On Generalization of Weyl Group Orbit Functions

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joint work with Agnieszka Tereszkiewicz

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- Infinite families of multivariable complex functions.
- Useful tool in the representation theory of Lie algebras (Lie groups).
- Continuous and discrete transforms based on orbit functions provide a Fourier-like analysis.

Outline

1. Weyl group orbit functions
   - Weyl group and affine Weyl group
   - Orbit functions
2. Irreducible characters and related functions
   - Irreducible characters of Weyl groups
   - Properties
   - Examples

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- Weight lattice: $P = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_n$, where $2\frac{\langle \alpha_i, \omega_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}$.

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- Weyl group W - generated by reflections r_i with respect to the hyperplanes orthogonal to α_i, i.e., $r_i x = x - 2\frac{\langle \alpha_i, x \rangle}{\langle \alpha_i, \alpha_i \rangle}$.

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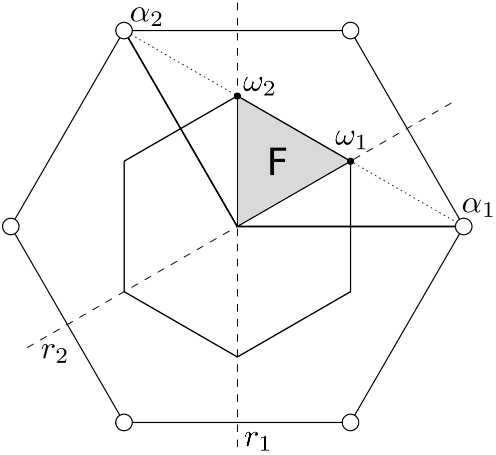
- Its fundamental domain F - $n-$dimensional simplex.

Weyl group of A_2



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- Orbit functions - for every $x \in \mathbb{R}^n$ and every $\lambda \in P$,

$$\varphi_\lambda^\sigma = \sum_{w \in W} \sigma(w) \mathrm{e}^{2\pi\imath\langle w\lambda, x\rangle}.$$

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- Notation - $C-, S-, S^s-, S^l-$ functions.

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- Forming sets orthogonal with respect to a continuous and discrete inner product.

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- For every element $g \in G$, $\chi(g^{-1}) = \overline{\chi(g)}$ - if every $g \in G$ is conjugated to its inverse then all the characters have real values.This is the case of characters of Weyl groups.

Character tables of A_2, C_2 and G_2

$[\rho]$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

$[\rho]$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

$[\rho]$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2	2	-1	-1	0	0
χ_6	2	-2	-1	1	0	0

Functions related to irreducible characters

Definition

Let W be a Weyl group of rank n with irreducible characters $\chi_1, \dots, \chi_r$. For every $x, \lambda \in \mathbb{R}^n$ and $i = 1, \dots, r$ we define

$$\phi_\lambda^k(x) = \sum_{w \in W} \chi_k(w) \mathrm{e}^{2\pi\imath\langle w\lambda, x\rangle}.$$

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- Sign homomorphisms σ^s and σ^l are the only two other possible linear characters.
- The definition includes all four families of orbit functions for B_n, C_n and G_2.

General properties

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$$\phi^k_\lambda(x) = \phi^k_x(\lambda)$$
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- Product of two functions ϕ^k, ϕ^l:

$$\phi^k_\lambda(x)\phi^l_\mu(x) = \sum_{w,\tilde{w}\in W} \chi_k(w)\chi_l(\tilde{w})\mathrm{e}^{2\pi\mathrm{i}\langle\, w\lambda+\tilde{w}\mu,x\rangle}.$$

Orthogonality

Theorem

For every $0 \neq \lambda, \mu \in P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \ldots + \mathbb{Z}^{\geq 0}\omega_n$ *and every pair of irreducible characters* χ_k, χ_l *the following relation holds,*

$$\int_{\widetilde{F}} \phi_\lambda^k(x)\overline{\phi_\mu^l(x)}dx = |W|^2|F|\frac{1}{d_k}\sum_{w\in \mathrm{stab}\, W(\lambda)} \chi_k(w)\delta_{\lambda\mu}\delta_{kl} \ ,$$

where d_k *is the degree of the character* χ_k *and* $\widetilde{F} = \bigcup_{w\in W} wF$.

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In particular, for $\lambda, \mu \in P^{++} = \mathbb{N}\omega_1 + \ldots + \mathbb{N}\omega_n$ *it holds that*

$$\int_{\widetilde{F}} \phi_\lambda^k(x)\overline{\phi_\mu^l(x)}dx = |F||W|^2\delta_{kl}\delta_{\lambda\mu} \quad .$$

Proof

- The proof uses the orthogonality of characters, the orthogonality of $C-$functions,

$$\int_F \phi^1_\lambda(x)\overline{\phi^1_\mu(x)}dx = |F||W||\operatorname{stab}_W(\lambda)|\delta_{\lambda\mu},$$

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- In a similar way, the discrete orthogonality relations are proven.

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- The functions $\phi^l_{w\lambda}$, $w \in W$, fulfil $r-1$ linearly independent relations: for each $k \neq l$,

$$\sum_{w \in W} \chi_k(w) \phi^l_{w\lambda}(x) = 0.$$

Weyl group of A_2

- Conjugacy classes:

$$W(A_2) = \{id\} \cup \{r_1, r_2, r_1 r_2 r_1\} \cup \{r_1 r_2, r_2 r_1\}.$$

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- $\phi^3_\lambda(x) = 2\mathrm{e}^{2\pi\imath\langle\lambda,x\rangle} - \mathrm{e}^{2\pi\imath\langle r_2 r_1\lambda,x\rangle} - \mathrm{e}^{2\pi\imath\langle r_1 r_2\lambda,x\rangle}$

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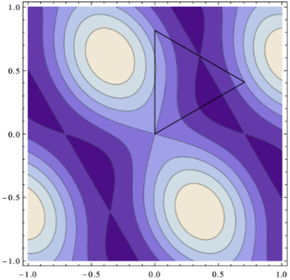
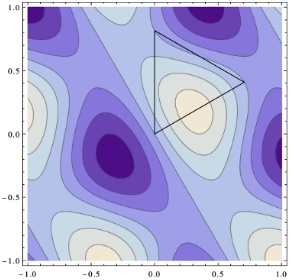
- Relations:

$$\phi^3_\lambda(x) + \phi^3_{r_1 r_2\lambda}(x) + \phi^3_{r_2 r_1\lambda}(x) = 0$$
$$\phi^3_{r_1\lambda}(x) + \phi^3_{r_2\lambda}(x) + \phi^3_{r_1 r_2 r_1\lambda}(x) = 0$$

Example

The contour plot of the real part (left) and the imaginary part (right) of the function $\phi^3_{(1,0)}(x)$. The triangle denotes the fundamental domain F of the affine Weyl group $W(A_2)$.





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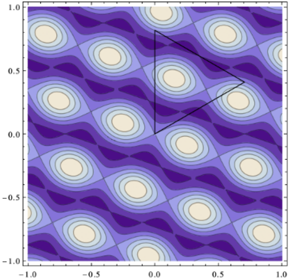

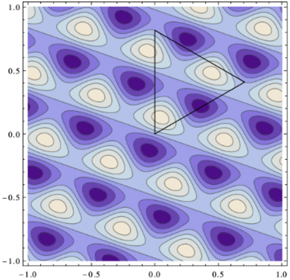

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