Jet schemes of nilpotent orbit closures

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(joint work with Rupert Yu)

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- If $X \subset \mathbb{A}^N$ is affine, defined by equations $f_1, \ldots, f_r$, then $\mathcal{J}_m(X) \subset \mathbb{A}^{N(m+1)}$ is affine, defined by equations expressing that

$$\forall j = 1, \ldots, r, \quad f_j(x_0 + x_1 t + \cdots + x_m t^m) \equiv 0 \mod [t^{m+1}].$$

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- $\mathcal{J}_m(X)$ smooth $\iff$ X smooth (if so, $\dim \mathcal{J}_m(X) = (m+1)\dim X$),
- $\boxed{\mathcal{J}_m(X) \text{ irreducible} \implies X \text{ irreducible}}$,
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▶ This was conjectured by D. Eisenbud and E. Frenkel in order to apply it to the nilpotent cone of a reductive Lie algebra...

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is the *generalized Takiff Lie algebra*, with Lie bracket

$$\forall\, x, y \in \mathfrak{g}, \forall\, k, l \in \mathbb{Z}_{\geqslant 0}, \quad [x \otimes t^k, y \otimes t^l] = [x, y] \otimes t^{k+l}.$$

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Theorem (Mustaţă / Eisenbud-Frenkel, 2001).

$\forall\, m \geqslant 0$, $\mathcal{J}_m(\mathcal{N})$ is irreducible. Moreover, $\mathbb{C}[\mathcal{J}_m(\mathfrak{g})]$ is free over $\mathbb{C}[\mathcal{J}_m(\mathfrak{g})]^{\mathcal{J}_m(G)}$.

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- If $\boldsymbol{\lambda} = (3^2)$, $n = 6$, then $\mathcal{J}_1(\overline{\mathcal{O}_{\boldsymbol{\lambda}}})$ is irreducible. But we still can use jets to conclude that $\overline{\mathcal{O}_{(3^2)}}$ is not a c.i. (more complicated).

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- If $\mathfrak{g}$ has type A, then $\mathcal{O}_{(2^p,1^q)}$ is little for $p > 0$ and $q > 0$.

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- Idea for $m = 1$. $\overline{\mathcal{O}}$ is defined by equations $f_1, \ldots, f_r$ with $\deg f_i \geqslant 2$. So, for all i,

$$f_i(0 + tx) = t^2 f_i(x) \equiv 0 \mod [t^2],$$

whence $\pi_1^{-1}(0) \cong \mathfrak{g}$ and $\dim \pi_1^{-1}(0) \geqslant 2 \dim \mathcal{O} = \dim \pi_m^{-1}(\mathcal{O})$...

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There exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ of $\mathfrak{g}$ such that $\mathcal{O}_{\mathfrak{g}} \cap (\mathcal{O}_{\mathfrak{l}} + \mathfrak{n})$ is dense in $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$. Moreover, and $\text{codim}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) = \text{codim}_{\mathfrak{l}}(\mathcal{O}_{\mathfrak{l}})$. We write $\mathcal{O}_{\mathfrak{g}} = \text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$.

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In particular, for any nilpotent orbit $\mathcal{O}$ induced from a little orbit, $\mathcal{J}_m(\overline{\mathcal{O}})$ is reducible for any m.

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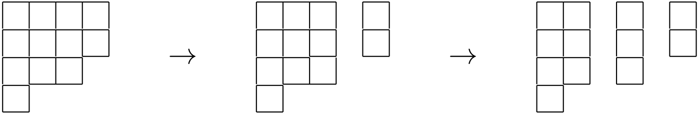

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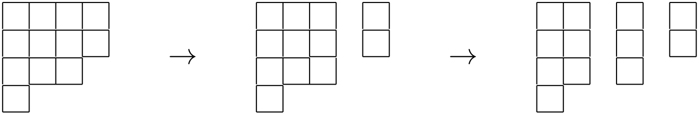

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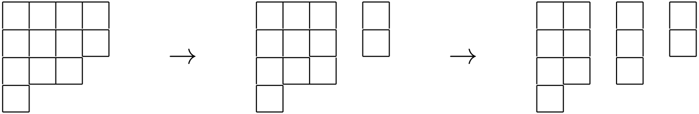

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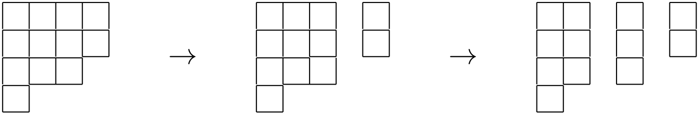

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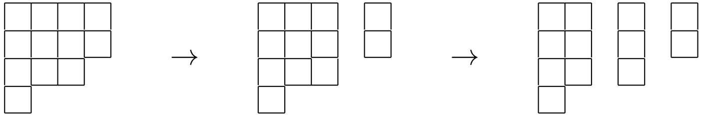

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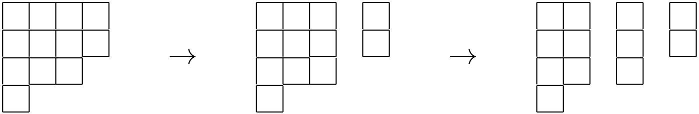

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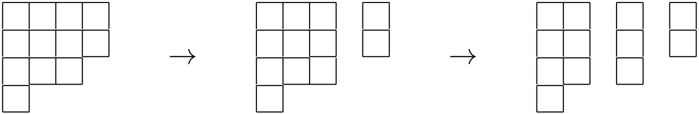

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Let $m \in \mathbb{Z}_{\geqslant 0}$. Then, $\mathcal{J}_m(\mathcal{N})$ is normal $\iff m = 0$.
More precisely, for $m \geqslant 1$, the singular locus of $\mathcal{J}_m(\mathcal{N})$ is $\mathcal{J}_m(\mathcal{N}) \setminus \pi_m^{-1}(\mathcal{O}_{\mathrm{reg}})$ and it has codimension 1 in $\mathcal{J}_m(\mathcal{N})$.

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∗ In turn, for $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$ (type $C_2 \cong B_2$) and $\mathcal{O} = \mathcal{O}_{(2^2)}$, one can show that $\mathcal{J}_1(\overline{\mathcal{O}_{(2^2)}})$ is irreducible and that

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Thank you !