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$*-$Lie algebras canonically associated with probability measures on $\mathbb{R}$

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Conclusion We prove that, each probability meassure on $\mathbb{R}$, with all moments, is canonically associated with:
– (i) a $*$–Lie algebra;
– (ii) a complexity index labeled by pairs of natural integers.

The measures with complexity index $(0, K)$ consist of two disjoint classes:
– that of all measures with finite support
– the semi–circle–arcsine class (this name will be motivated later).

The class $(1, 0)$ includes the Gaussian and Poisson measures and the associated $*$–Lie algebra is the Heisenberg algebra.

The class $(2, 0)$ includes the non standard (i.e. neither Gaussian nor Poisson) Meixner distributions and the associated $*$–Lie algebra is

$sl(2,\mathbb{R})$.

Starting from $n = 3$, the $*$–Lie algebra associated to the class $(n, 0)$ is infinite dimensional.

Notations

The polynomial algebra

$\mathcal{P} := \{$polynomial functions $\mathbb{R} \to \mathbb{C}\}$ (a $*$–algebra)

– $e_1 = 1$ the canonical basis of $\mathbb{R}$.

X coordinate function on $(\mathbb{R}, \mu)$

$$X : xe_1 \in \mathbb{R} \ \to \ X(x) := x \in \mathbb{R}$$

Then:
$\mathcal{P} \equiv *$–algebra of complex valued polynomial functions of X.

Probability measures on $\mathbb{R}$ and states on $\mathcal{P}$

$$\text{Prob}(\mathbb{R}) := \{\text{probability measures on } \mathbb{R}\}$$

$$\mu \in \text{Prob}(\mathbb{R}) \quad \text{probability distribution of } X$$

$$X := \begin{cases} \text{a } \mathbb{R}\text{–valued classical random variable} \\ \text{with all moments} \end{cases}$$

probability measure μ on $\mathbb{R}$ $\Rightarrow$ unique state μ on $\mathcal{P}$

$$\mu(P) := \int_{\mathbb{R}} P(x)\mu(dx) \qquad (1)$$

The inverse correspondence

state μ on $\mathcal{P}$ $\Rightarrow$ probability measure μ on $\mathbb{R}$

is many–to–one.

States and pre–scalar products on $\mathcal{P}$

state μ on $\mathcal{P}$ $\Rightarrow$ pre–scalar product on $\mathcal{P}$ $\langle \cdot , \cdot \rangle$

$$\mu(P^*Q) := \langle P, Q\rangle := \int_{\mathbb{R}} \overline{P(x)}Q(x)\mu(dx) \quad (2)$$

$\| \cdot \|$ pre–norm associated to $\langle \cdot , \cdot \rangle$.

Theorem A pre–scalar product $\langle \cdot , \cdot \rangle$, on $\mathcal{P}$ has the form (2) for some $\mu \in \text{Prob}(\mathbb{R})$, if and only if

$$X = X^*$$

(the multiplication operators by the coordinate function is $\langle \cdot , \cdot \rangle$–symmetric).

In the following:

– elements $P(X) \in \mathcal{P} \;\equiv$ multiplication operators on $\mathcal{P}$;

for $P(X)$ considered as a vector, we write

$$P(X) \cdot 1 =: P(X) \cdot \Phi$$

monomial gradation on $\mathcal{P}$:

$$\mathcal{P} = \dot{\sum_{n \in \mathbb{N}}} \mathbb{C} X^n \cdot 1$$

$\dot{\sum}$:= vector space direct sum.

Notation.

– $\mathcal{K}$ pre–Hilbert space
– $\xi \in \mathcal{K}$ unit vector

$$\xi^*(\eta) := \langle \xi, \eta \rangle \qquad ; \qquad \forall \xi, \eta \in \mathcal{K}$$

Dirac notation

$$\xi^* = \langle \xi | \text{ bra} \qquad ; \qquad \xi = | \xi \rangle \text{ ket}$$

The orthogonal polynomial gradation

Define inductively:
– the **monic ortogonal polynomials** Φ_n ,
– the **normalized** ortogonal polynomials

$$\|\tilde{\Phi}_n\| = 1 \qquad ; \qquad \forall n \in \mathbb{N}$$

as follows.

$$\tilde{\Phi}_0 := 1$$

$$P_{0]} := \tilde{\Phi}_0 \tilde{\Phi}_0^*$$

Having defined the sequence $(\tilde{\Phi}_m, P_{m]})$ $(m \in \{1, \ldots, n\})$, define

$$\Phi_{n+1} := X^{n+1} - P_{n]}(X^{n+1}) \tag{3}$$

$$\tilde{\Phi}_{n+1} := \begin{cases} \frac{\Phi_{n+1}}{\|\Phi_{n+1}\|} & , \text{ if } \|\Phi_{n+1}\| \neq 0 \\ \Phi_{n+1} & , \text{ if } \|\Phi_{n+1}\| = 0 \end{cases} \tag{4}$$

$$P_{n+1]} := \sum_{j \in \{1,\ldots,n+1\}} \tilde{\Phi}_j \tilde{\Phi}_j^*$$

$\tilde{\Phi}_{n+1}$ (resp. Φ_{n+1}) is a polynomial (resp. monic polynomial) of degree $n+1$ with real coefficients and orthogonal to all polynomials of degree $\leq n$.

$$\|\Phi_{n+1}\| = 0 \;\Rightarrow\; \tilde{\Phi}_{n+1} \perp \mathcal{P}$$

$$\Rightarrow\; \|\Phi_k\| = 0 \quad , \quad \forall k \geq n+1$$

Therefore the operator $\tilde{\Phi}_{n+1}\tilde{\Phi}_{n+1}^*$ is the zero operator and the operator defined by (5) is a projection in any case.

By construction, $\forall n \in \mathbb{N}$, the of the set

$$\text{linear span}\left\{\tilde{\Phi}_j \ : \ j \in \{1, \ldots, n\}\right\} \equiv \mathcal{P}_{n]} \cdot \tilde{\Phi}_0 :=$$

$$:= \{P \in \mathcal{P} \ : \ \text{degree}(P) \leq n\} \subseteq \mathcal{P} \cdot \tilde{\Phi}_0 :$$

$$P_{n]} \ : \ \mathcal{P} \cdot \tilde{\Phi}_0 : \to \ \mathcal{P}_{n]} \cdot \tilde{\Phi}_0 : \text{orthog. proj.}$$

$$P_n := P_{n+1]} - P_{n]}$$

$$\mathcal{P}_n := \text{range of}(P_n) :=$$

$$:= \text{space of orthog. polynomials of degree } n$$

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n \tag{6}$$

:= orthogonal polynomial gradation

The CAP operators

Theorem (the tri–diagonal Jacobi relation)

$$XP_n = P_{n+1}XP_n + P_nXP_n + P_{n-1}XP_n$$

Define

$$a_n^+ := P_{n+1}XP_n \;:\; \mathcal{P}_n \;\to\; \mathcal{P}_{n+1}$$

$$a_n^0 := P_nXP_n \;:\; \mathcal{P}_n \;\to\; \mathcal{P}_n$$

$$a_n^- := P_{n-1}XP_n \;:\; \mathcal{P}_n \;\to\; \mathcal{P}_{n-1}$$

$$a_n^\varepsilon := \sum_{n\in\mathbb{N}} P_{n+\varepsilon}XP_n \quad ; \quad \varepsilon \in \{+1, 0, -1\}$$

Then the tri–diagonal Jacobi relation becomes

$$XP_n = a_n^+ + a_n^0 + a_n^-$$

Summing over $n \in \mathbb{N}$

$$X = a^+ + a^0 + a^-$$

the quantum decomposition of X.

The 1–dimensional case

$$\mathcal{P} = \bigoplus_{n\in\mathbb{N}} \mathbb{C} \cdot \tilde{\Phi}_n \qquad (7)$$

$$a^0 = \sum_{n\in\mathbb{N}} \alpha_n \tilde{\Phi}_n \tilde{\Phi}_n^* \qquad (8)$$

$$a^+ = \sum_{n\in\mathbb{N}} \beta_{n+1} \tilde{\Phi}_{n+1} \tilde{\Phi}_n^* \qquad (9)$$

$$a^- = (a^+)^* = \sum_{n\in\mathbb{N}} \beta_{n+1} \tilde{\Phi}_n \tilde{\Phi}_{n+1}^* \qquad (10)$$

– (α_n) := secondary Jacobi sequence of μ; $\alpha_n \in \mathbb{R}$.
– (β_n) := principal symmetric Jacobi sequence of μ; $\beta_n \in \mathbb{R}$.
– (ω_n) := (β_n^2) := principal monic Jacobi sequence of μ.

$$\omega_n \geq 0 \quad ; \quad \forall n \in \mathbb{N}^* \quad ; \quad \omega_n = 0 \Rightarrow \omega_p = 0 \, , \forall p \geq n \qquad (11)$$

It took 18 years of hard work to understand what are the multi–dimensional analogues of the sequences (α_n) and (ω_n).

The canonical commutation relations of a classical probability measure on $\mathbb{R}$

Define

$$\Lambda := \sum_{n\in\mathbb{N}} n\tilde{\Phi}_n\tilde{\Phi}_n^* = \Lambda^* \text{ number operator} \quad (12)$$

Theorem

$$[a^-, a^+] = (\beta^2_{\Lambda+1} - \beta^2_{\Lambda}) =: \sum(\omega_{n+1} - \omega_n)\tilde{\Phi}_n\tilde{\Phi}_n^* \quad (13)$$

$$= \omega_{\Lambda+1} - \omega_{\Lambda}$$

The smallest $*$–Lie algebra canonically associated to μ

Definition 1 The $*$–Lie algebra generated by the adjointable operators on the pre–Hilbert space $\mathcal{P}$

$$a^{+} \qquad , \qquad a^{-}$$

i.e. the smallest $*$–Lie algebra containing these operators, will be denoted $\mathcal{L}_X^0$.

Remark. $\mathcal{L}_X^0$ is the smallest $*$–Lie algebra canonically associated to the measure μ.
The natural $*$–Lie algebra canonically associated to μ is the one generated by

$$a^{+} \qquad , \qquad a^{0} \qquad , \qquad a^{-}$$

In the present paper we restrict our attention to $\mathcal{L}_X^0$. From the measure theoretical point of view, this is equivalent to restrict one's attention to symmetric μ (all odd moment vanish).

Our goal is to describe $\mathcal{L}_X^0$.

Lemma 1 Let T be the left–shift operator, defined on Borel functions
$F : \mathbb{R} \to \mathbb{C}$ by:

$$TF_\Lambda := F_{\Lambda-1} \qquad ; \qquad \forall k \in \mathbb{N} \tag{14}$$

with the convention

$$F_h = 0 \qquad ; \qquad \forall h < 0 \tag{15}$$

Define the difference operators

$$\partial^{(1)} F_\Lambda := \partial F_\Lambda := F_\Lambda - TF_\Lambda = F_\Lambda - F_{\Lambda-1} \tag{16}$$

$$\partial^{(k)} F_\Lambda := F_\Lambda - T^k F_\Lambda = F_\Lambda - F_{\Lambda-k} \quad ; \quad \forall k \in \mathbb{N}\,,\ k \geq 2 \tag{17}$$

Then, for any $k \in \mathbb{N}$ and for any function $F :$
$n \in \mathbb{N} \to F_n \in \mathbb{C}$

$$[a^k, F_\Lambda] = a^k \partial^{(k)} \bar{F}_\Lambda \tag{18}$$

$$[a^{+k}, F_\Lambda] = -\partial^{(k)} F_\Lambda a^{+k}$$

$\bar{F}$ – the complex conjugate of F.

The dimension of $\mathcal{L}_X^0$

If $\dim(L^2(\mathbb{R},\mu)) < +\infty$ then also $\mathcal{L}_X^0$ will be finite–dimensional.
Therefore the problem to distinguish between finite and infinite dimensional $\mathcal{L}_X^0$ is non–trivial only if $\dim(L^2(\mathbb{R},\mu)) = +\infty$ and this is the case if and only if

$$\omega_n > 0 \qquad ; \qquad \forall n \in \mathbb{N}^* \tag{19}$$

or equivalently

$$\|\Phi_n\| \neq 0 \qquad ; \qquad \forall n \in \mathbb{N} \tag{20}$$

$\mathcal{L}_X^0$ must contain the $*$–Lie algebra generated by

$$a^- \ , \ a^+ \ , \ [a^-, a^+] = \partial\omega_\Lambda$$

Therefore, a sufficient condition for $\mathcal{L}_X^0$ to be infinite dimensional is that this algebra is infinite dimensional.
If X is a symmetric random variable, i.e. $a^0 = 0$, this condition is also necessary.

Theorem 1 For a random variable X, a necessary condition for $\mathcal{L}_X^0$ to be finite dimensional is that there exists $n, K \in \mathbb{N}$ such that

$$(\partial^n \omega)_m = 0 \qquad ; \qquad \forall m \geq K + 1 \tag{21}$$

i.e., starting from $K + 1$, the sequence (ω_n) satisfies the difference equation (21).

Indices of information complexity of μ

In the following we do not assume that $\omega_n > 0$ for each $n \in \mathbb{N}$.

Definition 2 The **index of information complexity** (or simply **information complexity**) of a probability measure $\mu \in \mathrm{Prob}(\mathbb{R})$, with principal Jacobi sequence (ω_n), is the pair $C(\mu) \in \mathbb{N}^2$ defined as follows:

$$C(\mu) := \tag{22}$$

$$\begin{cases} (k, K) \ , \ \text{if } k = \min\{n \in \mathbb{N}^* \ : \\ (\partial^{n+1}\omega)_m = 0 \ , \ \forall m \geq K+1\} \\ +\infty \ , \ \text{if no pair } (n, K) \\ \text{with the above property exists} \end{cases}$$

and K is the smallest natural integer satisfying (23).

Remark. Notice that the relation

$$\mu \sim \nu \iff C(\mu) = C(\nu) \qquad ; \qquad \mu, \nu \in \mathrm{Prob}(\mathbb{R}) \tag{23}$$

is an equivalence relation and that it involves only the principal Jacobi sequence $\omega \equiv (\omega_n)$.

The class $C(\mu) = (0, K)$ ($K \in \mathbb{N}$)

According to Definition (2) the probability measures belonging to the information complexity class $(0, K)$ are those for which there exists $K \in \mathbb{N}$ such that

$$(\partial\omega)_m = 0 \qquad ; \qquad \forall m \geq K + 1 \qquad (24)$$

and K is smallest number with respect to the property (24).

Theorem 2 Let $(\mu, (\omega_n))$ be a probability measure with principal Jacobi sequence (ω_n) and information complexity $C(\mu) = (0, K)$, $(K \in \mathbb{N})$. Then, with the convention that

$$x \leq 0 \;\Rightarrow\; \{1, \ldots, x\} := \emptyset \tag{25}$$

and for K as in (24), one of the following alternatives takes place:

(i) $|\text{supp}(\mu)| = K + 1$ and (ω_n) has the form

$$\omega_n = \begin{cases} \text{arbitrary } > 0 \;, \text{ if } n \in \{1, \ldots, K-1\} \\ 0 \;, \text{ if } n \geq K \end{cases} \tag{26}$$

Moreover all measures $\mu \in \text{Prob}(\mathbb{R})$ are in this class.

(ii) $|\text{supp}(\mu)| = \infty$ and (ω_n) has the form

$$\omega_n = \begin{cases} \text{arbitrary } > 0 \;, \text{ if } n \in \{1, \ldots, K\} \\ \omega > 0 \;, \text{ if } n \geq K + 1 \end{cases} \tag{27}$$

Examples of measures in the class $C(\mu) = (0, K)$ $(K \in \mathbb{N})$.

$C(\mu) = (0,0)$, $\omega = 0$: **the δ–measures**

If $C(\mu) = (0,0)$ and $\omega = 0$, then $\partial\omega_n = 0$ for all $n \geq 1$. In particular
$\partial\omega_1 = \omega_1 - \omega_0 = \omega_1 = 0$.
This condition characterizes the δ–**measures**, i.e. those μ such that

$$\mu = \delta_c \qquad \text{for some } c \in \mathbb{R}$$

This is easily seen directly because

$$0 = \omega_1 = \mu(X^2) - \mu(X)^2 \tag{28}$$

it follows that X has constant range.
Inversely, suppose that X has constant range equal to $\{c\}$ $(c \in \mathbb{R})$, i.e.

$$\mu = \delta_c$$

Then (28) holds.

$C(\mu) = (0, 0)$, $\omega > 0$: the semi–circle laws

In this case $\omega_n = \omega_1 =: a > 0$ for all $n \geq 1$. It is known that this class coincides with the class of **Semi–circle** distributions, the *Gaussian* of free probability.

$C(\mu) = (0, 1)$, $\omega > 0$: the arcsine laws

In this case

$$\omega_n = \begin{cases} a > 0 \text{ if } n = 1 \\ 0 < b \neq a \text{ if } n \geq 2 \end{cases}$$

and it is known that this class coincides with the class of **Arcsine laws**, the *Gaussian* of monotone probability..

$C(\mu) = (0, K)$, $\omega > 0$, $K \geq 3$: the extended semi–circle–arcsine laws

Remark. The structure of the measures in this class is a natural extension of the semi–circle and arcsine laws.
Almost nothing is known about the specific structure of this class.

$C(\mu) = (1, K)$

The probability measures belonging to the information complexity class $(1, K)$ are those for which there exists $K \in \mathbb{N}$ such that

$$\partial^2 \omega_n = 0 \qquad ; \qquad \forall n \geq K + 1 \tag{29}$$

and both the exponent 2 and the number K are the smallest ones with respect to property (29).

Theorem 3 The class of probability measures on $\mathbb{R}$ with information complexity $C(\mu) = (1, K)$, $(K \in \mathbb{N})$ is characterized by the fact that their principal Jacobi sequence (ω_n) has the following structure:
there exists $b \in \mathbb{R}_+^*$ and $c \in \mathbb{R}$ such that, with the convention (25), (ω_n) has the form

$$\omega_n = \begin{cases} \text{arbitrary } > 0 \ , \ \text{if } n \in \{1, \ldots, K\} \\ bn + c > 0 \ , \ \text{if } n \geq K + 1 \end{cases} \tag{30}$$

In particular, if c is positive, then it can be arbitrary while, if negative, it must satisfy

$$|c| \leq b(K + 1) \tag{31}$$

$C_\omega(\mu) = (1,0)$: Gaussian and Poisson

Complexity class $C(\mu) = (1,0)$:

$$\omega_n = bn + c \qquad ; \qquad \forall n \geq 1$$

with $b \in \mathbb{R}_+^*$, $c \in \mathbb{R}_+$.
Take $c = 0$, then

$$\omega_n = bn > 0 \qquad ; \qquad \forall n \geq 1$$

This class includes both the Gaussian distribution with mean 0 and variance b and the Poisson distribution with intensity b (see **[AcKuoSta07]**).

The $*-$Lie algebra of the class $C_\omega(\mu) = (1,0)$ is the Heisenberg algebra

Theorem 4 Let $(\mu, (\omega_n))$ be any probability measure with information complexity $C(\mu) = (1,0)$, then the 3–dimensional linear space $\mathcal{L}_X^0$ generated by the operators

$$\left\{a^-, a^+, \partial\omega_\Lambda\right\}$$

is a $*$-Lie algebra isomorphic to the Heisenberg Lie algebra.

Proof.
If $C(\mu) = (1,0)$, then (ω_n) is the solution of a second order difference equation.
Therefore it has the form

$$bn + c \qquad ; \qquad \forall n \geq 1$$

Therefore the commutation relations become

$$[a^-, a^+] = \omega_\Lambda - \omega_{\Lambda-1} = b \cdot 1$$

$$[a^-, b \cdot 1] = [a^+, b \cdot 1] = 0$$

which are the defining relations of the Heisenberg $*$-Lie algebra.

The classes $C_\omega(\mu) = (2, K)$

The probability measures belonging to the information complexity class $(2, K)$ are those for which there exists $K, n \in \mathbb{N}$ such that

$$\partial^3 \omega_n = 0 \qquad ; \qquad \forall n \geq K + 1 \tag{32}$$

and both the exponent 3 and the number K are minimal with respect to the property (32).

Theorem 5 The class of probability measures on $\mathbb{R}$ with information complexity $C(\mu) = (2, K)$, $(K \in \mathbb{N})$ is characterized by the fact that their principal Jacobi sequence (ω_n) has the following structure:
there exists $b, c, d \in \mathbb{R}$ such that $b > 0$ and, with the convention (25) (ω_n) has the form

$$\omega_n = \begin{cases} \text{arbitrary } > 0 \text{ , if } n \in \{1, \ldots, K\} \\ bn^2 + cn + d > 0 \text{ , if } n \geq K + 1 \end{cases} \quad (33)$$

In particular, if c, d are positive, then they can be arbitrary while, if one of them is negative, then their choice is constrained by the fact that the right hand side of (33) must be positive.

The class $C(\mu) = (2,1)$

In this case $(\omega_n)_n$ has the form

$$bn^2 + cn + d \qquad ; \qquad \forall n \geq 1$$

with $b \in \mathbb{R}_+^*$, $c, d \in \mathbb{R}$.
In particular, if $d = 0$ then $\omega_n = bn^2 + cn > 0$ for all $n \geq 1$ and it is known that this class coincides with the class of non–standard (i.e. neither Gaussian nor Poisson) Meixner distributions.

The $*$–Lie algebra of the class $C_\omega(\mu) = (2, 0)$ is $sl(2, \mathbb{R})$

Theorem 6 Let $(\mu, (\omega_n))$ be a probability measure with information complexity $C(\mu) = (2, 0)$, then the 3–dimensional linear space $\mathcal{L}_X^0$ generated by the operators

$$\left\{a^-, a^+, \partial\omega_\Lambda\right\} \tag{34}$$

a $*$-Lie algebra isomorphic to the $sl(2, \mathbb{R})$ Lie algebra.

Proof

If $C(\mu) = (2, 0)$ then (ω_n) has the form

$$\omega_n = bn^2 + cn + d$$

for all $n \geq 1$, with $b > 0$. This implies that

$$\partial\omega_\Lambda = 2b\Lambda + c \qquad ; \qquad \partial^2\omega_\Lambda = 2b$$

Hence, from Lemma (1) we have the commutation relations

$$[a^-, a^+] = \partial\omega_\Lambda = 2b\Lambda + c$$

$$[a^-, \partial\omega_\Lambda] = [a^-, 2b\Lambda + c] = 2b[a^-, \Lambda]$$

$$= 2ba^-\partial\Lambda = 2ba^-$$

Consequently

$$[a^+, \partial\omega_\Lambda] = -2ba^+$$

The statement then follows from the definition of the $sl(2, \mathbb{R})$ $*$-Lie algebra.

The class $C(\mu) = (3, 0)$

Theorem 7 Let $(\mu, (\omega_n))$ be a probability measure with information complexity $C(\mu) = (3, 0)$, then the linear space $\mathcal{L}_X^0$ generated by the operators

$$\left\{a^-, a^+, \partial\omega_\Lambda\right\}$$

is a ∞–dimensional $*$-Lie algebra.

Proof If $C(\mu) = (3, 0)$, then (ω_n) has the form

$$bn^3 + cn^2 + dn + en$$

This implies,

$$\partial\omega_\Lambda = 3b\Lambda^2 + (2c - 3b)\,\Lambda + (2d - c) \qquad (35)$$

and

$$\partial^2\omega\Lambda = 6b\Lambda + 2c - 5b \quad ; \quad \partial^3\omega_\Lambda = 6b$$

By lemma (1) we have

$$[a^+, \Lambda] = -\partial\Lambda a^+ = -a^+ \in \mathcal{L}_X \qquad (36)$$

By relation (35) and (36) one has

$$\begin{aligned}[a^+, \partial\omega_\Lambda] &= [a^+, 3b\Lambda^2 + (2c-3b)\Lambda + (2d-c)] \\ &= -3b(a^+\Lambda + \Lambda a^+) + (3b-2c)a^+ \in \mathcal{L}_X\end{aligned}$$

Since $3b \neq 0$ and $a^+ \in \mathcal{L}_X$ this implies

$$a^+\Lambda + \Lambda a^+ \in \mathcal{L}_X$$

Moreover, by relation (36) one has

$$[a^+, a^+\Lambda + \Lambda a^+] = a^+[a^+, \Lambda] + [a^+, \Lambda]a^+$$

$$= -2(a^+)^2 \in \mathcal{L}_X$$

Then one proves that for any $n \geq 1$, $(a^+)^n \in \mathcal{L}_X$ the family of operators

$$\{(a^+)^n \ : \ n \in \mathbb{N}\}$$

is linearly independent and this implies that $\mathcal{L}^0_X$ is infinite dimensional.

Probabilistic extensions of quantum mechanics: the free evolution of a classical random variable

Consider the symmetric case:

$$a^0 = 0$$

Start from the probabilistic commutation relations in monic form:

$$[a^-, a^+] \ = \ (\omega_\Lambda - \omega_{\Lambda-1})$$

$$[a^+, a^+] = [a^-, a^-] = 0$$

The Schrödinger equation (with $\hbar = 1$) is:

$$\partial_t \psi_t = -iH\psi_t \tag{37}$$

Taking

$$H := ca^{+}a^{-} \qquad ; \qquad c > 0$$

the corresponding Heisenberg evolution for $a^{\pm}$ is

$$a_t := u_t(a^{\pm}) := \tag{38}$$

$$= e^{itH}a^{\pm}e^{-itH} = e^{itca^{+}a^{-}}a^{\pm}e^{-itca^{+}a^{-}}$$

From

$$[a^{-}, a^{+}a^{-}] = [a^{-}, a^{+}]a^{-} = (\omega_{\Lambda} - \omega_{\Lambda-1})a^{-}$$

one deduces that

$$\frac{d}{dt}a_t = \frac{d}{dt}e^{itca^{+}a^{-}}a^{-}e^{-itca^{+}a^{-}} =$$

$$= ite^{itca^{+}a^{-}}[ca^{+}a^{-}, a^{-}]e^{-itca^{+}a^{-}}$$

$$= -ite^{itca^{+}a^{-}}c(\omega_{\Lambda} - \omega_{\Lambda-1})a^{-}e^{-itca^{+}a^{-}}$$

$$= -itc(\omega_{\Lambda} - \omega_{\Lambda-1})e^{itca^{+}a^{-}}a^{-}e^{-itca^{+}a^{-}}$$

Since ca^+a^- leaves the orthogonal gradation invariant, it commutes with Λ,

$$[a^+a^-, \Lambda] = 0$$

and, since a^+a^- and Λ have discrete spectrum, with all its Borel functions. Therefore, denoting

$$a^-(t) := e^{itca^+a^-} a^- e^{-itca^+a^-}$$

since $a^-(0) = a^-$ one obtains

$$\frac{d}{dt} a^-(t) = -it(\omega_\Lambda - \omega_{\Lambda-1}) a^-(t)$$

$$a^-(0) = a^-$$

whose unique solution is

$$a^-(t) = e^{itca^+a^-} a^- e^{-itca^+a^-} =$$

$$= e^{-itc(\omega_\Lambda - \omega_{\Lambda-1})} a^-$$

Therefore

$$a^+(t) = a^+ e^{itc(\omega_\Lambda - \omega_{\Lambda-1})}$$

This implies that

$$[a(t), a^+(t)] =$$

$$= e^{-itc(\omega_\Lambda - \omega_{\Lambda-1})} a^- a^+ e^{itc(\omega_\Lambda - \omega_{\Lambda-1})}$$

$$- a^+ e^{itc(\omega_\Lambda - \omega_{\Lambda-1})} e^{-itc(\omega_\Lambda - \omega_{\Lambda-1})} a^-$$

$$= a^- a^+ - a^+ a^- = [a, a^+]$$

Thus the map

$$t \in R \ \mapsto \ a^\pm(t) = \begin{cases} a^+ e^{it(\omega_\Lambda - \omega_{\Lambda-1})} \\ e^{-it(\omega_\Lambda - \omega_{\Lambda-1})} a^- \end{cases}$$

called **the generalized free evolution**, is a $*$–Lie–algebra isomorphism, hence it extends to an (associative) $*$–algebra isomorphism of the universal enveloping algebra of $(a^+, a^-, 1)$.

This means that the $*$–algebra

$$\text{Pol}(a^{\pm}) := \text{algebraic span of} \{a^{\pm}\}$$

is left invariant by its unique $*$–automorphism extending the generalized free evolution.

Equilibriumand local equilibrium states of the Lie algebra

From the expression of the generalized free evolution

$$u_t(X) = e^{it(\omega_\Lambda - \omega_{\Lambda-1})} X e^{-it(\omega_\Lambda - \omega_{\Lambda-1})} \qquad ; \qquad t \in R$$

by analytic continuation at $t = -i\beta$ one obtains

$$u_{-i\beta}(X) = \tag{39}$$

$$= e^{\beta(\omega_\Lambda - \omega_{\Lambda-1})} X e^{-\beta(\omega_\Lambda - \omega_{\Lambda-1})}$$

The KMS equilibrium condition at inverse temperature β, for a given density matrix W, is:

$$\mathrm{Tr}(W u_{t-i\beta}(X) Y) = \mathrm{Tr}(W Y u_t(X))$$

Putting $t = 0$ this becomes

$$\mathrm{Tr}(W u_{-i\beta}(X) Y) = \mathrm{Tr}(W Y X)$$

Using the explicit form (39) of $u_{-i\beta}(X)$, this is equivalent to

$$\mathrm{Tr}(WYX) = \mathrm{Tr}(Wu_{-i\beta}(X)Y)$$

$$= \mathrm{Tr}(We^{\beta(\omega_\Lambda - \omega_{\Lambda-1})}Xe^{-\beta(\omega_\Lambda - \omega_{\Lambda-1})}Y)$$

$$= \mathrm{Tr}(XWY) \Leftrightarrow$$

$$\Leftrightarrow We^{\beta(\omega_\Lambda - \omega_{\Lambda-1})}Xe^{-\beta(\omega_\Lambda - \omega_{\Lambda-1})} = XW \Leftrightarrow$$

$$\Leftrightarrow We^{\beta(\omega_\Lambda - \omega_{\Lambda-1})}X = XWe^{\beta(\omega_\Lambda - \omega_{\Lambda-1})} \Leftrightarrow$$

$$We^{\beta(\omega_\Lambda - \omega_{\Lambda-1})} =: \frac{1}{Z_\beta} \cdot 1 \Leftrightarrow$$

$$\Leftrightarrow W = \frac{e^{-\beta(\omega_\Lambda - \omega_{\Lambda-1})}}{Z_\beta}$$

The covariance of a (β, ω)–KMS state in a representation $(\mathcal{H}, \Phi)$ where $\mathcal{P}(a^+, a)$ has a cyclic unit vector Φ.

$$a^{+m} a^n \Phi \quad \text{total in } \mathcal{H}$$

$$\varphi(u_{-i\beta}(a^+)a)$$

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Identification of the theory of orthogonal polynomials in d–indeterminates with the theory of 3–diagonal symmetric interacting Fock spaces on $\mathbb{C}^d$,
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