Hexagons for short Operators:

A hands on approach

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Summery

- $\mathcal{N}=4$ integrable CFT in the planar limit (Kostya's talk).
We want to able to compute anomalous dimensions and structure constants.

- Spectral problem, obtaining anomalous dimensions, solved by QSC (Simon's talk). We are able to obtain results in all regimes

$$
\begin{aligned}
\gamma_{11} \;=\;& -242508705792+107663966208\zeta_3+70251466752\zeta_3^2-12468142080\zeta_3^3 \\
&+1463132160\zeta_3^4-71663616\zeta_3^5+180173002752\zeta_5-16655486976\zeta_3\zeta_5 \\
&-24628230144\zeta_3^2\zeta_5-2895575040\zeta_3^3\zeta_5+19278176256\zeta_5^2-9619845120\zeta_3\zeta_5^2 \\
&+2504494080\zeta_3^2\zeta_5^2+\tfrac{882108048384}{175}\zeta_5^3+45602231040\zeta_7+14993482752\zeta_3\zeta_7 \\
&-12034759680\zeta_3^2\zeta_7+1406730240\zeta_3^3\zeta_7+30605033088\zeta_5\zeta_7+21217637376\zeta_3\zeta_5\zeta_7 \\
&-\tfrac{1309941061632}{275}\zeta_5^2\zeta_7-13215327552\zeta_7^2-4059901440\zeta_3\zeta_7^2-69762034944\zeta_9 \\
&+23284599552\zeta_3\zeta_9-3631889664\zeta_3^2\zeta_9-11032374528\zeta_5\zeta_9-6666706944\zeta_3\zeta_5\zeta_9 \\
&-23148129024\zeta_7\zeta_9-10024051968\zeta_9^2-54555179184\zeta_{11}+\tfrac{10048541184}{5}\zeta_3\zeta_{11} \\
&-726029568\zeta_3^2\zeta_{11}-8975463552\zeta_5\zeta_{11}-22529041920\zeta_7\zeta_{11}-\tfrac{1437993422496}{175}\zeta_{13} \\
&+\tfrac{1504385419392}{35}\zeta_3\zeta_{13}-30324602880\zeta_5\zeta_{13}-\tfrac{151130039581392}{875}\zeta_{15}-41375093760\zeta_3\zeta_{15} \\
&-\tfrac{196484147423712}{275}\zeta_{17}+309361358592\zeta_{19}-1729880064Z_{11}^{(2)}-\tfrac{1620393984}{5}\zeta_3 Z_{11}^{(2)} \\
&-131383296\zeta_5 Z_{11}^{(2)}+\tfrac{138107420928}{175}Z_{13}^{(2)}+\tfrac{3543865344}{35}\zeta_3 Z_{13}^{(2)}-\tfrac{5716780416}{7}Z_{13}^{(3)} \\
&-\tfrac{674832384}{7}\zeta_3 Z_{13}^{(3)}+\tfrac{48227088384}{175}Z_{15}^{(2)}+\tfrac{3581880576}{25}Z_{15}^{(3)}+754974720Z_{15}^{(4)} \\
&-\tfrac{854924544}{11}Z_{17}^{(2)}+\tfrac{4963244544}{55}Z_{17}^{(3)}+\tfrac{818159616}{275}Z_{17}^{(4)}+\tfrac{175363688448}{1925}Z_{17}^{(5)}\,. \qquad \text{(A.2)}
\end{aligned}
$$

$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_{S=2}(\lambda)$	$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_{S=2}(\lambda)$
0.1	4.115506377945221056840042671851	0.2	4.418859880802350962250362876243
0.3	4.826948662284842304671283425271	0.4	5.271565182595898008221528540034
0.5	5.712723424787739030626966875973	0.6	6.133862814488691819595425762346
0.7	6.531606077852440195886557953690	0.8	6.907504206024567515828872789717
0.9	7.2641695874391127748396398539	1	7.60407071704738848334286555
1.1	7.9292942641568451632186264	1.2	8.241563441147703542676050
1.3	8.54230287229506674486342	1.4	8.8326999393163090494514
1.5	9.11375404891588560886	1.6	9.386314656368554140399
1.7	9.65111042653013781471	1.8	9.9087717085593508789
1.9	10.1598480131615473641	2	10.4048217434405061127
2.1	10.6441190951617575972	2.2	10.878118797537726796
2.3	11.107159189584305149	2.4	11.331544000504529107
2.5	11.551547111042160297	2.6	11.76741650605722239
2.7	11.97937757952067741	2.8	12.18763591669137588
2.9	12.3923796509149519	3	12.5937814717988565
3.1	12.7920003457144898	3.2	12.9871829973986392
3.3	13.1794651919629055	3.4	13.368972849208144
3.5	13.555823016292914	3.6	13.740124720157966
3.7	13.921979717391474	3.8	14.101483156227149
3.9	14.278724162943763	4	14.45378636296056
4.1	14.62674834530641	4.2	14.79768407780976
4.3	14.96666327925592	4.4	15.13375175384302
4.5	15.29901169250472	4.6	15.4625019450274
4.7	15.6242782663505	4.8	15.7843935399844
4.9	15.942897981092	5	16.099839321454

Table 1: Conformal dimension of Konishi operator

What about C_{ijk}?

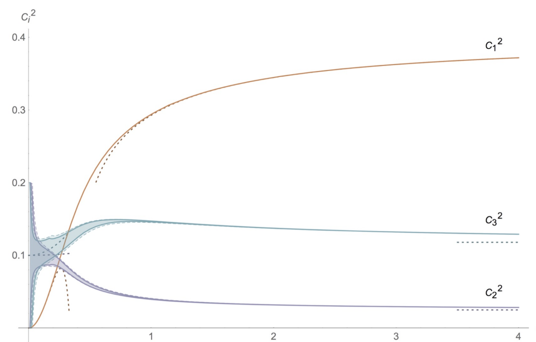
- Some exciting new results combining bootstrap + integrability (bootstrability)

Kolya's talk (defect CFT)

$$\lambda_K^2(g = 0.3) \in [0.24, 0.33]$$

$\Downarrow$

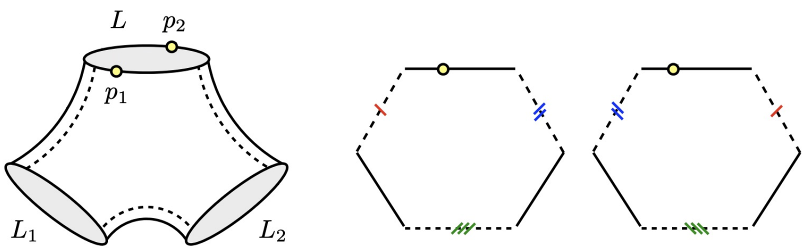
bound on Konishi

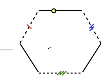


- SoV approach in spin-chains (see Paul's talk)

What about C_{ijk}?

— Integrability based approach: Hexagons



Fundamental object is the hexagon FF! 

Can also be used to tessellate higher p.f.

We will be interested in structure constants of two protected and one unprotected operator. More general spinning C_{ijk} can also be considered but Matrix part is hard! (Also from higher point func.)
Enrico's talk

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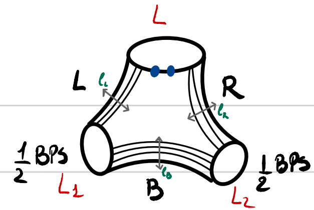
Asymptotic Hexagons



$\ell_L = \frac{L_1+L-L_2}{2}$

$\ell_R = L-\ell_L \qquad \ell_B = \frac{L_1+L_2-L}{2}$

- To compute C_{IJK} we need to glue back together the two hexagons

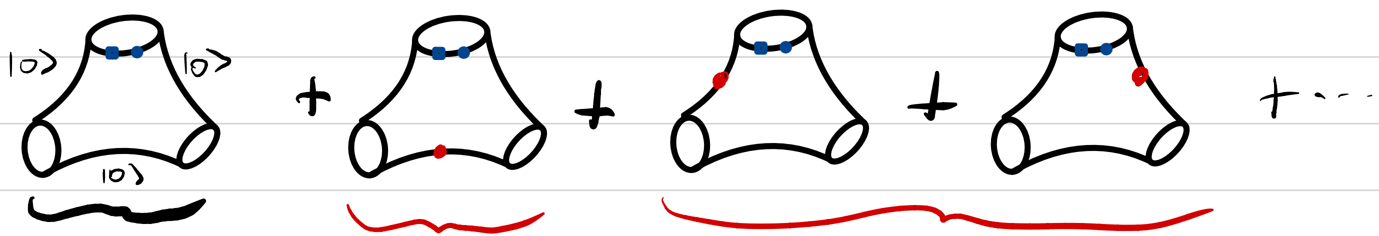
We will be interested in $\langle \mathrm{Tr}[\underset{\circ}{Z_1}^{L_1}]\,\mathrm{Tr}[\underset{\circ}{Z_2}^{L_2}]\,\mathrm{Tr}[\underset{\bullet}{D^S Z^L}]\rangle$

$$C^{\bullet\circ\circ} = N \times \sum\!\!\!\!\!\!\int_L \sum\!\!\!\!\!\!\int_R \sum\!\!\!\!\!\!\int_B e^{-\ell_L E_L}\, e^{-\ell_R E_R}\, e^{-\ell_B E_B}\, |\mathcal{H}|^2$$

where $\sum\!\!\!\!\!\!\int = \sum_{N=0}^{\infty} \prod_{i=1}^{N} \sum_{a_i=1}^{\infty} \int \frac{du_i}{2\pi}\, \mu_{a_i}(u_i) \prod_{k<j} P_{a_i a_j}(u_i,u_j)$ describe a magnon with energy $E_a = \log[x^{[+a]} x^{[-a]}]$

and the integrand $|\mathcal{H}|^2$ is obtained by studying the interactions between magnons and bethe roots

Gluing Back

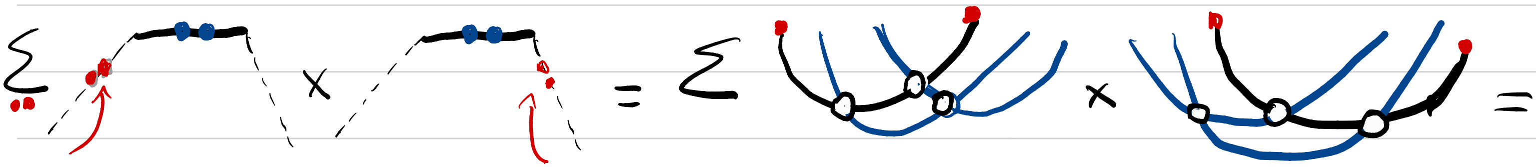


asymptotic contribution

g^4 $\Rightarrow$ For shortest operator $\Leftarrow$ g^6

$\ell_B = \ell_R = \ell_L \gg 1$

How does the single magnon insertions behave?

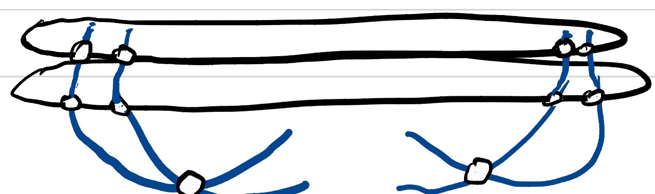


$\propto T(u^{+\gamma})$

↳ SU(2|2) transfer matrices

...

For multi magnons the result factorizes into product of T



$$|\mathcal{H}|^2 = \prod_{i,j,k}^{N_L N_R N_B} \frac{\mathbb{W}^L_{a_i}(u_i)\,\mathbb{W}^R_{b_j}(v_j)\,\mathbb{W}^B_{c_k}(w_k)}{P_{a b_j}(u_i, v_j)}$$

multi particle product $\sim \delta_{ab}\,\mu_a(u)\,(u-v)^2$ $\Rightarrow$ pole!

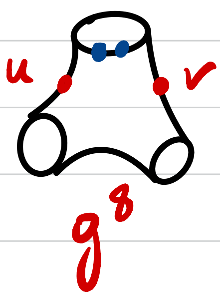
The single weights are:

$$\mathbb{W}^R_a(u) = \frac{T_a(u^\gamma)}{h_{1a}(z, u^\gamma)} \qquad \mathbb{W}^L_a(u) = h_{a1}(u^\gamma, z)\, T_a(u^\gamma)$$

$$\mathbb{W}^B_a(u) = \frac{T(u^\gamma)}{h_{a1}(u^\gamma, z)}$$

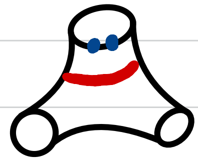
What happens if we have 2 mirror excitations?

Wrapping

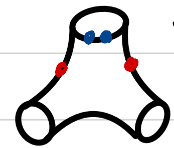
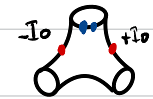


pole $u=v$ $\Longrightarrow$

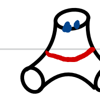
badly divergent!



how to fix this! deform contour and extract contact term (chord)

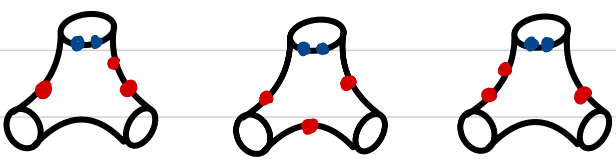
$\Big[$  $\Big]$ = 

bulk integral with shifted contour

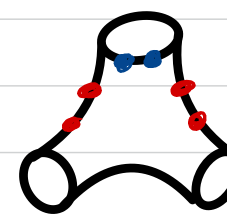
\+ 

1-folded contact term

Many more terms at higher loops

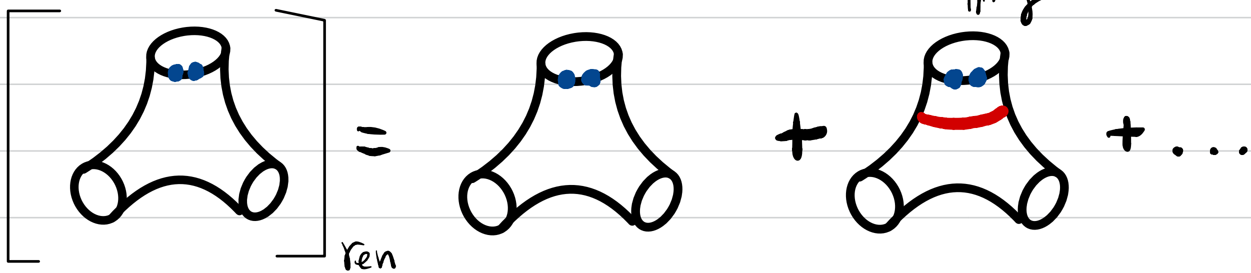


or even double wrapping effects!



Idea: compute as many as we can and try to extract some pattern!

The new integrand can be also packed into a factorizable form!
Now each weight $\mathbb{W} \Rightarrow \mathbb{W}_{\text{asympt}} + \underbrace{\delta \mathbb{W}}_{\text{wrapping}}$



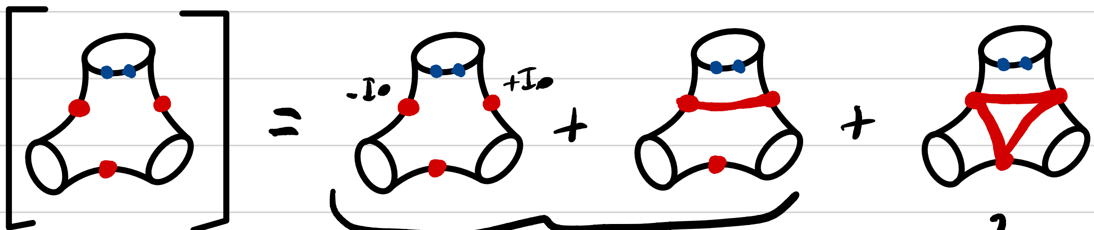
After some carefull extrapolation we can write relations that should be satisfied by the "complete" weights.

$$\boxed{\begin{aligned} &e^{-L\mathcal{E}_a(u)}\mathbb{W}^L_a(u)\mathbb{W}^R_a(u) = \frac{Y_{a,0}}{1+Y_{a,0}} \\ &\frac{\mathbb{W}^L_a(u)}{\mathbb{W}^R_a(u)} = p_{a1}(u,\mathbf{z})\, e^{i\sum_b \oint \frac{dv}{2\pi} L_b(v)\partial_v \log p_{ba}(v,u)} \end{aligned}}$$

$$\left.\begin{aligned} Y_{a,s} &= T_{a,s+1}T_{a,s-1}/T_{a+1,s}T_{a-1,s} \\ L_a &= \log(1+Y_{a,0}) \end{aligned}\right\} \text{TBA data}$$

Bottom bridge

Let us study the corrections to the bottom bridge

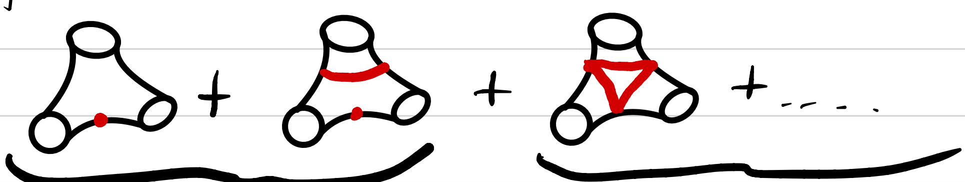


fit into picture ?

$\Longrightarrow$ new effect uncovered @ 5-loops

We can now fix the B-Bridge!

$\mathbb{W}^B \simeq$



$$\frac{e^{ip\frac{L}{2}}\, T(y^{\gamma})}{h_{1a}(z,u^{r})} \qquad\qquad \frac{e^{-ip\frac{L}{2}}\, \bar{T}(y^{\gamma})}{h_{1a}(z,u^{r})}$$

Conjecture

$$\mathbf{T}^{\pm}_{a,s} = \mathbf{T}_{a,s}\left(u \pm \tfrac{i}{2}\right)$$

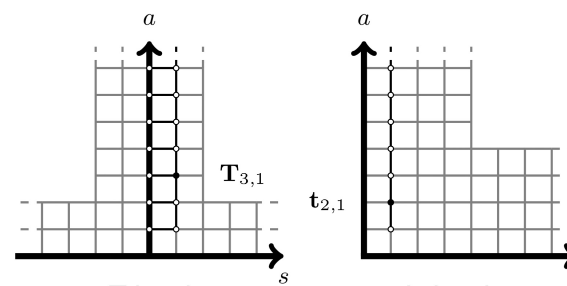
They are solutions of the hirota eq

$$\mathbf{T}^{+}_{a,s}\mathbf{T}^{-}_{a,s} = \mathbf{T}_{a+1,s}\mathbf{T}_{a-1,s} + \mathbf{T}_{a,s+1}\mathbf{T}_{a,s-1}$$

T matrices are defined up to gauge transformations

$$\mathbf{T}_{a,s} = g_1^{[a+s]}\, g_2^{[a-s]}\, g_3^{[s-a]}\, g_4^{[-s-a]}\, \mathbf{T}_{a,s}$$

$$\boxed{\begin{aligned} \mathbb{W}^{L}_{a}(u) &= e^{\frac{1}{2}L\mathcal{E}_a(u)}\,\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^{-}_{a,0}(u)} \\ \mathbb{W}^{R}_{a}(u) &= e^{\frac{1}{2}L\mathcal{E}_a(u)}\,\frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^{+}_{a,0}(u)} \\ \mathbb{W}^{B}_{a}(u) &= e^{-\frac{1}{2}L\mathcal{E}_a(u)}\,\mathbf{t}_{a,1}(u) \end{aligned}}$$



T-hook L-hook

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- Present conjecture for Hexagons of short operators
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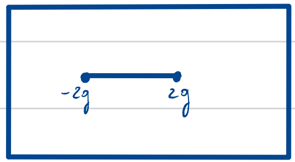
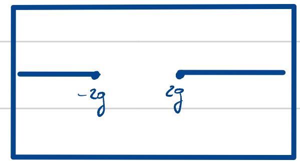
Perturbative QSC 101

- 8 basic Q-functions $(P_1, P_2, P_3, P_4 \,|\, Q_1, Q_2, Q_3, Q_4)$





- Information about the state is encoded in the asymptotics.
- This is a R-H problem

  QQ-Relations + Analiticity

- We can obtain T matrices from combinations of Q-functions.

- We are interested in the perturbative solution of the QSC

space of functions: $u, \frac{1}{u}, \eta_{a_1 \cdots a_n}, \bar{\eta}_{a_1 \cdots a_n}$

$$\eta_{a_1 \cdots a_n}(u) = \sum_{0 \leq k_1 < \cdots < k_n}^{\infty} \frac{1}{(u+ik_1)^{a_1} \cdots (u+ik_n)^{a_n}} \qquad \eta_{a_1 \cdots a_n}(i) = (i)^{a} \zeta_{a_1 \cdots a_n}$$

They also satisfy some stuffle relations

$$\eta_a(u)\,\eta_b(u) = \eta_{a+b}(u) + \eta_{a,b}(u) + \eta_{b,a}(u)$$

- In principle we are ready to compute the integrand for the hexagon

$$T_{a,1} = -Q_{12|i}^{[+a]} \left(Q^{12|i}\right)^{[-a]} \qquad T_{a,0} = \frac{1}{2} Q_{[12|i]}^{[+a]} Q^{[12|i][-a]} \qquad t_{a,1} = -Q_i^{[+a]} Q^{i[-a]}$$

- Dima's implementation gives only the P functions. We need to generate Q with single e multi indices.

- To do so we can use

$$Q_{a|i}(u+\tfrac{i}{2}) - Q_{a|i}(u-\tfrac{i}{2}) = P_a(u)\,Q_i(u)$$

$$- P^a Q_{a|i}$$

$$Q_i = -P^a Q_{a|i}$$

$$Q' = -P_a Q^{a|i}$$

$$Q_{a|i} Q^{a|J} = -\delta_i^J$$

if we know the tree level value of $Q_{a|i}$ we can make an ansetz for the next order.

$$Q_{a|i} = Q^{(0)}_{a|i} + b_i^J(u+\tfrac{i}{2})\,Q^{(0)}_{a|J}(u) + \text{higher order}$$

This gives us a finite difference eq. for b!

$$b_i^J(u+i) - b_i^k(u) = -\delta S_{a|i}\,Q^{(0)a|k}(u-\tfrac{i}{2}) + O(\text{higher})$$

ex

$$f(u+i) - f(u) = \frac{1}{u^2} \qquad f(u) = \eta_2(u)$$

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What can we compute?

- So far integrating the adjacent bridge is still difficult, but the bottom contribution can be "automatized"

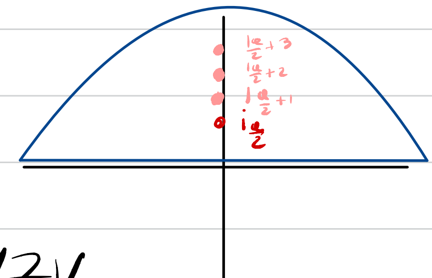
- $$R = \frac{C^{\bullet\circ\circ}}{\lim\limits_{l_B \to \infty} C^{\circ\circ\circ}} = 1 + \sum_{a=1}^{\infty} \int \frac{du}{2\pi}\, \mu_a(u)\, e^{-l_B E_a(u)}\, \mathbb{W}_a^B(u) + \dots$$

This requires the integration of $t_{a,1} = - \sum_{i}^{4} Q_i^{[+a]} \tilde{Q}^{i,[-a]}$

Let us focus on a specific case $S = L = 2$

What is the structure of the integrand?

$$\mu_a(u)\, e^{-\ell_B \mathcal{E}_a(u)}\, \mathbb{W}^B = \sum c_i\, a\, u^{v}\, \frac{\eta^{[+a]}_{a_1\dots a_n}(u)\, \bar{\eta}^{[-a]}_{a_1\dots a_n}(u)}{(u-\frac{ia}{2})^{\ell}(u+\frac{ia}{2})^{\ell}}$$



One can compute the integrand by residues

$$\sum c_i \frac{S_{n_1\dots n_n}(a)}{a_i^{p}}\, \zeta_{n_1\dots n_n} \Longrightarrow \text{combination of MZV}$$

$$S_{n_1\dots n_n}(x) = \sum_{k_1=1}^{x} \frac{S_{n_2\dots n_n}(k_1)}{k_1^{n_1}} \qquad S_n(x) = \sum_{k=1}^{x} \frac{1}{k^n}$$

We can compute $R(\ell_B)$ for different values of ℓ_B

Matches with PT

$\underline{\ell_B = 1}\Big|$ g^{10} $\quad 8064\,\zeta_3 + 864\,\zeta_3^2 - 720\,\zeta_5 - 9936\,\zeta_3\zeta_5 + 8022\,\zeta_7 - 54432\,\zeta_9$

$\underline{\ell_B = 2}\Big|$ g^{12}

$$-1728\,\zeta_3^3 + 26880\,\zeta_5 - 36360\,\zeta_5^2 - 9360\,\zeta_3\zeta_5 - 64512\,\zeta_3\zeta_7 + 24738\,\zeta_7 + 64992\,\zeta_9 - 471240\,\zeta_{11}$$

$\underline{\ell_B = 3}\Big|$ g^{14}

$$\frac{20088}{5}\,\zeta_{3,5,3} - \frac{20088}{5}\,\zeta_{5,3}\zeta_3 - 5184\,\zeta_3^3 - 40392\,\zeta_3^2\zeta_5 + 33120\,\zeta_5^2 + 44060\,\zeta_7 - 82152\,\zeta_3\zeta_7 - 380736\,\zeta_5\zeta_7 + 159012\,\zeta_9 - 319680\,\zeta_3\zeta_9 + \frac{2158038}{5}\,\zeta_{11} + 3418272\,\zeta_{13}$$

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Conclusions

- New formalism to construct hexagons integrands.
  How far can we push it for automatizing perturbative computations?
- Higher spin? (Needs Regulator)
- Can we do finite coupling in some approximation?
- Relation to SoV? Other theories?

Благодаря!

(Thenk you)