Structure of $\mathcal{W}$-algebras with non-admissible levels

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The Deligne exceptional series

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- The Deligne dimension formulas

$$\dim \mathfrak{g} = \frac{2(h^\vee + 1)(5h^\vee - 6)}{h^\vee + 6},$$

$$\dim L(2\theta) = \frac{5h^{\vee 2}(2h^\vee + 3)(5h^\vee - 6)}{(h^\vee + 12)(h^\vee + 6)},$$

etc.

2-character rational conformal field theories (RCFTs)

Consider the $SL_2(\mathbb{Z})$-invariant (modular invariant) differentail equation of the form

$$\left(q\frac{d}{dq}\right)^2 f(\tau) + 2E_2(\tau)\left(q\frac{d}{dq}\right) f(\tau) + \mu \cdot E_4(\tau) f(\tau) = 0. \qquad (1)$$

Here μ is a numerical constant, τ a complex number in the complex upper half-plane $\mathbb{H}$ with $q = e^{2\pi i\tau}$, and $E_k(\tau)(k = 2, 4, 6, \ldots)$ the Eisenstein series.

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[Mathur-Mukhi-Sen, 1988]: By studying the above equations, they showed that the characters of RCFTs (C_2-cofinite rational $\mathbb{Z}_+$-graded VOAs) with 2 independent irreducible characters must be the characters of the level one WZW-models (simple affine VOAs)

$$L_1(A_1) \subset L_1(A_2) \subset L_1(G_2) \subset L_1(D_4) \subset L_1(F_4) \subset L_1(E_6) \subset L_1(E_7) \subset L_1(E_8)$$

associated with the Deligne exceptional Lie algebras.

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- Here, for fixed $\mathfrak{g}$, a number $k \in \mathbb{C}$ is called *admissible* if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{>0}, \quad (p, q) = 1, \quad p \geq \begin{cases} h^\vee & (r^\vee, q) = 1, \\ h & (r^\vee, q) = r^\vee. \end{cases}$$

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  (cf. Kac-Wakimoto conjecture: a simple affine VOA $L_k(\mathfrak{g})$ has the modular invariance property if and only if the level k is admissible))

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The simple quotient is denoted by $\mathcal{W}_k(\mathfrak{g}, f)$ and called the simple *affine $\mathcal{W}$-algebra*.

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Suppose that

- $\mathfrak{g}$ is not of type A_l;
- the maximal affine subVOA V and certain Virasoro subVOA U of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ are RCFTs (C_2-cofinite and rational $\mathbb{Z}_+$-graded VOAs);
- $\mathcal{W}_k(\mathfrak{g}, f_\theta) \cong V \otimes U \oplus N \otimes M$ with *simple current* V, U-modules N, M.

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Theorem (K. 2015)

The complete list of the pair $(\mathfrak{g}, k)$ *satisfying the above assumptions is given by the following pairs:*

1. $(\mathfrak{g}, k) = (sp_4, 1/2)$;
2. $\mathfrak{g} = G_2, D_4, F_4, E_6, E_7, E_8$ *with* $k = -h^\vee/6$.

Main theorem (explicit branching rules)

We have the following isomorphisms:

$$\mathcal{W}_{1/2}(C_2, f_\theta) \cong L_1(A_1) \otimes L(-25/7, 0) \oplus L_1(A_1; \alpha/2) \otimes L(-25/7, 5/4),$$

$$\mathcal{W}_{-2/3}(G_2, f_\theta) \cong L_3(A_1) \otimes L(-3/5, 0) \oplus L_3(A_1; \alpha/2) \otimes L(-3/5, 3/4),$$

$$\mathcal{W}_{-1}(D_4, f_\theta) \cong L_1(A_1)^{\otimes 3} \otimes L(-3/5, 0) \oplus L_1(A_1; \alpha/2)^{\otimes 3} \otimes L(-3/5, 3/4),$$

$$\mathcal{W}_{-3/2}(F_4, f_\theta) \cong L_1(C_3) \otimes L(-3/5, 0) \oplus L_1(C_3; \varpi_3) \otimes L(-3/5, 3/4),$$

$$\mathcal{W}_{-2}(E_6, f_\theta) \cong L_1(A_5) \otimes L(-3/5, 0) \oplus L_1(A_5; \varpi_3) \otimes L(-3/5, 3/4),$$

$$\mathcal{W}_{-3}(E_7, f_\theta) \cong L_1(D_6) \otimes L(-3/5, 0) \oplus L_1(D_6; \varpi_6) \otimes L(-3/5, 3/4),$$

and

$$\mathcal{W}_{-5}(E_8, f_\theta) \cong L_1(E_7) \otimes L(-3/5, 0) \oplus L_1(E_7; \varpi_7) \otimes L(-3/5, 3/4).$$

Here, $L(c, h)$ denotes the simple Virasoro VOA (or module) of central charge c and L_0-weight h.

Remarks

- For $\mathfrak{g} = A_1, A_2$ with $k = -h^\vee/6 = -1/3, -1/2$, we have

$$\mathcal{W}_{-1/3}(A_1, f_\theta) \cong L(-3/5, 0),$$

$$\mathcal{W}_{-1/2}(A_2, f_\theta) \cong V_{\sqrt{3}A_1} \otimes L(-3/5, 0) \oplus V_{\sqrt{3}A_1 + \sqrt{3}\alpha/2} \otimes L(-3/5, 3/4).$$

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- For $\mathfrak{g} = D_4, E_6, E_7, E_8$, the levels $k = -h^\vee/6 = -1, -2, -3, -5$ are not admissible numbers.
- Since each $\mathcal{W}$-algebra has been believed to be C_2-cofinite and rational only if the level k is admissible (cf. [Kac-Wakimoto, 2008]), they are new examples of C_2-cofinite rational $\mathcal{W}$-algebras.