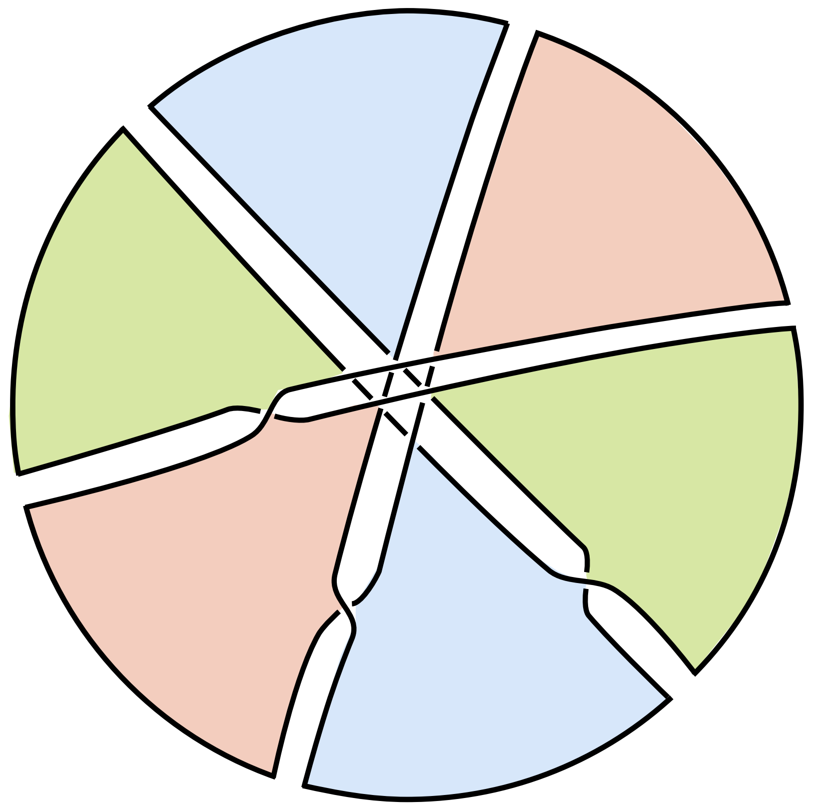
N=4 SYM on $\mathbb{RP}^4$ with charge conjugation

- Claim: $\langle \mathcal{O} \rangle$ given by the crosscap overlap
  - Crosscap worldsheet from the probe orientifold
  - Spin chain: antipodal contraction

$\langle \mathcal{O} \rangle \sim$ 

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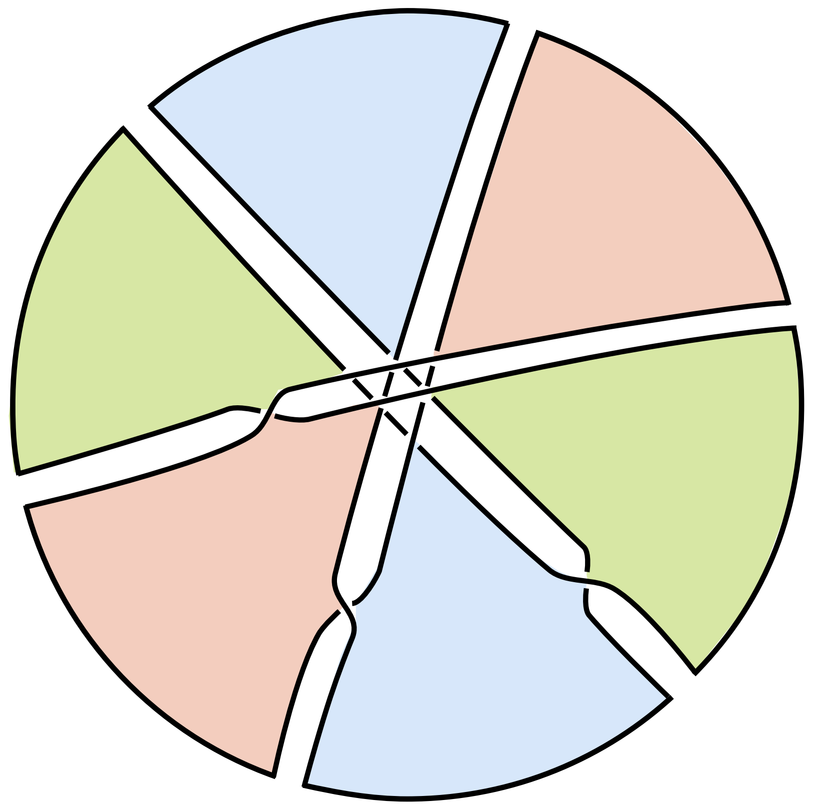
$\langle \mathcal{O} \rangle \sim$ 

- Conjecture for the asymptotic formula

$$\langle \mathcal{O} \rangle = \sqrt{\frac{\det G_{+}^{(\text{Gaudin})}}{\det G_{-}^{(\text{Gaudin})}}} \quad \text{for } L \gg 1$$

Derivatives of log of Bethe equations

- Analogous to the boundary overlap (e.g. D3-D5 system) but without non-universal prefactors

[Bajnok, Gombor, de Leeuw, Komatsu, Kristjansen, Lindardopoulos, Pozsgay, Wang, Wilhelm, Zarembo etc.]

Outlook

- Localization of N=4 SYM on $\mathbb{RP}^4$ without charge conjugation leads to 2D Yang-Mills on $\mathbb{RP}^2$. Similar story with charge conjugation? [Wang' 20]

- New classical background dual to N=4 SYM on $\mathbb{RP}^4$ without charge conjugation contains singularities which are geometrically of the orientifold type (in flat space). How does the gauge theory help to resolve them?

- Precision holography with matrix model from localisation?

- Sigma-model proof of integrability for the crosscap? [Linardopoulos, Zarembo]

- Other models in $\mathbb{RP}^d$? ABJM (oriented though) ?

- Fishnet theories on $\mathbb{RP}^4$? Nonplanar version of Basso-Dixon diagrams?

Outlook

- Bootstrap for the two point-functions on $\mathbb{RP}^4$: involve conformal dimensions + three point couplings + one-pt functions (known from integrability). No new boundary operators! Bootstrability?

[Cavaglià, Gromov, Julius, Preti]

[Caron-Huot, Coronado, Trinh, Zahraee]

- Antipodal map defining $\mathbb{RP}^4$ in embedding coordinates is given by

$$(X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, -X_1, -X_2, -X_3, -X_4, -X_5)$$

- But we can imagine doing:

$$(X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, X_1, X_2, -X_3, -X_4, -X_5)$$

- In N=4 SYM, are these new higher codimension versions of $\mathbb{RP}^4$? 1/2 BPS?

- Presumably they define new types of crosscaps on the worldsheet which are also integrable

Thank you

Backup slides

Holographic Dual of $\mathcal{N} = 4$ SYM on $\mathbb{RP}^4$

(without charge conjugation)

- New (euclidean) 1/2-BPS solution of 10D IIB supergravity (asymptotically AdS)

$$ds^2_{\text{10D}} = \Delta^{1/4}\left(\underbrace{ds^2_{\text{5D}}}_{dr^2 + e^{2A}ds^2_{\mathbb{RP}^4}} + \frac{4}{g^2}\left(d\theta^2 + \frac{\cos^2\theta}{1+\mathcal{K}_+\cos^2\theta}\underbrace{d\Omega^2_{S^2}}_{SO(3)} + \frac{\sin^2\theta}{1+\mathcal{K}_-\sin^2\theta}\underbrace{d\Omega^2_{dS_2}}_{SO(2,1)}\right)\right)$$

Explicit functions of r and $\mathcal{J}$

$$e^{2A} = \frac{\boxed{\mathcal{J}^3}}{4}\sinh 2r + \frac{1}{4}(2-\boxed{\mathcal{J}^3})\cosh 2r - \frac{1}{2}$$

Parametric family of backgrounds. To be fixed by comparison to the gauge theory.

$\mathcal{J} \to 0$: Standard (euclidean) $AdS_5 \times S^5$

- Solution contains analytic expressions for non-trivial dilaton, B_2, C_2 and C_4

Holographic Dual of $\mathcal{N} = 4$ SYM on $\mathbb{RP}^4$

(without charge conjugation)

- Singular in the IR

$$ds^2_{\text{10D}} \sim h^{-3/4} d\Omega^2_{dS_2} + h^{1/4}\left(\mathcal{J} dz^2 + \mathcal{J} ds^2_{\mathbb{RP}^4} + d\xi^2 + \xi^2 d\Omega^2_{S^2}\right)$$

$$e^{-\Phi} \sim h^{-1/2}$$

$$C_2 \sim h^{-1}\, dV_{dS_2}$$

- Resembles an $\mathcal{O}1_-$ plane in flat space!

- However $\mathcal{O}$-planes are **incompatible** with the previous large N counting: this setup does not involve gauging of worldsheet parity ($\leftrightarrow$ charge conjugation)

- What's the embedding in string theory?