Duality + integrability of quantum Q-systems for classical root systems

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Varna, August 17 2022

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Plan

Fusion relations as discrete dynamical systems

Quantization

Macdonald-Koornwinder operators in q-Whittaker limit

Heisenberg spin chain: T-system $\rightsquigarrow$ Q-system for $\mathfrak{g} = \mathfrak{sl}_N$

Commuting transfer matrices $\{T_{i,j,k} = T_{V(k\omega_i)(q^j\zeta)}\}$

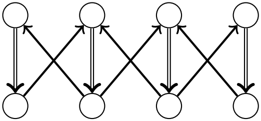
Fusion: $T_{i,j,k+1}T_{i,j,k-1} = T_{i,j+1,k}T_{i,j-1,k} - T_{i-1,j,k}T_{i+1,j,k}, \quad T_{0,j,k} = 1 = T_N$

Asymptotics $\zeta \to \infty$, $T_{i,j,k} \to \mathsf{Q}_{i,k}$:

Q-system: $\mathsf{Q}_{i,k+1}\mathsf{Q}_{i,k-1} = \mathsf{Q}_{i,k}^2 - \mathsf{Q}_{i-1,k}\mathsf{Q}_{i+1,k}, \quad \mathsf{Q}_{0,k} = 1 = \mathsf{Q}_{N+1,k}.$

Discrete evolution equations in k:

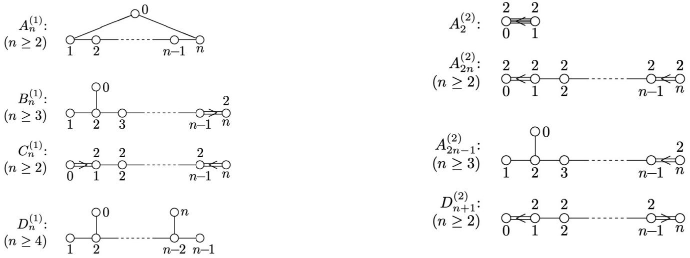
1. **Integrable:** Hamiltonians $\{H_a(Q_{i,k}, Q_{i,k+1})\}_{a=1}^{N-1}$ independent of k;
2. **Cluster algebra:**



3. Has canonical **Quantization**

Other classical root systems

Fusion relations, Q-systems for each affine root system, classical ones are



Quantum **Q-systems:** $\mathfrak{g} \to (R, R^*)$ (Macdonald notation)

$\mathfrak{g}$	$A_N^{(1)}$	$B_N^{(1)}$	$C_N^{(1)}$	$D_N^{(1)}$	$A_{2N-1}^{(2)}$	$A_{2N}^{(2)}$	$D_{N+1}^{(2)}$
R	A_N	B_N	C_N	D_N	C_N	BC_N	B_N
R^*	A_N	C_N	B_N	D_N	C_N	BC_N	B_N

Q-systems for the classical root systems

Characters of finite-dimensional KR-modules satisfy the Q-system

$$\mathsf{Q}_{a,k+1}\mathsf{Q}_{a,k-1} = \mathsf{Q}_{a,k}^2 - \prod_{b\neq a} \mathsf{Q}_{b,k}^{-C_{ba}}, \qquad 1 \leq a \leq r, \quad k \in \mathbb{Z}$$

($C =$ Cartan matrix of R)

Except for

$$B_N^{(1)} \quad : \quad \mathsf{Q}_{N-1,k+1}\mathsf{Q}_{N-1,k-1} = \mathsf{Q}_{N-1,k}^2 - \mathsf{Q}_{N-2,k}\mathsf{Q}_{N,2k};$$

$$\mathsf{Q}_{N,k+1}\mathsf{Q}_{N,k-1} = \mathsf{Q}_{N,k}^2 - \mathsf{Q}_{N-1,\lfloor \frac{k}{2} \rfloor}\mathsf{Q}_{N-1,\lfloor \frac{k+1}{2} \rfloor};$$

$$C_N^{(1)} \quad : \quad \mathsf{Q}_{N-1,k+1}\mathsf{Q}_{N-1,k-1} = \mathsf{Q}_{N-1,k}^2 - \mathsf{Q}_{N-2,k}\mathsf{Q}_{N,\lfloor \frac{k}{2} \rfloor}\mathsf{Q}_{N,\lfloor \frac{k+1}{2} \rfloor};$$

$$\mathsf{Q}_{N,k+1}\mathsf{Q}_{N,k-1} = \mathsf{Q}_{N,k}^2 - \mathsf{Q}_{N-1,2k};$$

$$A_{2n}^{(2)} \quad : \quad \mathsf{Q}_{N,k+1}\mathsf{Q}_{N,k-1} = \mathsf{Q}_{N,k}^2 - \mathsf{Q}_{N-1,k}\mathsf{Q}_{N,k}.$$

cluster algebra mutations relations*

Our goals:

- Integrable structure for any $\mathfrak{g}$
- Quantization, solutions as q-difference operators

Unifying framework: Macdonald-Koornwinder theory

Quantization of Q-systems $\mathrm{Q}_{a,k+1}\mathrm{Q}_{a,k-1} = \mathrm{Q}_{a,k}^2 - M_{a,k}$

Canonical quantization of the associated cluster algebras*: $\mathrm{Q} \rightsquigarrow \mathcal{Q}$

Noncommuting variables:

$$\mathcal{Q}_{a,t_a k+i}\mathcal{Q}_{b,t_b k+j} = q^{\Lambda_{ab} j - \Lambda_{ba} i}\mathcal{Q}_{b,t_b k+j}\mathcal{Q}_{a,t_a k+i}, \quad i,j = 0,1.$$

$\Lambda_{a,b} = \omega_a^* \cdot \omega_b$, ω_a, ω_a^* fundamental weights of R, R^*
$t_a = 2$ for short roots, $t_a = 1$ for long roots

Noncommutative evolutions:

$$q^{\Lambda_{aa}}\mathcal{Q}_{a,k+1}\mathcal{Q}_{a,k-1} = \mathcal{Q}_{a,k}^2 - :\!\mathcal{M}_{a,k}\!:, \quad a \in \Pi, k \in \mathbb{Z}$$

$:\!\mathcal{M}_{a,k}\!:=$normal/Weyl ordered product

Example: $R = R^* = \mathfrak{sl}_N$

$$\begin{aligned} \mathcal{Q}_{a,k}\mathcal{Q}_{b,k+i} &= q^{\min(a,b) i}\mathcal{Q}_{b,k+i}\mathcal{Q}_{a,k}, \qquad i = 0,1; \\ q^a \mathcal{Q}_{a,k+1}\mathcal{Q}_{a,k-1} &= \mathcal{Q}_{a,k}^2 - \mathcal{Q}_{a-1,k}\mathcal{Q}_{a+1,k}. \qquad \mathrm{Q}_{0,k} = 1, \mathrm{Q}_{N+1,k} = 0 \end{aligned}$$

The program

For each affine root system, we want to:

1. Prove integrability of quantum Q-systems
2. Find N quantum relativistic Toda Hamiltonians
3. Find solutions of quantum Q-systems as q-difference operators

The following program in type $A_{N-1}^{(1)}$ generalizes to the classical affine root systems $X_N^{(r)}$, $X = ABCD$, $r = 1, 2$.

Ingredients:

1. Duality (bi-spectrality) in Macdonald-Koornwinder theory
2. "Fourier transform" and q-Whittaker limit
3. $\tau_+ \in \mathrm{SL}_2(\mathbb{Z})$-action on DAHA commutes with Toda Hamiltonians, Baxter Q-operator, acts as time-translation

Macdonald's equations and their universal solutions

Macdonald eigenvalue equation: $\mathcal{D}_1(X)P_\lambda(X) = e_1(S)P_\lambda(X)$

- The variable $S = (s_1, ..., s_N)$ encodes λ: $s_i = q^{\lambda_i} t^{N-i}$.
- $e_a(S)$ are the elementary symmetric functions in s_i.

Macdonald's commuting difference operators act on the variables $X = (x_1, ..., x_N)$:

$$\mathcal{D}_a(X) = \sum_{\substack{I \subset [1,N] \\ |I|=a}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \Gamma_I, \qquad \Gamma_I \; x_j = q^{\delta_{j\in I}} x_j \Gamma_I.$$

- Macdonald polynomials: Unique monic, polynomial solution $P_\lambda(X; q, t)$ for λ dominant integral $\mathfrak{sl}_N$-weight

Universal Macdonald functions

Write $P_\lambda(X;q,t) \rightsquigarrow P(X|S)$ with $S=(s_1,...,s_N)$, $s_i = t^{N-i}q^{\lambda_i}$

Fact: Up to normalization, the eigenvalue equation

$$\mathcal{D}_1(X)P(X|S) = e_1(S)P(X|S)$$

has a unique solution as a series in $\{X^{-\alpha_i} = x_{i+1}/x_i\}$ of the form

$$P(X|S) = q^{\mu\cdot\lambda}\sum_{\beta\in \mathsf{Q}_+} c_\beta(S)X^{-\beta}, \qquad\qquad x_i = t^{N-i}q^{\mu_i}.$$

(Expand $\mathcal{D}_1(X)$ as a power series in x_{i+1}/x_i, then Eigenvalue equation is a triangular system for coefficients)

[Shiraishi-Noumi, Stockman's basic Harish-Chandra series]

Two normalizations for universal solution $P(X|S)$:

1. **Macdonald polynomials:**

$$P^{(1)}(X|S) = q^{\lambda\cdot\mu} \sum_{\beta\in\mathsf{Q}_+} c^{(1)}_\beta(S) X^{-\beta}, \quad c^{(1)}_0 = 1.$$

When $\lambda \in P_+$ the series truncates:

$$t^{\lambda\cdot\rho} P^{(1)}(X|S)\Big|_{\lambda\in P_+} = P_\lambda(X), \qquad \rho_i = N - i.$$

Any Macdonald polynomial is a specialization of the universal solution $P^{(1)}(X|S)$.

2. **Self-dual solutions:**

$$P^{(2)}(X|S) = q^{\lambda\cdot\mu} \sum_{\beta\in\mathsf{Q}_+} c^{(2)}_\beta(S) X^{-\beta}, \quad c^{(2)}_0(S) = \prod_{\substack{\alpha\in R_+\\ n>0}} \frac{1 - q^n S^{-\alpha}}{1 - t^{-1} q^n S^{-\alpha}}$$

Theorem [Macdonald's Duality for the universal function]:

$$P^{(2)}(X|S) = P^{(2)}(S|X).$$

Duality $\Longrightarrow$ Pieri rules:

Starting from the Macdonald eigenvalue equations

$$\mathcal{D}_a(X)P^{(2)}(X|S) = e_a(S)P^{(2)}(X|S) \tag{1}$$

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$$\mathcal{D}_a(X)P^{(2)}(X|S) = e_a(S)P^{(2)}(X|S) \qquad (1)$$

Rename the variables $X \leftrightarrow S$,

$$\mathcal{D}_a(S)P^{(2)}(S|X) = e_a(X)P^{(2)}(S|X)$$

Duality $\Longrightarrow$ Pieri rules:

Starting from the Macdonald eigenvalue equations

$$\mathcal{D}_a(X)P^{(2)}(X|S) = e_a(S)P^{(2)}(X|S) \qquad (1)$$

Rename the variables $X \leftrightarrow S$,

$$\mathcal{D}_a(S)P^{(2)}(S|X) = e_a(X)P^{(2)}(S|X)$$

Use duality $P^{(2)}(S|X) = P^{(2)}(X|S)$:

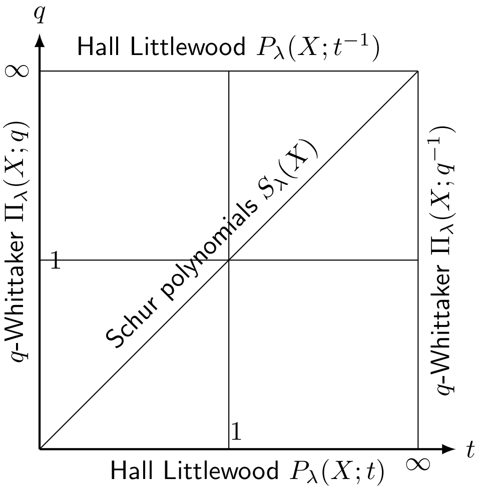
$$\mathcal{D}_a(S)P^{(2)}(X|S) = e_a(X)P^{(2)}(X|S). \qquad (2)$$

Pieri rule – Specialize to $\lambda \in P_+$: $\quad \mathcal{H}_a(S)P_\lambda(X) = e_a(X)P_\lambda(X)$,

Pieri operators = Hamiltonians $\mathcal{H}_a(S) = t^{\rho\cdot\lambda}\Delta^{-1}(S)\ \mathcal{D}_a(S)\ \Delta(S)t^{-\rho\cdot\lambda}$
q-difference operators in S

Theorem: Equation (2) has a unique solution, up to normalization, as a series in $\{S^{-\alpha_i}\}$. The two series, in $X^{-\alpha_i}$ and $S^{-\alpha_i}$, must be equal!

The q-Whittaker limit $t \to \infty$



Symmetry of Macdonald polynomials $P_\lambda(X; q, t) = P_\lambda(X; q^{-1}, t^{-1})$

Duality, EV equations in q-Whittaker limit $t \to \infty$

Duality: We lose symmetry: $(s_i = t^{N-i} q^{\lambda_i})$

$$\Delta(S) = c_0^{(2)}(S) \to 1, \quad \Delta(X) \to \overline{\Delta}(X) = \prod_{\substack{\alpha\in\Pi \\ n>0}} (1 - q^n X^{-\alpha}).$$

$$P^{(2)}(X|S) = P^{(2)}(S|X) \quad \underset{t\to\infty}{\longrightarrow} \quad \Pi(X|\Lambda) = \overline{\Delta}(X) K(\Lambda|X), \qquad \Lambda_i = q^{\lambda_i}.$$

Eigenvalue equations: $e_a(S) \to q^{\lambda_1+\cdots+\lambda_a} = \Lambda^{\omega_a}$ dominant term when $t \to \infty$

$$D_a(X)\Pi(X|\Lambda) = \Lambda^{\omega_a}\Pi(X|\Lambda), \qquad D_a(X) = \sum_{\substack{I\subset[1,N] \\ |I|=a}} \prod_{\substack{i\in I \\ j\notin I}} \frac{x_i}{x_i - x_j}\Gamma_I$$

The unique series solution $\Pi(X|\Lambda) = X^\lambda \sum_{\beta\in\mathsf{Q}_+} c_\beta(\Lambda) X^{-\beta}$ with $c_0 = 1$ truncates when $\lambda \in P_+$ to a class-I q-Whittaker polynomial.

Pieri rules in the q-Whittaker limit

$$H_a(\Lambda)K(\Lambda|X) = e_a(X)K(\Lambda|X)$$

Commuting operators $H_a(\Lambda) =$ relativistic quantum Toda Hamiltonians

$$H_1(\Lambda) = \sum_{i=0}^{N-1}(1-\Lambda^{-\alpha_i})T_i, \qquad \Lambda^{-\alpha_0} = 0, \quad T_i\Lambda_i = q\Lambda_i T_i.$$

- q-difference operator in $q^\lambda = \Lambda$

$$H_a(\Lambda) = \sum_{\substack{I\subset[1,N]\\|I|=a}} \prod_{\substack{i\in I,\\ i-1\notin I}} (1-\Lambda^{-\alpha_{i-1}})T_I.$$

Pieri rule has unique series solution of the form

$$K(\Lambda|X) = X^\lambda \sum_{\beta\in\mathsf{Q}_+} \tilde{c}_\beta(X)\Lambda^{-\beta}$$

($\tilde{c}_0 = 1$ gives fundamental q-Whittaker functions, weight μ if $X = q^{\mu+\rho}$)

$\mathrm{SL}_2(\mathbb{Z})$-action: Time translation operator

$\tau_+ \in \mathrm{SL}_2(\mathbb{Z})$ acts on the functional rep of DAHA via adjoint action of Cherednik's Gaussian γ:

$$\gamma(X) = \prod_{a=1}^{N} e^{\frac{(\log x_a)^2}{2\log q}}, \qquad \mathrm{Ad}_{\gamma}(x_i) = x_i, \quad \mathrm{Ad}_{\gamma^{-1}}(\Gamma_i) = q^{1/2} x_i \Gamma_i.$$

Define

$$D_{a,k}(X) := q^{-\frac{ak}{2}} \gamma^{-k} D_a(X) \gamma^k.$$

Theorem (Di Francesco-K, 19)

The q-difference operators $D_{a,k}(X)$ satisfy the $A_{N-1}^{(1)}$ quantum Q-system.

$$\text{(1)} \qquad D_{a,i} D_{b,i+1} = q^{\min(ab)} D_{b,i+1} D_{a,i}, \quad \text{(q-Commutation relations)}$$

$$\text{(2)} \qquad q^a D_{a,k+1} D_{a,k-1} = D_{a,k}^2 - D_{a+1,k} D_{a-1,k}, \quad \text{(Recursion relations)}$$

The following proof is generalizable to other root systems.

"Fourier Transform:" From X to Λ variables

Useful trick: The Universal q-Whittaker functions are "complete":
If $\{f_i(X)\}_i$ are q-difference operators in X satisfying relations R, with

$$f_i(x)\Pi(X|\Lambda) = \hat{f}_i(\Lambda)\Pi(X|\Lambda),$$

then $\{\hat{f}_i(\Lambda)\}_i$ satisfy relations R^{op} with opposite multiplication.

Define $\widehat{D}_{a,k}$: Solutions of the quantum Q^{op}-system with initial data:

1. $\widehat{D}_{a,0} = \widehat{D}_a = \Lambda^{\omega_a}$;
2. $\widehat{D}_{a,1} = \Lambda^{\omega_a} T^{\omega_a}$, where $T^{\omega_a}\Lambda^{\alpha_b} = q^{\delta_{ab}}\Lambda^{\alpha_b}T^{\omega_a}$;
3. $\widehat{D}_{a,k}$, $k \in \mathbb{Z}$ defined from Q^{op}-system
   $q^a\widehat{D}_{a,k-1}\widehat{D}_{a,k+1} = \widehat{D}^2_{a,k} - \widehat{D}_{a+1,k}\widehat{D}_{a-1,k}$.

We already know $D_{a,0}(X)\Pi(X|\Lambda) = \widehat{D}_{a,0}(\Lambda)\Pi(X|\Lambda)$ (eigenvalue equation).
We want to prove $D_{a,k}(X)$ is the FT of $\widehat{D}_{a,k}(\Lambda)$ for all k.

FT of γ=Time translation operator=Baxter Q op

Theorem

1. *There exists a unique q-difference operator* $g(\Lambda)$ *such that* $g\widehat{D}_{a,k}(\Lambda)g^{-1} = q^{a/2}\widehat{D}_{a,k+1}(\Lambda)$*:*

$$g(\Lambda) = \prod_{a=1}^{N} e^{\frac{(\log T_a)^2}{2\log q}} \prod_{a=1}^{N-1}\prod_{n\geq 0}(1-q^n\Lambda^{-\alpha_a})^{-1}.$$

2. $g(\Lambda)$ *commutes with the quantum Toda Hamiltonians* $H_a(\Lambda)$*.*
3. $g(\Lambda)$ *is the Fourier transform of the Gaussian* $\gamma(X)$*:*

$$\gamma(X)\Pi(X|\Lambda) = g(\Lambda)\Pi(X|\Lambda).$$

Proof:
(1) Is by explicit calculation using Q^{op}-system; (2) by using explicit form of Hamiltonians.

Proof of (3): $\gamma(X)\Pi(X|\Lambda) = g(\Lambda)\Pi(X|\Lambda)$.

1. Pieri rule $H_1(\Lambda)K(\Lambda|X) = e_1(X)K(\Lambda|X)$ has unique solution.
2. Act on Pieri with $g(\Lambda)$, using $[g(\Lambda), H_1(\Lambda)] = 0$:

$$g(\Lambda)H_1(\Lambda)K(\Lambda|X) = H_1(\Lambda)g(\Lambda)K(\Lambda|X) = e_a(X)g(\Lambda)K(\Lambda|X),$$

$$\Longrightarrow g(\Lambda)K(\Lambda|X) = \text{const } K(\Lambda|X).$$

3. Use expression of $K(\Lambda|X)$ as a series in $\Lambda^{-\alpha_a}$, with leading term x^λ, and the relation

$$g(\Lambda)x^\lambda g(\Lambda)^{-1} = \gamma(x)(1 + \text{lower}) \Longrightarrow \text{const} = \gamma(X). \qquad \square$$

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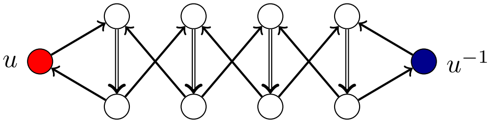
Corollary:

1. FT of $\widehat{D}_{a,k}(\Lambda) = q^{-\frac{ak}{2}} g(\Lambda)\widehat{D}_{a,0} g(\Lambda)^{-1}$ is $D_{a,k}(X) = q^{-\frac{ak}{2}} \gamma(X)^{-1} D_{a,0}\gamma(X)$.
2. $D_{a,k}(X)$ satisfy the quantum Q-system.
3. $[g(\Lambda), H_a(\Lambda)] = 0$: Conserved quantities of qQ-system are Toda Hamiltonians.

Baxter Q-operator and $g(\Lambda)$

The exponential generating function of Toda Hamiltonians is the Baxter Q-operator: A sequence of mutations in the extended cluster algebra



Mutations in a quantum cluster algebra act by adjoint action of q-dilogarithms:

$$Q(u) = Q_+(u)Q_-(u^{-1}), \qquad g(\Lambda) = Q(1)$$

The generating function for $H_a(\Lambda)$:

$$T(u) = Q_+(q^{\frac{1}{2}}u)Q_+(q^{-\frac{1}{2}}u)^{-1} = \sum_{a=0}^{N} u^a H_a(\Lambda)$$

Generalization to BC-root systems

Koornwinder operators + higher q-difference operators (+van Diejen, Macdonald, Rains...) depend on parameters $\alpha = (a, b, c, d)$

1. Specialize α for each classical affine root system:

$\mathfrak{g}$	a	b	c	d	R	R^*
$D_N^{(1)}$	1	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	D_N	D_N
$B_N^{(1)}$	t	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	B_N	C_N
$C_N^{(1)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	$-t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	C_N	B_N
$A_{2N-1}^{(2)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	C_N	C_N
$D_{N+1}^{(2)}$	t	-1	$t\,q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	B_N	B_N
$A_{2N}^{(2)}$	t	-1	$t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	$-t^{\frac{1}{2}}\,q^{\frac{1}{2}}$	BC_N	BC_N

2. Act on BC-symmetric Laurent polynomials, invariant under permutations of variables + inversions.

Universal Koornwinder function: Eigenvalue equations

Koornwinder operator: $\mathcal{D}_1^{(\alpha)}(X) = \sum_{\epsilon=\pm 1} \sum_{i=1}^{N} \Phi_{i,\epsilon}(X)\Gamma_i^{\epsilon} + \varphi(X),$

$$\Phi_{i,\epsilon}(X) = \frac{\prod_{\alpha=a,b,c,d} (1-\alpha x_i^{\epsilon})}{(1-x_i^{2\epsilon})(1-q x_i^{2\epsilon})} \prod_{\substack{j\neq i \\ \epsilon'=\pm 1}} \frac{t x_i^{\epsilon}/x_j^{\epsilon'} - 1}{x_i^{\epsilon}/x_j^{\epsilon'} - 1}$$

Eigenvalue equation

$$D_1^{(\alpha)}(X) P^{(\alpha)}(X|S) = \widehat{e}_1(S) P^{(\alpha)}(X|S), \qquad \widehat{e}_1(S) = \sum_{i=1}^{N} (s_i + s_i^{-1})$$

Unique solution, up to normalization $c_0(S)$, of the form

$$P^{(a,b,c,d)}(X|S) = q^{\lambda\cdot\mu} \sum_{\beta \in \mathsf{Q}_+(B_N)} c_\beta(S) X^{-\beta}.$$

Two natural normalizations of universal function

1. **Koornwinder polynomials**: $c_0(S) = 1$

$$P^{(\alpha)}(X|S) \to P^{(\alpha)}_\lambda(X)$$

If $\lambda \in P_+(C_N)$, $s_i = \sqrt{\frac{abcd}{q}} t^{N-i} q^{\lambda_i}$: Weyl-symmetric Laurent polynomial in X.

2. "Self-Dual":

$$c_0(S) = \Delta^{(\alpha^*)}(S) = \prod_{i=1}^{N} \frac{(\frac{q}{x_i^2};q)_\infty}{\prod_{\alpha=a,b,c,d}(\frac{q}{\alpha^* x_i};q)_\infty} \prod_{\substack{i<j \\ \epsilon=\pm 1}} \frac{(\frac{q x_j^\epsilon}{x_i};q)_\infty}{(\frac{q x_j^\epsilon}{t x_i};q)_\infty}$$

$$(a,b,c,d)^* = \left(\sqrt{\frac{abcd}{q}}, -\sqrt{\frac{qac}{bd}}, \sqrt{\frac{qac}{bd}}, \sqrt{\frac{qad}{bc}}\right).$$

Duality/bispectrality theorem for Koornwinder functions:

$$P(X|S) = P^*(S|X)$$

Toda Hamiltonians

- Eigenvalue equation + duality $\Longrightarrow$ Pieri rules:

$$\begin{aligned}
D_i^{(\alpha)}(X)P^{(\alpha)}(X|S) &= \widehat{e}_i(S)P^{(\alpha)}(X|S) && \text{(Eigenvalue eq.)}\\
D_i^{(\alpha)}(S)P^{(\alpha)}(S|X) &= \widehat{e}_i(X)P^{(\alpha)}(S|X) && (X \leftrightarrow S)\\
D_i^{(\alpha)}(S)P^{(\alpha^*)}(X|S) &= \widehat{e}_i(X)P^{(\alpha^*)}(X|S) && \text{(Duality)}\\
D_i^{(\alpha^*)}(S)P^{(\alpha)}(S|X) &= \widehat{e}_i(X)P^{(\alpha)}(S|X) && (\alpha \leftrightarrow \alpha^*)
\end{aligned}$$

- Specialize $\alpha = (a, b, c, d)$ to $\mathfrak{g}$, q-Whittaker limit $t \to \infty$:
  Commuting Toda Hamiltonians $H_i^{(\mathfrak{g})}(\Lambda)$

$$H_i^{(\mathfrak{g})}(\Lambda) = \lim_{t\to\infty} t^{\rho^*\cdot\lambda}\Delta^{(\mathfrak{g}^*)}(S)^{-1}D_i^{(\mathfrak{g}^*)}(S)\Delta^{(\mathfrak{g}^*)}(S)t^{-\rho^*\cdot\lambda}$$

  Pieri operators for q-Whittaker functions:

$$H_i^{(\mathfrak{g})}(\Lambda)\Pi_\lambda^{(\mathfrak{g})}(X) = \widehat{e}_i(X)\Pi_\lambda^{(\mathfrak{g})}(X)$$

  Eigenvalue equation in q-Whittaker limit:

$$D_i^{(\mathfrak{g})}(X)\Pi_\lambda^{(\mathfrak{g})}(X) = \Lambda^{\omega_i^*}\Pi_\lambda^{(\mathfrak{g})}(X)$$

Time translation operators for all classical $\mathfrak{g}$

1. q-Whittaker limit $D_{i,0}^{(\mathfrak{g})}(X) = \lim_{t\to\infty} \mathcal{D}_i^{(\mathfrak{g})}(X;q,t)$;
2. Define $D_{i,k}^{(\mathfrak{g})}(X) = q^{-k\omega_i\cdot\omega_i^*/2}\gamma(X)^{-1}D_i^{(\mathfrak{g})}\gamma(X)$ (i long label, $+$...)
3. **Def:** $\widehat{D}_{i,k}^{(\mathfrak{g})}(\Lambda)$=solutions of quantum Q^{op}-system, initial data

$$\widehat{D}_{i,0}^{(\mathfrak{g})} = \Lambda^{\omega_i^*} \qquad \widehat{D}_{i,1}^{(\mathfrak{g})} = \Lambda^{\omega_i^*}T^{\omega_i}$$

4. Construct unique time-translation operator $g^{(\mathfrak{g})}(\Lambda)$ such that

$$\widehat{D}_{i,k}^{(\mathfrak{g})}(\Lambda) = q^{-k\omega_i\cdot\omega_i^*/2}g^{(\mathfrak{g})}(\Lambda)\widehat{D}_{i,0}^{(\mathfrak{g})}(\Lambda)g^{(\mathfrak{g})}(\Lambda)^{-1}$$

   Theorem: The functions $g^{(\mathfrak{g})}(\Lambda)$ exist, unique, commute with the Toda Hamiltonians.

5. **Corollary:** The Fourier transforms $D_{i,k}^{(\mathfrak{g})}(X)$ satisfy the quantum Q-systems (functional representation).
6. Thm/Conj: $g^{(\mathfrak{g})}(\Lambda)$ are evaluations of Baxter Q-operators.

Summary

1. Solutions of quantum Q-systems are τ_+-translated (special choice of) Macdonald-Koornwinder operators in q-Whittaker limit with specialized (a, b, c, d).
2. Elements in the spherical DAHA: Cluster algebra structure in sDAHA.
3. Integrability: Hamiltonians $=$ Pieri operators in q-Whittaker limit.
4. Time translation operators $g(\Lambda) =$ "Baxter Q" at specialized spectral parameter
5. At finite t, Translated Macdonald operators $D_{a,k}(X; q, t)$ satisfy (quotient of) quantum toroidal algebra in type A. Algebra in other types?
6. Exceptional types?

Exchange matrices for the Q-systems

For $\mathfrak{g} = X_N^{(1)}$, the quiver/exchange matrix is skew-symmetric:

$$B = \left[\begin{array}{c|c} C^t - C & -C^t \\ \hline C & 0 \end{array}\right]$$

with C the Cartan matrix of R.

For $\mathfrak{g} = A_{2N-1}^{(2)}$ or $D_{N+1}^{(2)}$,

$$B = \left[\begin{array}{c|c} 0 & -C \\ \hline C & 0 \end{array}\right].$$

For $\mathfrak{g} = A_{2N}^{(2)}$: Not a cluster algebra.

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