Minimal nilpotent orbit closures

and

lisse minimal W-algebras

Tomoyuki Arakawa (RIMS)

joint work with Anne Moreau

ref arXiv:1506.00710

motivated by Kawasetsu's result.

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$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}k$ affine KM-alg

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$k + h^\vee \in \mathbb{Q}_{>0}$

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$\cap$ || (KW-conj)

$\{\text{modular inv. reps}\}$

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Strong FF conj

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$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

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admissible (known by [A, IMRN'15])

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Let $\mathfrak{g}$ belong to the Deligne ex. series outside type A,

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Let $\mathfrak{g}$ belong to the Deligne ex. series outside type A,

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$\Rightarrow$ get classification of simple h.w. $L(k\Lambda_0)$-modules

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(recovers earlier results of Axtell-Lee for type G_2

Perše for types D_4, F_4)

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W-alg of type $W(1^{\dim \mathfrak{g}^\natural}, \frac{3}{2}^{\dim \mathfrak{g}_{\frac{1}{2}}}, 2)$

where $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1$, $\mathfrak{g}_i := \{x \in \mathfrak{g} \mid [h_\theta, x] = 2i\,x\}$

$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C} h_\theta$

$$F : \mathcal{O}_k \longrightarrow \{ W^k(\mathfrak{g}, f_{min})\text{-modules} \}$$
$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \\ M & \longmapsto & F(M) = H^{\frac{\infty}{2}+0}_{f_\theta}(M) \end{array}$$

qDS reduction functor

Thm [A, DMJ'05, IMRN'15]

(1) F is exact.

$$F : \mathcal{O}_k \longrightarrow \{ W^k(\mathfrak{g}, f_{min})\text{-modules} \} \qquad \text{qDS reduction functor}$$

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Thm [A, DMJ'05, IMRN'15]

(1) F is exact. $\Leftarrow H_{f_\theta}^{\frac{\infty}{2}+i}(M) = 0 \quad {}^\forall i \neq 0 \quad {}^\forall M \in \mathcal{O}_k$

$$F : \mathcal{O}_k \longrightarrow \{ W^k(\mathfrak{g}, f_{min})\text{-modules} \}$$
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$$F : \mathcal{O}_k \longrightarrow \{ W^k(\mathfrak{g}, f_{min})\text{-modules} \}$$
$$\quad \ \ \rotatebox{90}{$\in$} \hspace{10em}$$
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Thm [A, DMJ'05, IMRN'15]

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Moreover any simple $W^k(\mathfrak{g}, f_{min})$-module appears in this way

$\Longrightarrow$ get character formula of all simple modules $(\forall k)$

$$\mathcal{F} : \mathcal{O}_k \longrightarrow \{W^k(\mathfrak{g}, f_{min})\text{-modules}\}$$
$$\quad\ \ \rotatebox{90}{$\in$}$$
$$M \longmapsto \mathcal{F}(M) = H^{\frac{\infty}{2}+0}_{f_\theta}(M)$$

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(3) For any quotient of V of $V(k\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k$

$\downarrow$ $L(k\Lambda_0)$

universal affine VA.

$W^k(\mathfrak{g}, f_{min}) = \mathcal{F}(V(k\Lambda_0))$

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ass. variety of V

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$\mathcal{S}_{min} \supset \underset{\substack{\rotatebox{90}{$\in$} \\ \text{ass. variety of } F(V)}}{\underline{X_{F(V)}}} = X_V \cap \underline{\mathcal{S}_{min}}$ $\quad$ $\mathbb{C}^*\text{-inv}$

$'' f_\theta + \mathfrak{g}^{e_\theta}$, Slodowy slice at f_θ

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Hence if $X_V = \overline{\mathbb{O}_{min}}$, then $F(V)$ is lisse (C_2-cofinite)

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(new series for types D_4, E_6, E_7, E_8)

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Rem For $k = -\frac{h^\vee}{6}$, this is a result of Kawasetsu.

Thank You!