Q-systems, QQ-relations and T-functions beyond $\mathfrak{gl}_N$

Simon Ekhammar

arXiv: 22xx.xxxx with Dmytro Volin, 2008.10597,2104.04539 with Volin,Shu

New mathematical methods in solvable models and gauge/string dualities, Varna

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- We specify model by imposing analytic properties.

  Example: Rational spin chain

twist

$$\begin{aligned} Q_1 &= \mathrm{x}_1^{\frac{u}{\hbar}} \sigma \prod_j (u - u_{1,j}) \\ Q_2 &= \mathrm{x}_2^{\frac{u}{\hbar}} \sigma \underbrace{\prod_j (u - u_{2,j})}_{\text{polynomial}} \end{aligned} \xrightarrow[\text{QQ}]{} \frac{\mathrm{x}_1}{\mathrm{x}_2} \prod_{j \neq i} \frac{(u_{1,i} - u_{1,j} + \hbar)}{(u_{1,i} - u_{1,j} - \hbar)} = \left(\frac{u + \frac{\hbar}{2}}{u - \frac{\hbar}{2}} \right)^L$$

dressing

$$\frac{1}{\sigma^+ \sigma^-} = u^L$$

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Analytic Q-system = Functional Relations + Analytic Properties

QQ-relations

Polynomial,
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  - $AdS_5 \times S^5$
    [Gromov,Kazakov,Leurent,Volin,'13'14]
  - $AdS_4 \times \mathbb{CP}^3$ (ABJM)
    [Bombardelli,Cavaglià,Fioravanti,Gromov,Tateo '17]
  - $AdS_3 \times S^3 \times T^4$
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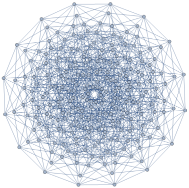

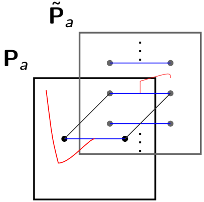

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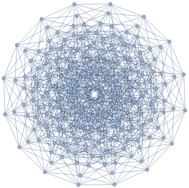
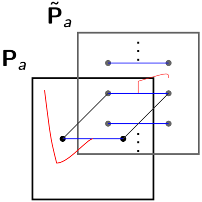
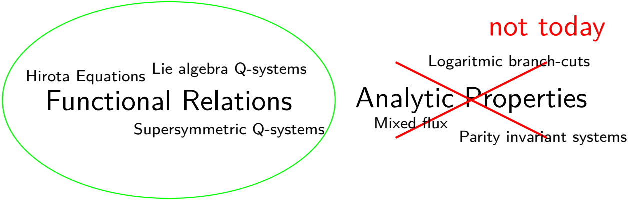




- Ultimate goal: Extend the QSC formalism as widely as possible! Concrete recipe for $\mathfrak{su}(2|2)$: Monodromy Bootstrap [SE,Volin '21]

Challenges and hopes

- Challenges extending QSC?



Hirota Equations

Lie algebra Q-systems

Functional Relations

Supersymmetric Q-systems

not today

Logaritmic branch-cuts

~~Analytic Properties~~

Mixed flux

Parity invariant systems

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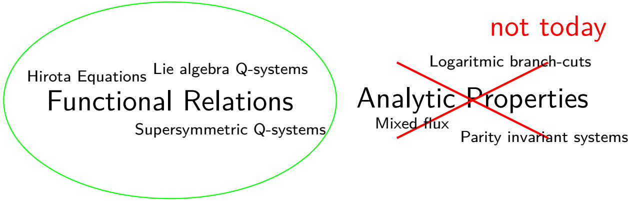

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- Study of functional relations appears in many areas. Important progress made in ODE/IM [Sun'12,Masoero,Raimondo,Valeri '15] for general Lie-algebras. D_r has recieved attention recently [Ferrando,Frassek,Kazakov '20, SE,Shu,Volin '20, SE Volin'21]

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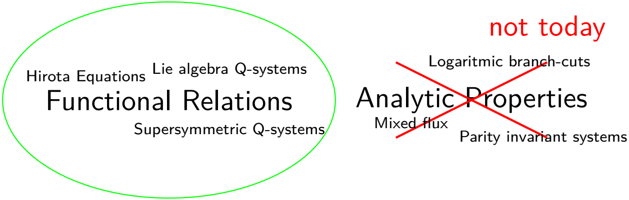

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- Except for new construction of QSC: What else can we hope to do?
  - Find powerful ways to solve BAE. [Marboe,Volin '16]
  - Fishnets in arbitrary dimensions? [Kazakov,Olivucci'18, Basso,Ferrando,Kazakov,Zhong '19]
  - ABJM fishchains ($B_2 \simeq C_2$ Q-systems) using QSC?
  - Other (1+1)D QFT's? [Balog, Hegedus '05, Gromov,Kazakov,Vieira '08,Kazakov,Leurent '10]

Plan of the talk

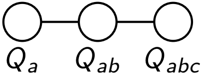
- Philosophy:
  Explicit examples before generalities, focus on rational spin chains.
- Q-systems and T-systems for A_r.
- Q-systems and T-systems for $B_2 \simeq C_2$.
- Bethe equations and T-functions.
- Solving $B_2 \simeq C_2$ spin chains.
- Extensions to higher rank

Reminder on A_r Q-systems

Warmup: Q-system for A_r

- We can attach a Q-function to every node on the A_r Dynkin diagram

A_3:



where $a = 1, \ldots, r+1$.

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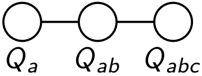

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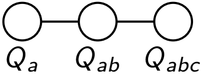

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- Bethe equations for a homogeneous rational spin chain can be found by setting

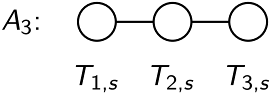
$$Q_a = \sigma_1 q_a\,, \quad Q_{ab} = \sigma_2 q_{ab}\,, \quad Q_{abc} = \sigma_3 q_{abc}$$

$$\left.\frac{q_1^{[2]}}{q_1^{[-2]}}\frac{q_{12}^{[-1]}}{q_1^{[+1]}}\right|_{q_1=0} = -\left(\frac{u+\frac{\hbar}{2}}{u-\frac{\hbar}{2}}\right)^L \nwarrow$$

$$\frac{\sigma_{(2)}}{\sigma^+_{(1)}\sigma^+_{(1)}} = u^L$$

Warmup: Hirota for A_r

- We can also attach T-functions to the Dynkin diagram



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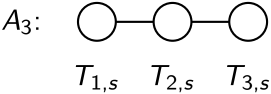

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$$T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1} \, .$$

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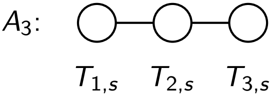

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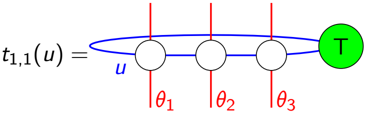

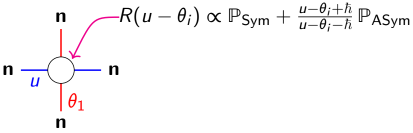

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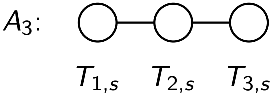

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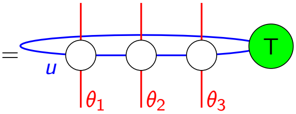
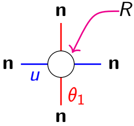
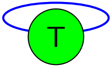
$t_{1,1}(u) =$





$R(u-\theta_i) \propto \mathbb{P}_{\mathrm{Sym}} + \frac{u-\theta_i+\hbar}{u-\theta_i-\hbar}\,\mathbb{P}_{\mathrm{ASym}}$

- Start with character solution, $T_{1,1} =$



$= \mathrm{x}_1 + \mathrm{x}_2 + \ldots \mathrm{x}_{r+1}$

ex A_2 :

$$T_{1,s} = \chi_{1,s} = \frac{1}{\Delta}\begin{vmatrix} \mathrm{x}_1^{s+2} & \mathrm{x}_1 & 1 \\ \mathrm{x}_2^{s+2} & \mathrm{x}_2 & 1 \\ \mathrm{x}_3^{s+2} & \mathrm{x}_3 & 1 \end{vmatrix}$$

Relating Q-functions and T-functions

- Q-functions nicely "quantize" the character formula:

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- Unpack determinant:

$$\Delta\chi_{1,s} = \mathtt{x}_1^{s+3}\left(\frac{\mathtt{x}_2}{\mathtt{x}_1} - \frac{\mathtt{x}_3}{\mathtt{x}_1}\right) - \mathtt{x}_2^{s+3}\left(\frac{\mathtt{x}_1}{\mathtt{x}_2} - \frac{\mathtt{x}_3}{\mathtt{x}_2}\right) + \mathtt{x}_3^{s+3}\left(\frac{\mathtt{x}_1}{\mathtt{x}_3} - \frac{\mathtt{x}_2}{\mathtt{x}_3}\right)$$

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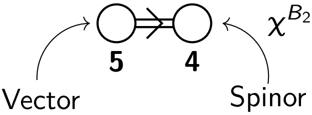
- A simple solution is the character solution .

Q-functions and Hirota for $B_2 \simeq C_2$

$B_2 \simeq C_2$ Spin chains

- Simplest non-simply laced case: $B_2 \simeq C_2$.



$$\chi^{B_2}([01]) = \underset{++}{\sqrt{\mathtt{x}_1\mathtt{x}_2}} + \underset{+-}{\sqrt{\frac{\mathtt{x}_1}{\mathtt{x}_2}}} + \underset{-+}{\sqrt{\frac{\mathtt{x}_2}{\mathtt{x}_1}}} + \underset{--}{\sqrt{\frac{1}{\mathtt{x}_1\mathtt{x}_2}}}$$

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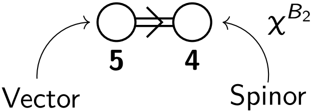

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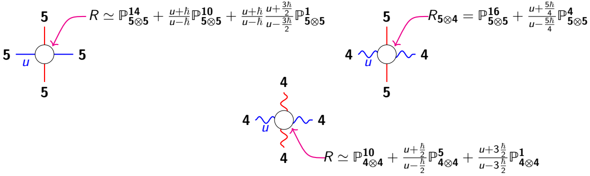
- 3 important types of scattering events



$$R \simeq \mathbb{P}^{\mathbf{14}}_{\mathbf{5}\otimes\mathbf{5}} + \frac{u+\hbar}{u-\hbar}\mathbb{P}^{\mathbf{10}}_{\mathbf{5}\otimes\mathbf{5}} + \frac{u+\hbar}{u-\hbar}\frac{u+\frac{3\hbar}{2}}{u-\frac{3\hbar}{2}}\mathbb{P}^{\mathbf{1}}_{\mathbf{5}\otimes\mathbf{5}}$$

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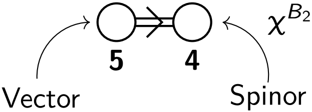
$B_2 \simeq C_2$ Spin chains

- Simplest non-simply laced case: $B_2 \simeq C_2$.



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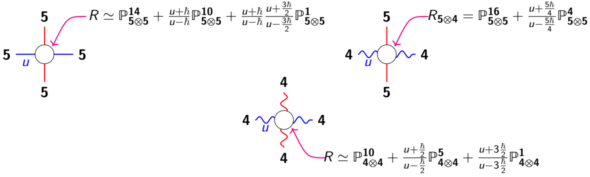

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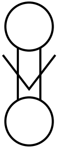
- Once again possible to construct transfer-matrices. Ex: $(\mathcal{H} = \mathbf{4}^3)$

T-system for $B_2 \simeq C_2$

- There exist many more T-functions, they form a T-system



$$T_{\mathbf{5},s}^{[-2]} T_{\mathbf{5},s}^{[2]} = T_{\mathbf{4},2s} + T_{\mathbf{5},s+1} T_{\mathbf{5},s-1}$$

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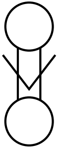

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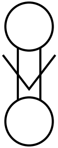

- Goal: Find character solution, find a bilinear formula and then quantize by introducing Q-functions.
- The character solution:

$$\left.\begin{array}{l} T_{\mathbf{4},0}{=}1 \\[2ex] T_{\mathbf{4},1}{=}\mathtt{x}_1{+}\mathtt{x}_2{+}\frac{1}{\mathtt{x}_2}{+}\frac{1}{\mathtt{x}_1} \\[2ex] T_{\mathbf{5},0}{=}1 \\[2ex] T_{\mathbf{5},1}{=}\mathtt{x}_1\mathtt{x}_2{+}\frac{\mathtt{x}_2}{\mathtt{x}_1}{+}1{+}\frac{\mathtt{x}_1}{\mathtt{x}_2}{+}\frac{1}{\mathtt{x}_1\mathtt{x}_2} \end{array}\right\} \quad T_{\mathbf{5},s} = \frac{1}{\tilde{\Delta}} \begin{vmatrix} \mathtt{x}_1^{s+2} - \frac{1}{\mathtt{x}_1^{s+2}} & \mathtt{x}_1^{s+1} - \frac{1}{\mathtt{x}_1^{+s+1}} \\ \mathtt{x}_2^{s+2} - \frac{1}{\mathtt{x}_2^{s+2}} & \mathtt{x}_2^{s+1} - \frac{1}{\mathtt{x}_2^{s+1}} \end{vmatrix}$$

$\chi^{C_2}([0s])$

Rewriting $\chi^{C_2}([0s])$

- Decompose determinant to look for starting point:

$$\tilde{\Delta}_{T_{5,s}} = \begin{vmatrix} x_1^{s+2} - \frac{1}{x_1^{s+2}} & x_1^{s+1} - \frac{1}{x_1^{s+1}} \\ x_2^{s+2} - \frac{1}{x_2^{s+2}} & x_2^{s+1} - \frac{1}{x_2^{s+1}} \end{vmatrix} = \left(\overset{++}{(\sqrt{x_1 x_2})^{2s+3}} - \overset{--}{\left(\frac{1}{\sqrt{x_1 x_2}}\right)^{2s+3}} \right) \left(\sqrt{\frac{x_1}{x_2}} - \frac{\sqrt{x_2}}{\sqrt{x_1}} \right)$$

$$- \left(\underset{+-}{\left(\sqrt{\frac{x_1}{x_2}}\right)^{2s+3}} - \underset{-+}{\left(\sqrt{\frac{x_2}{x_1}}\right)^{2s+3}} \right) \left(\sqrt{x_1 x_2} - \frac{1}{\sqrt{x_1 x_2}} \right)$$

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- Surprise: $T_{\mathbf{5},s}$ is nicely described by spinors!

$$T_{\mathbf{5},s} = (\Psi^{[2s+3]})^T \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Psi^{[-2s-3]} \equiv \bar{\Psi}^{[2s+3]} \Psi^{[-2s-3]}$$

$$\Psi = \begin{pmatrix} 1 \\ 1 \\ \frac{\sqrt{\frac{x_2}{x_1}} - \sqrt{\frac{x_1}{x_2}}}{\tilde{\Delta}} \\ \frac{\sqrt{\frac{1}{x_1 x_2}} - \sqrt{x_1 x_2}}{\tilde{\Delta}} \end{pmatrix} \begin{pmatrix} \sqrt{x_1 x_2} \\ \sqrt{\frac{x_1}{x_2}} \\ \sqrt{\frac{x_1}{x_2}} \\ \frac{1}{\sqrt{x_1 x_2}} \end{pmatrix}^{\frac{u}{\hbar}}$$

Quantizing the character solution

- Quantize by taking an ansatz

Functions of u

$$T_{\mathbf{5},s} = \bar{\Psi}^{[-2s-3]}\Psi^{[2s+3]} \,,$$

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- The γ functions appear naturally due to Fierz identities

$$\delta^\gamma_\alpha \delta^\delta_\beta = -\frac{1}{4} C_{\alpha\beta} C^{\gamma\delta} + \frac{1}{2}(\gamma_i)_{\alpha\beta}(\gamma^i)^{\gamma\delta} + \frac{1}{2}(\gamma_{ij})_{\alpha\beta}(\gamma^{ij})^{\gamma\delta}$$

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- We can solve the $B_2 \simeq C_2$ T-system using

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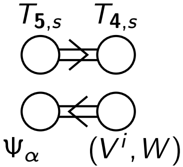
- The Q-functions are now not as easy to place on the Dynkin diagram. "Natural" attempt:



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$$V^i = \bar{\Psi}^{[-2]}\gamma^i\Psi^{[2]}\,, \quad \underset{\nwarrow}{V^{ij}} = \bar{\Psi}^{[-1]}\gamma^{ij}\Psi^{[1]}\,.$$

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$$\leftarrow (V^i)^+(V^j)^- - (V^i)^-(V^j)^+$$

- Consider the highest weight components:

$$\Psi_1^{[2]}\Psi_2^{[-2]} - \Psi_2^{[2]}\Psi_1^{[-2]} \propto V^1$$
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- Shifting and evaluating at zeros gives now:

$$\left.\frac{\Psi_1^{[4]}}{\Psi_1^{[-4]}}\frac{(V^1)^{[-2]}}{(V^1)^{[2]}}\right|_{\Psi_1=0} = -1$$
$$\left.\frac{(V^1)^{[2]}}{(V^1)^{[-2]}}\frac{\Psi_1^{[-2]}}{\Psi^{[2]}}{}_1\right|_{V^1=0} = -1$$

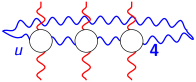
$$\prod_{a=1}^{2}\frac{q_a^{[B_{ab}]}}{q_a^{[-B_{ab}]}} = -1$$

where B_{ab} is the symmetrized C_2 Cartan matrix.

Fundamental T-functions $T_{\mathbf{4},1}(u)$

- Consider a spin chain, ex:

$t_{\mathbf{4},1}(u) =$



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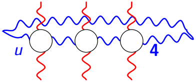
$t_{\mathbf{4},1}(u) =$ 

- The spectrum can be calculated using the following expression:

$$t_{\mathbf{4},1} = P_{\mathbf{4}}^{[3]} P_{\mathbf{4}}^{[-1]} \frac{q_{\mathbf{4}}^{[4]}}{q_{\mathbf{4}}^{[2]}} + P_{\mathbf{4}}^{[3]} P_{\mathbf{4}}^{[-3]} \frac{q_{\mathbf{5}}^{[4]}}{q_{\mathbf{5}}} \frac{q_{\mathbf{4}}}{q_{\mathbf{4}}^{[2]}}$$

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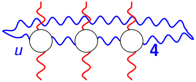

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- To reproduce these expressions we use the following identification:

$$V^1 = \sigma_{\mathbf{4}} q_{\mathbf{4}} \,, \quad \Psi_1 = \sigma_{\mathbf{5}} q_{\mathbf{5}} \,. \qquad t_{\mathbf{4},1} = \frac{1}{\sigma_{\mathbf{4}}^{[4]} \sigma_{\mathbf{4}}^{[-4]}} T_{\mathbf{4},1}$$

$$\frac{\sigma_{\mathbf{5}}^{+} \sigma_{\mathbf{5}}^{-}}{\sigma_{\mathbf{4}}^{+} \sigma_{\mathbf{4}}^{-}} = P_{\mathbf{4}} \,, \quad \frac{\sigma_{\mathbf{4}}}{\sigma_{\mathbf{5}}^{[2]} \sigma_{\mathbf{5}}^{[-2]}} = P_{\mathbf{5}}$$

Singlet and oscillating sign

- Can we improve the expression, for $T_{\mathbf{4},s}$?

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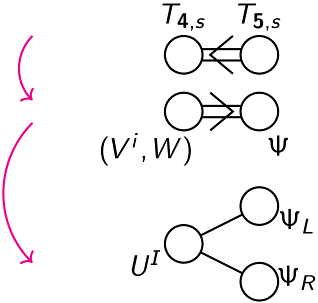

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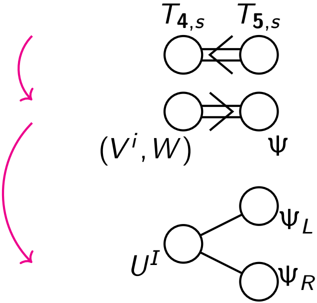

$$\bar{\Psi}_L^{[-1]} \Gamma^{\mathcal{I}} \Psi_L^{[1]} = (RU)^{\mathcal{I}}$$

$$\bar{\Psi}_R^{[-1]} \Gamma^{\mathcal{I}} \Psi_R^{[1]} = U^{\mathcal{I}}$$

Summarizing the $B_2 \simeq C_2$ Q-system

- A Q-system for $B_2 \simeq C_2$ can be built from spinors Ψ. There is also a vector, $V^i = \bar{\Psi}^{[-2]}\gamma^i\Psi^{[2]}$ and a scalar $W = \bar{\Psi}^{[-2]}\Psi^{[2]}$.

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$$U_I^+(RU)_J^- - U_I^-(RU)_J^- \propto \bar{\Psi}_R^-\Gamma_{IJ}\bar{\Psi}_L^+ \tag{2.1}$$

Summarizing the $B_2 \simeq C_2$ Q-system

- A Q-system for $B_2 \simeq C_2$ can be built from spinors Ψ. There is also a vector, $V^i = \bar{\Psi}^{[-2]}\gamma^i\Psi^{[2]}$ and a scalar $W = \bar{\Psi}^{[-2]}\Psi^{[2]}$.
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- Interesting similarities with $\mathrm{AdS}_4/\mathrm{CFT}_3$ QSC.

Wronskian Bethe Ansatz

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- Plan:
  - Example of alternating $\mathfrak{su}_4$
  - Back to $C_2 \simeq B_2$

The alternating $\mathfrak{su}_4$ spin chain (ABJM), Part 1

- The analytic ansatz amounts to setting

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$$u^{L_4} = \frac{\sigma_{(2)}}{\sigma_{(1)}^+ \sigma_{(1)}^-} \qquad u^{L_6} = \frac{\sigma_{(1)} \sigma_{(3)}}{\sigma_{(2)}^+ \sigma_{(2)}^-} \qquad u^{L_{\bar{4}}} = \frac{\sigma_{(2)}}{\sigma_{(3)}^+ \sigma_{(3)}^-}$$

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- Example: homogeneous alternating $\mathfrak{su}_4$ spin chain with sites in $\mathbf{4}, \bar{\mathbf{4}}$, then we set

$$L_{\mathbf{4}} = L\,, \qquad L_{\mathbf{6}} = 0\,, \qquad L_{\bar{\mathbf{4}}} = L\,.$$

The alternating $\mathfrak{su}_4$ spin chain (ABJM), Part 2

- The trick now is to *not* solve Bethe equations but instead the Wronskian equation $Q_{1234} = 1$;

$$W(q_1, q_2, q_3, q_4) = \frac{1}{\sigma_{(1)}^{[3]}\sigma_{1}^{[1]}\sigma_{(1)}^{[-1]}\sigma_{(1)}^{[-3]}} = \left(u^2(u+\mathtt{i})(u-\mathtt{i})\right)^L \ .$$

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- The degree of a q-functions fixed by distance from highest weight:

$$\begin{tikzcd} \bigotimes^2 ([100] \otimes [001]) = & {[202]} \arrow[rrrr, bend left] \arrow[rr, bend right] & \oplus & {[210]} & \oplus & {[012]} & \oplus \, [020] \oplus 4\,[101] \oplus 2\,[000] \end{tikzcd}$$

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- Can study for example determinant operators [Yang,Jiang,Komatsu,Wu 21'].

L	2	3	4	5
# Solutions	10 (0.4 s)	70 (5 s)	558 (3 min)	-
+ impose sym	5 (0.5 s)	17 (3.5 s)	59 (20 s)	119 (2min)

The $B_2 \simeq C_2$ case

- Once again take an ansatz

$$\Psi_\alpha = \sigma_{\mathbf{5}}\, \psi_\alpha\,, \qquad\qquad V^i = \sigma_{\mathbf{4}}\, v^i\,, \tag{3.1}$$

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$$\begin{vmatrix} \psi_1^{[3]} & 0 & \psi_1^{[-1]} & 0 \\ 0 & \psi_1^{[1]} & 0 & \psi_1^{[-3]} \\ \psi_3^{[3]} & \psi_3^{[1]} & \psi_3^{[-1]} & \psi_3^{[-3]} \\ \psi_2^{[3]} & \psi_2^{[1]} & \psi_2^{[-1]} & \psi_2^{[-3]} \end{vmatrix} = P_{\mathbf{4}}\, \underbrace{P_{\mathbf{5}}^+ P_{\mathbf{5}}^-}_{\text{Fused product}}\, \overbrace{\psi_0^+ \psi_0^-}^{\text{"Dynamic source"}}$$

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- For spin chains containing vector representations:

$$\text{Remainder}\left(\frac{\bar{\psi}\gamma^i\psi}{P_{\mathbf{5}}}\right) = 0 \tag{3.2}$$

Some explicit results for $B_2 \simeq C_2$

- Technical comments
  - It is always necessary to gauge-fix Q-function. Why?

    $$W(q_1, q_2) = f \implies W(q_1, q_2 + q_1) = f \, .$$

    Gauge-fixing is done by subtracting lower degree from higher degree q-functions.
  - The degree of Q-functions can be computed as

    $$\begin{aligned} \deg(q_a) &= \langle \text{Lowering Roots}, \epsilon_a \rangle \\ &\quad + \langle \text{Ground State}, (\omega_a - \epsilon_a) \rangle + \langle \text{Weyl vector}, (\omega_a - \epsilon_a) \rangle \, . \end{aligned}$$

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- Running the described algorithm:

L	3	4	5	6	7
# Sol	6 (0.3 s)	20 (1.03 s)	50 (1.7)	175 (11 s)	490 (3 min)

Table. Homogeneous spin chain with spinors

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Table. Homogeneous spin chain with spinors

- The method also works for twisted spin chains.

Generalisations

Example of B_3

- Two examples to show how to generalize:

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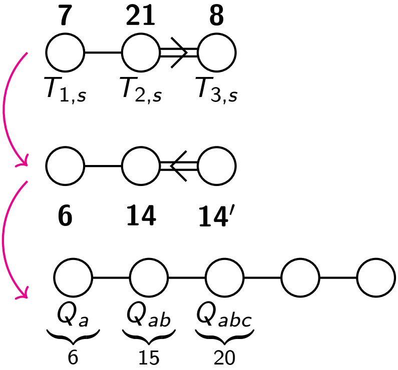

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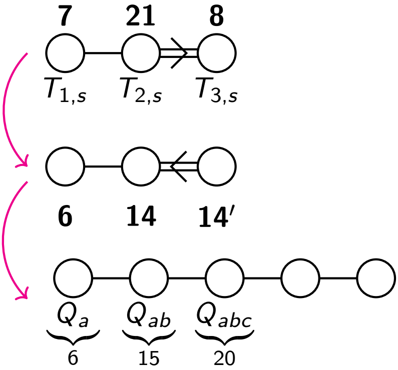


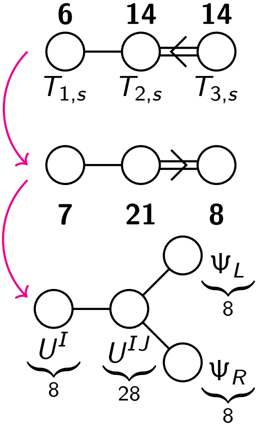
- The T-system can now be solved by

$$T_{1,s} = Q_a^{[-s-\frac{5}{2}]}\kappa^{ab}Q_b^{[s+\frac{5}{2}]}\,, \quad T_{2,s} = \frac{1}{2}Q_{ab}^{[-s-\frac{5}{2}]}\kappa^{ac}\kappa^{bd}Q_{cd}^{[s+\frac{5}{2}]}$$

$$T_{3,s} = \tfrac{1}{6}Q_{abc}^{[-s-\frac{5}{2}]}\kappa^{ad}\kappa^{be}\kappa^{cf}R(Q_{def}^{[s+\frac{5}{2}]}) \leftarrow \mathbf{20{=}14{\otimes}6}$$

Example of C_3

- For C_3:



$$U^I = (V^i, W) \quad U^{IJ} = (V^{ij}, W^i)$$

$$V^i \propto \bar{\Psi}^{[-2]}\gamma^i\Psi^{[2]}, W \propto \bar{\Psi}^{[-2]}\Psi^{[2]}$$

$$V^{ij} \propto \bar{\Psi}^{[-2]}\gamma^i\Psi^{[2]}, W \propto \bar{\Psi}^{[-2]}\Psi^{[2]}$$

Conclusions and outlook

Conclusions

- We can solve various different T-systems using bilinear formulas for T-functions
- The symmetry of the Q-system does not in general coincide with that of the integrable model.
- From the T-system we can construct Wronskian Bethe Equations. These equations can be used to solve for the spectrum.

Outlook

- On-going work:
    - Exceptional non-simply laced algebras, expected to follow a similar pattern. ($D_4^{(3)}$ should give G_2)
    - Supersymmetric Q-systems, still mysterious. (Some help from ABJM QSC)
    - Personal goal: $D(2,1|\alpha)$
- Would be very interesting to extend beyond rational spin chains.
- Long-term goal: Extend the QSC and, hopefully, find a rich family of "AdS/CFT" like systems.