Cluster structures on open Richardson varieties and their quantizations

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Cluster Algebras

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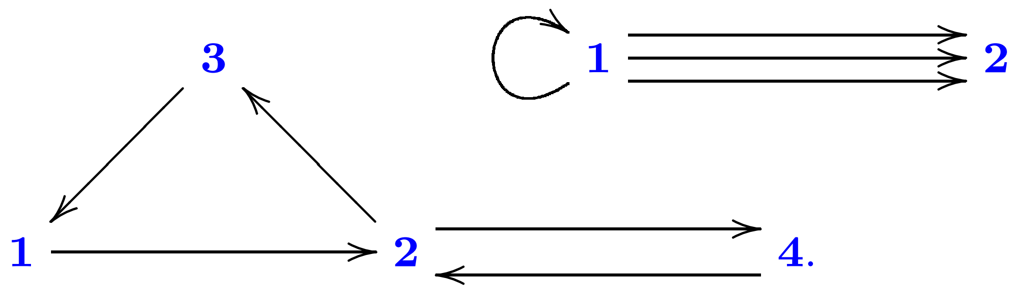
Classically, algebras are defined via generators and relations. Cluster algebras are defined via an infinite combinatorial procedure of quiver mutation. This creates many subsets (cluster monomials) that are used to study bases of these inf dim algebras.

Combinatorial side of quiver mutation, many other deeper interpretations in alg geometry and math physics (Calabi-Yau manifolds, scattering), representation theory (Auslander-Reiten translation).

Cluster Algebras

Input: A quiver (a directed graph) without loops and 2-cycles. Its vertices are indexed by $1, \ldots, m$.

Example. The following quivers are not allowed:



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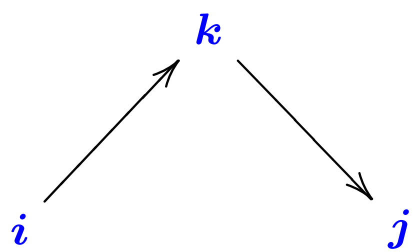
Step 2: Complete

the 2-paths



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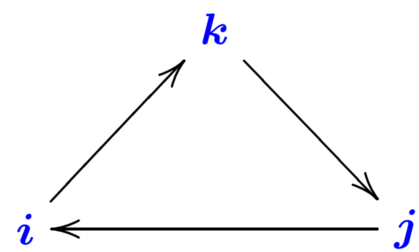

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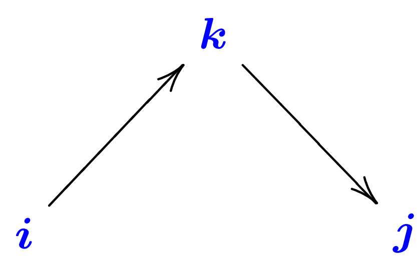

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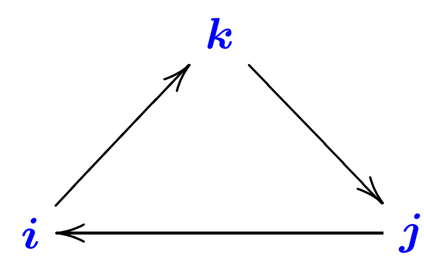
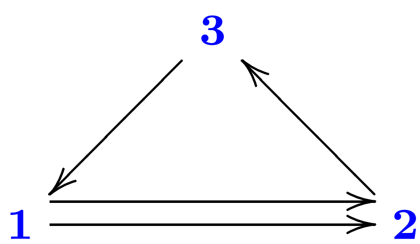


Step 3: Cancel out pairs of opposite arrows.

An example: $\mu_3(Q)$

$Q :=$



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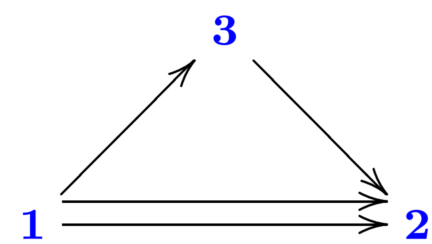

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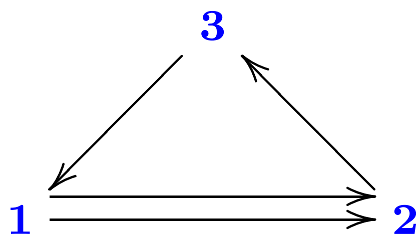

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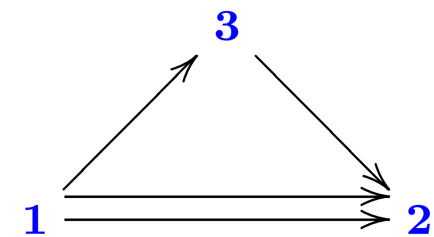
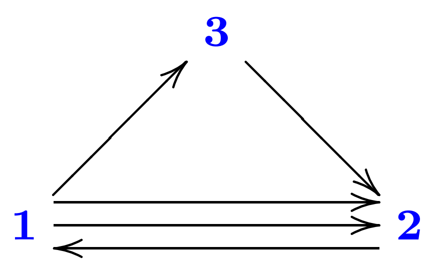
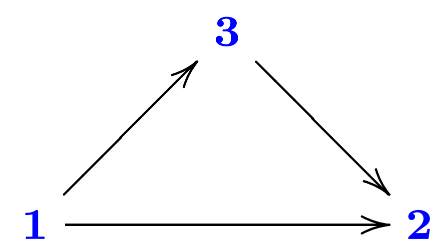


Step 2 (2-path through 3):



Step 3: $\mu_3(Q) =$



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$$\mu_k(\Sigma) = (y_1, \ldots y_{k-1}, y'_k, y_{k+1}, \ldots, y_m; \mu_k(Q)) \text{ where } y'_k := \frac{1}{y_k} \left(\prod_{j \to k} y_j + \prod_{i \leftarrow k} y_i \right).$$

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Choose $n \leq m$, call $1, \dots, n$ mutable vertices and $n+1, \dots, m$ frozen vertices.

Mutate the original seed Σ infinitely many times in all mutable directions:

$$\mu_{k_1} \dots \mu_{k_l}(\Sigma), \quad k_1, \dots, k_l \in [1, n], \ \ l = 1, 2, \dots$$

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Definition. The cluster algebra $\mathcal{A}(Q)$ is the subalgebra of $\mathbb{K}$ generated by the cluster variables in all such seeds (optional: and $y_{n+1}^{-1}, \ldots, y_m^{-1}$).

Cluster algebras can also have tropical part but this is not related to the addressed problems.

Cluster Algebras II

Example. The space of 2×2 matrices. Its coordinate ring $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}]$ has cluster structure with only 2 clusters:

$$(x_{11}, x_{12}, x_{21}, \Delta), \quad \left(x_{22} = \frac{x_{12}x_{21} + \Delta}{x_{11}}, x_{12}, x_{21}, \Delta \right).$$

The variables x_{12}, x_{21}, Δ are frozen. The variables x_{11} and x_{22} are mutable. Easy to see that in this example

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Quantum cluster algebras $\mathcal{A}_q(Q)$ introduced by Berenstein and Zelevinsky in 2004. Idea: Replace all Laurent polynomial rings by quantum tori:

$$\mathcal{T} := \frac{\mathbb{C}\langle y_1^{\pm 1}, \ldots, y_m^{\pm 1}\rangle}{(y_j y_k - q_{jk} y_k y_j)}$$

for some $q_{jk} \in \mathbb{C}^*$.

Cluster Algebras in Lie Theory

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3. Many geometric and combinatorial "descriptions" of canonical bases but hard to compute explicitly. If $\mathbb{C}[M]$=a cluster algebra, then all cluster monomials belong to the canonical basis and form a very large part of it.

Cluster monomials: fix a cluster $(y_1, \ldots, y_m)$. A cluster monomial:

$$y_1^{k_1} \ldots y_m^{k_m}, \qquad k_1, \ldots, k_m \in \mathbb{N}.$$

Take the union over all clusters, but do not mix variables from different clusters in the same monomial.

By a theorem of Keller, Cerulli-Irelli, Labardini-Fragoso, Plamondon this is a linearly independent subset of $\mathcal{A}(Q)$.

The Problem

Let G be a symmetrizable Kac–Moody group with Weyl group W and $B_\pm$ be a pair of opposite Borel subgroups of G. Schubert cell decompositions

$$G/B_+ = \bigsqcup_{w \in W} B_+ \cdot wB_+ = \bigsqcup_{u \in W} B_- \cdot uB_+.$$

The open Richardson varieties are

$$R_{u,w} = B_+ \cdot wB_+ \cap B_- \cdot uB_+ \neq \varnothing \text{ iff } u \leq w.$$

Dimension $= \ell(w) - \ell(u)$. Decomposition $G/B_+ = \bigsqcup_{u \leq w \in W} R_{u,w}$.

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For $\mathfrak{g}$ symmetric and all $u \leq w$ Leclerc [2014] constructed a cluster algebra

$$\mathcal{A}(\Sigma_{u,w}) \subseteq \mathbb{C}[R_{u,w}]$$

of the same dimension and conjectured that they are equal. Open even for $G = SL_n$. Methods: rep theory of fin dim algebras, cluster categories.

The algebras $\mathcal{U}^-[w]$ and their primes

To $w \in W$, one associates the quantum Schubert cell algebra

$$\mathcal{U}^{\pm}[w] = \mathcal{U}_q(\mathfrak{n}_\pm \cap w(\mathfrak{n}_\mp))$$

[De Concini–Kac–Procesi, Lusztig, 1990]. The quantum R-matrix

$$\mathcal{R}^w \in \mathcal{U}^+[w] \widehat{\otimes} \mathcal{U}^-[w].$$

Let T be the maximal torus of G (acts on $\mathcal{U}^{\pm}[w]$ by algebra automorphisms).

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Theorem. [Cauchon-Mériaux, Y, 2009] The T-invariant prime ideals of $\mathcal{U}^-[w]$ are parametrized by $u \in W$, $u \leq w$,

$$I_w(u) := \{(c_{\xi,\lambda} \otimes \mathrm{id})(\mathcal{R}^w) \mid \xi \perp V_u(\lambda)\}$$

Notation: $V(\lambda)$ irreducible module with highest weight λ,

$$V_u(\lambda) = \mathcal{U}_q^-(\mathfrak{g}) T_u v_\lambda$$

Demazure module, $c_{\xi,\lambda} \in (\mathcal{U}_q(\mathfrak{g}))^*$ matrix coefficient for $\xi \in V(\lambda)^*$ and h.w. vector v_λ.

Idea

The quantizatized coordinate rings of the Richardson varieties are

$$R_q[R_{u,w}] = (\mathcal{U}^-[w]/I_w(u))[E^{-1}]$$

localizations by normal elements (frozen variables), $ua = \varphi(a)u,\ \varphi \in \mathrm{Aut}(A)$. Denote

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Consider chains of subalgebras

$$A_1 \subset A_2 \subset \ldots \subset A_{\ell(w)} = \mathcal{U}^-[w]$$

such that $\mathrm{GKdim} A_k - \mathrm{GKdim} A_{k-1} = 1$. Project

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Construct quantum clusters

$$\Sigma := \{y_k \in \pi(A_k) \ \text{normal, for those } k \text{ such that } \mathrm{GKdim}\pi(A_k) - \mathrm{GKdim}\pi(A_{k-1}) = 1\}$$

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Construct sufficiently many quantum clusters and connect them by mutation to control the size of Leclerc's cluster algebra from below.

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Two chains of subalgebras of $\mathcal{U}^-[w]$

Fix a reduced expression

$$w = s_{i_1} \dots s_{i_l}.$$

The roots of $\mathfrak{n}_+ \cap w(\mathfrak{n}_-)$ are

$$\beta_k := w_{\leq k-1}(\alpha_{i_k}), \qquad k = 1, \dots, l.$$

The algebra $\mathcal{U}^-[w]$ is generated by Lusztig's root vectors

$$F_{\beta_1}, \dots, F_{\beta_l}, \qquad F_{\beta_k} = T_{w_{\leq k-1}}(F_{i_k}).$$

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Two chains of subalgebras:

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adjoining F's in the direct order;

$$A_{1,\mathrm{opp}} \subset A_{2,\mathrm{opp}} \subset \dots \subset A_{l,\mathrm{opp}} = \mathcal{U}^-[w]$$

adjoining F's in the opposite order.

Positive subexpressions

Index sets of subexpressions $D \subseteq [1, l]$; denote

$$s_k^D := \begin{cases} s_{i_k}, & \text{if } k \in D, \\ 1, & \text{if } k \notin D \end{cases}$$

Subexpression $s_1^D \dots s_l^D$. Let

$$w_{\leq k}^D := s_1^D \dots s_k^D, \quad w_{\geq k}^D := s_k^D \dots s_N^D.$$

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Definition. Right positive subexpression if

$$w_{\leq k}^D < w_{\leq k}^D s_{i_{k+1}}, \quad \forall k.$$

Left positive subexpression if

$$w_{\geq k}^D < s_{i_{k-1}} w_{\geq k}^D, \quad \forall k.$$

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Proposition [Deodhar, Marsh–Rietsch]. For every $u \leq w$, there are unique right and left positive subexpressions of $s_{i_1} \ldots s_{i_l}$ with total products u. Denote their index sets by

$$\mathcal{RP}_w(u), \qquad \mathcal{LP}_w(u).$$

Clusters I

Example. $u = s_1 < s_1 s_2 s_1$. Right and left positive subexpressions

$$s_1 s_2 \textcolor{red}{s_1}, \qquad \textcolor{red}{s_1} s_2 s_1.$$

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Theorem (A) [Lenagan-Y]. For all k,

$$\pi(A_k) \cong \mathcal{U}^-[w_{\leq k}]/I_{w_{\leq k}}(\vec{u}_{\leq k}).$$

We have the equality of sets

$$\{k \mid \mathrm{GKdim}\pi(A_k) - \mathrm{GKdim}\pi(A_{k-1}) = 1\} = [1, l] \backslash \mathcal{RP}_w(u).$$

The following is a sequence of normal elements that produces a cluster of the quantum Richardson algebra $R_q[R_{u,w}] = (\mathcal{U}^-[w]/I_w(u))[E^{-1}]$:

$$\Sigma := \{\pi(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) \mid k \in [1, l] \backslash \mathcal{RP}_w(u)\}.$$

Clusters II

Theorem (B). For all k,

$$\pi(A_{k,\mathrm{opp}}) \cong \mathcal{U}^-[w_{\geq k}]/I_{w_{\geq k}}(\overleftarrow{u}_{\geq k}).$$

We have the equality of sets

$$\{k \mid \mathrm{GKdim}\pi(A_{k,\mathrm{opp}}) - \mathrm{GKdim}\pi(A_{k-1,\mathrm{opp}}) = 1\} = [1,l]\backslash\mathcal{LP}_w(u).$$

The following is a sequence of normal elements that produces a cluster of the quantum Richardson algebra $R_q[R_{u,w}] = (\mathcal{U}^-[w]/I_w(u))[E^{-1}]$:

$$\Sigma := \{\pi(T_{w_{\leq k-1}}\Delta_{\overleftarrow{u}_{\geq k}u^{-1}\varpi_{i_k}, w_{\geq k}u^{-1}\varpi_{i_k}}) \mid k \in [1,l]\backslash\mathcal{LP}_w(u)\}.$$

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$$\pi(A_{k,\mathrm{opp}}) \cong \mathcal{U}^-[w_{\geq k}]/I_{w_{\geq k}}(\overleftarrow{u}_{\geq k}).$$

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Ingredients:

1. Representation theory – Demazure modules [Joseph],

2. Noncommutative ring theory – [Cauchon, Goodearl–Letzter] prime ideals of iterated skew polynomial extensions.

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Relate Theorems (A) and (B), via a "twist map"

$$\Theta_w = T_w \tau S \tau \omega \colon R_q[R_{u,w}] \to R_q[R_{u^{-1},w^{-1}}],$$ an algebra anti-isomorphism.

Clusters III

Define the set

$$\Xi_l = \{\sigma \in S_l \mid \sigma([1,k]) \text{ is an interval for all } k\}.$$

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$$F_{\sigma(1)}, F_{\sigma(2)}, \ldots, F_{\sigma(l)}.$$

This gives rise to a chain of subalgebras

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Theorem (C). All elements of Ξ_l give rise to clusters of the quantum Richardson algebras $R_q[R_{u,w}]$ by constructing sequences of normal elements. These clusters are related by mutations.

Theorem (D) [Leclerc's Conjecture]. For all symmetric Kac–Moody algebras $\mathfrak{g}$ and Weyl group elements $u \le w \in W$:

$$\mathbb{C}[R_{u,w}] = \mathcal{A}(\Sigma_{u,w}), \quad R_q[R_{u,w}] = \mathcal{A}_q(\Sigma_{u,w})$$

where $\mathcal{A}(\Sigma_{u,w})$ is Leclerc's cluster agebra.

Applications to Dixmier program

Dixmier program: Let $\mathfrak{b}$ be a solvable Lie algebra. In the late 60's, Dixmier constructed a map

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Construction. T-equivariant bijection

$$\mathrm{Dix}_{G,w} : \mathrm{Symp}(B_+ \cdot wB_+, \pi_{DS}) \to \mathrm{Prim}\,\mathcal{U}^-[w]$$ and conjecture that it is a homeomorphism.

Thank you for the attention!