Spectral metrics on quantum projective spaces

Joint work with Max Holst Mikkelsen

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The state space of a unital C^*-algebra

Definition

Let A be a unital C^*-algebra. A **state** *on A is a unital postive map $\mu : A \to \mathbb{C}$. The set of states is denoted by $S(A)$ and becomes a* **compact Hausdorff space** *when equipped with the* **weak-$*$-topology** *(pointwise convergence).*

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Example

Let X be a compact Hausdorff space. It holds that $C(X)$ is a commutative unital C^*-algebra and the **state space** *$S(C(X))$ can be identified with the* **Radon probability measures** *on X.*

Compact quantum metric spaces

Definition

Let $\mathcal{A}$ be a dense unital $*$-subalgebra of a unital C^*-algebra A. Suppose that $L : \mathcal{A} \to [0, \infty)$ is a $*$-invariant seminorm with $L(1) = 0$.

- *The* **Monge-Kantorovich metric** *is the (extended) metric on the state space $\rho_L : S(A) \times S(A) \to [0, \infty]$ defined by*

$$\rho_L(\mu, \nu) := \sup \{ |\mu(x) - \nu(x)| : L(x) \leq 1 \}.$$

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Definition (Rieffel)

The pair $(\mathcal{A}, L)$ is a **compact quantum metric space** *if the metric ρ_L* **metrizes** *the weak-$*$-topology on the state space $S(A)$.*

Compact metric spaces

Example

Let X be a compact metric space with metric d and equip the Lipschitz functions $\mathrm{Lip}(X) \subseteq C(X)$ *with the seminorm*

$$L_d(f) := \text{“Lipschitz constant of } f\text{”}.$$

It then holds that $(\mathrm{Lip}(X), L_d)$ *is a compact quantum metric space and* $d(p,q) = \rho_{L_d}(\mathrm{ev}_p, \mathrm{ev}_q)$ *for all* $p, q \in X$.

Unital spectral triples

Definition

Let A be a unital C^*-algebra. A **unital spectral triple** $(\mathcal{A}, H, D)$ *on A is given by*

1. *A dense unital $*$-subalgebra $\mathcal{A} \subseteq A$, called the* **coordinate algebra**;
2. *A separable Hilbert space H equipped with a left action of A given by a unital $*$-homomorphism $\pi : A \to \mathbb{L}(H)$;*
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- *The resolvent $(i + D)^{-1} : H \to H$ is a* **compact** *operator.*

Compact spin manifolds

Example

Let M be a **compact spin manifold** *and let* $D\!\!\!/ : \Gamma^\infty(M, S) \to \Gamma^\infty(M, S)$ *denote the* **Dirac operator** *acting on the smooth sections of the spinor bundle S.*

- *It holds that the triple $(C^\infty(M), L^2(M, S), D\!\!\!/)$ is a* **unital spectral triple** *on $C(M)$.*

Spectral metric spaces

Observation

Let $(\mathcal{A}, H, D)$ *be a unital spectral triple. The seminorm* $L_D : \mathcal{A} \to [0, \infty)$ *defined by* $L_D(a) := \|\delta(a)\|$ *is* $*$*-invariant and satisfies* $L_D(1) = 0$.

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Definition (Bellisard, Marcolli and Reihani)

A unital spectral triple $(\mathcal{A}, H, D)$ *is a* **spectral metric space***, if the pair* $(\mathcal{A}, L_D)$ *is a compact quantum metric space.*

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Definition (Bellisard, Marcolli and Reihani)

A unital spectral triple $(\mathcal{A}, H, D)$ is a **spectral metric space**, *if the pair $(\mathcal{A}, L_D)$ is a compact quantum metric space.*

Theorem (Connes)

If M is a **compact spin manifold**, *then the unital spectral triple $(C^\infty(M), L^2(M, S), \not{D})$ is a* **spectral metric space** *and the Monge-Kantorovich metric recovers the metric on M (induced by the Riemannian metric).*

Quantum $SU(n)$: Coordinates

Definition (Woronowicz)

Let $q \in (0,1)$. The **coordinate algebra for quantum** *$SU(n)$ is a Hopf $*$-algebra $\mathcal{O}(SU_q(n))$ generated by n^2 elements u_{ij} subject to quadratic relations*

- $u_{ik} u_{jk} = q \cdot u_{jk} u_{ik}$, $u_{ki} u_{kj} = q \cdot u_{kj} u_{ki}$ $\quad i < j$
- $[u_{il}, u_{jk}] = 0$, $[u_{ik}, u_{jl}] = (q - q^{-1}) \cdot u_{il} u_{jk}$ $\quad i < j$, $k < l$

together with the determinant relation saying that the **quantum determinant equals** *1.*

Quantum $SU(n)$: Vector fields

Definition/Theorem (Drinfeld-Jimbo)

Let $q \in (0,1)$. The **quantized enveloping algebra of** $\mathfrak{su}(n)$ *is a Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{su}(n))$ which can be explicitly defined in terms of generators and relations.*

- *The quantized enveloping algebra acts on the coordinate algebra $\mathcal{O}(SU_q(n))$ via a* **non-degenerate dual pairing** *of Hopf $*$-algebras*

$$\langle \cdot, \cdot \rangle : \mathcal{U}_q(\mathfrak{su}(n)) \times \mathcal{O}(SU_q(n)) \to \mathbb{C}.$$

Quantum projective space

Definition

The **coordinate algebra for quantum projective space** $\mathcal{O}(\mathbb{C}P_q^{n-1})$ *is the unital* $*$*-subalgebra of* $\mathcal{O}(SU_q(n))$ *generated by* $u_{ni} \cdot u_{nj}^*$ *for* $i, j \in \{1, \ldots, n\}$.

Dolbeault operator

Definition

The q-deformed **Dolbeault operator**

$$\overline{\partial} + \overline{\partial}^{\dagger} : \Omega_q \to \Omega_q$$

acts on the q-deformed **antiholomorphic forms***. It can be explicitly defined in terms of the action of* $\mathcal{U}_q(\mathfrak{su}(n))$ *on* $\mathcal{O}(SU_q(n))$*.*

Remark

The Dolbeault operator induces an **essentially selfadjoint** *unbounded operator*

$$\overline{\partial} + \overline{\partial}^{\dagger} : \Omega_q \to L^2(\Omega_q, h)$$

on the Hilbert space completion of the antiholomorphic forms wrt. the **Haar state** $h : \mathcal{O}(SU_q(n)) \to \mathbb{C}$. *We put* D_q *equal to the selfadjoint closure of the Dolbeault operator.*

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Theorem (D'Andrea, Dabrowski (and Landi for $n = 3$))

The data $(\mathcal{O}(\mathbb{C}P_q^{n-1}), L^2(\Omega_q, h), D_q)$ *is a* **unital spectral triple**.

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Theorem (Mikkelsen and K.)

The unital spectral triple $\big(\mathcal{O}(\mathbb{C}P_q^{n-1}), L^2(\Omega_q, h), D_q\big)$ *is a* **spectral metric space***.*

An open question

Question

- *Can we understand the* **continuity properties** *of the spectral metric spaces* $\big(\mathcal{O}(\mathbb{C}P_q^{n-1}), L^2(\Omega_q, h), D_q\big)$ *under variations of the* **deformation parameter** q*?*

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- *One possibility is to consider this question with respect to the* **quantum Gromov-Hausdorff distance** *introduced by Marc Rieffel.*
- *Another, and more challenging option, is to apply the* **spectral propinquity** *which has recently been invented by Frédéric Latrémolière.*

The extension theorem

Lemma

The **spectrum** *of* $y := 1 - u_{nn}u_{nn}^*$ *agrees with the* **quantized interval** $I_q = \{q^{2k} \mid k \in \mathbb{N}_0\} \cup \{0\}$*. Hence all polynomial expressions in* y *can be identified with a unital* $*$*-subalgebra* $\mathcal{O}(I_q)$ *of* $C(I_q)$*.*

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Theorem (Mikkelsen and K.)

It holds that $\big(\mathcal{O}(\mathbb{C}P_q^{n-1}), L^2(\Omega_q, h), D_q\big)$ *is a* **spectral metric space** *if and only if* $\big(\mathcal{O}(I_q), L^2(\Omega_q, h), D_q\big)$ *is a* **spectral metric space**.

Metric properties of the quantized interval and main theorem as a corollary

Definition

Define a **metric** d *on the* **quantized interval** I_q *by the formula*

$$d(q^{2k}, q^{2m}) = \sum_{j=k}^{m-1} \frac{q^j(1-q^2)}{\sqrt{1-q^{2(j+1)}}} \quad \textit{for } m > k$$

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Theorem (Mikkelsen and K.; Gotfredsen, K. and Kyed)

It holds that $L_d(f) \leq L_{D_q}(f)$ *for all* $f \in \mathcal{O}(I_q)$.

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Corollary

A combination of the **extension theorem** *and the* **above estimate** *shows that* $\big(\mathcal{O}(\mathbb{C}P_q^{n-1}), L^2(\Omega_q, h), D_q\big)$ *is a* **spectral metric space**.