Institute of Mathematics of National Academy of Sciences of Ukraine
3 Tereshchenkivs'ka Str., Kiev, Ukraine vaneeva@gmail.com

Equivalence transformations in the study of integrability

Olena Vaneeva

Joint work with Roman Popovych and Christodoulos Sophocleous

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According to F. Calogero integrable equations can be divided into those that are linearizable by an appropriate *Change of variables* (C-integrable equations) and equations integrable by inverse scattering transform method (*Spectral transform* technique) (S-integrable equations).

Among the C-integrable equations there are, e.g., the famous Burgers equation

$$u_t + uu_x = \nu u_{xx}, \quad \nu = \mathrm{const},$$

that can be linearized to the heat equation by the Hopf–Cole transform., the Sharma–Tasso–Olver equation

$$u_t + u_{xxx} + 3u^2 u_x + 3u_x^2 + 3uu_{xx} = 0,$$

that is the second member of the Burgers hierarchy,
the u^{-2}-diffusion equation (named also Fujita–Storm equation)

$$u_t = (u^{-2} u_x)_x + au,$$

the Fokas–Yortsos equation

$$u_t = (u^{-2} u_x)_x + au^{-2} u_x.$$

S-integrable equations in (1+1)-dimensions include:

the KdV and modified KdV equations

$$u_t + uu_x + u_{xxx} = 0 \quad \text{and} \quad u_t + u^2 u_x + u_{xxx} = 0,$$

the Gardner equation (the combined KdV–mKdV equation)

$$u_t + uu_x + u^2 u_x + u_{xxx} = 0,$$

the cylindrical KdV equation

$$u_t = u_{xxx} + 6uu_x - \frac{1}{2t}u,$$

the Dym equation

$$u_t = u^3 u_{xxx},$$

the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0.$$

Many papers devoted to the study of variable-coefficient equations were published, especially, in recent years. Usual topics of these papers are the application of Painlevé test in order to single out subclasses of integrable equations within wider classes of variable-coefficient equations, the construction of conservation laws, Lax pairs and bilinear representations and finding exact soliton solutions by the Hirota bilinear method. Since sometimes variable-coefficient models are quite complicated and the number of variable coefficients varies from one to five or even to ten in some cases, packages of symbolic computations are widely used to complete these tasks.

At the same time, the equivalence between equations in a class under study is neglected in many works. Though even in pioneering works on exactly solvable models it was shown that if an integrable PDE is related to another PDE by certain change of variables (point or non-point), then the latter PDE is also integrable. The classical examples are the connection between the KdV and mKdV equation via the Miura transformation, the reducibility of the Gardner equation to the mKdV (Miura, 1968) and of the cylindrical KdV equation to the classical KdV (Lugovtsov&Lugovtsov, 1969; Johnson, 1979).

The KdV and cubic Schrödinger equations with time-dependent coefficients,

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0 \quad \text{and} \quad iu_t + f(t)u_{xx} + g(t)|u|^2u = 0, \quad (1)$$

appear in different applications. Here f and g are nonvanishing smooth functions of t.

It was shown in (Joshi, 1987) that the KdV and nonlinear Schrödinger equations with time-dependent coefficients (1) pass the Painlevé test if and only if the coefficients f and g satisfy the conditions

$$g(t) = f(t)\Big(a_1\int^t f(s)\,\mathrm{d}s + a_0\Big) \quad \text{and} \quad g(t) = f(t)\Big/\Big(a_1\int^t f(s)\,\mathrm{d}s + a_0\Big),$$

respectively. Here a_1 and a_0 are constants with $a_1^2 + a_0^2 \neq 0$. These conditions coincide with those of reducibility of equations (1) to their constant-coefficient counterparts, which were obtained in (Grimshaw, 1979).

Another way for construction of variable-coefficient integrable models from constant-coefficient members of integrable hierarchies was presented in (Fuchssteiner, 1993).

Any "linear superposition", with arbitrary time-dependent coefficients, of members of an integrable evolution hierarchy that correspond to mutually commuting flows proved to be again integrable.

For example, the equation

$$u_t + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x + u_{xxxxx} + \psi(t)(6uu_x + u_{xxx}) = 0,$$

is S-integrable.

Consider the class of variable-coefficient fifth-order KdV-like equations of the form

$$u_t + a(t)uu_{xxx} + b(t)u_x u_{xx} + c(t)u^2 u_x + f(t)uu_x + g(t)u_{xxxxx} \\ + h(t)u_{xxx} + m(t)u + n(t)u_x + k(t)xu_x = 0,$$

where the functions a, b, c, f, g, h, m, n, and k are arbitrary smooth functions of the time variable t with $g(a^2 + b^2 + c^2) \neq 0$. Recently certain subclasses of this class were studied in many papers, e.g., in

- X. Yu, Y.-T. Gao, Z.-Y. Sun and Y. Liu, *N*-soliton solutions, Bäcklund transformation and Lax pair for a generalized variable-coefficient fifth-order Korteweg–de Vries equation, *Phys. Scr.* **81** (2010), 045402, 6 pp.
- X. Yu, Y.-T. Gao, Z.-Y. Sun and Y. Liu, Investigation on a nonisospectral fifth-order Korteweg–de Vries equation generalized from fluids, *J. Math. Phys.* **53** (2012), 013502, 8 pp.
- G.-Q. Xu, Painlevé integrability of a generalized fifth-order KdV equation with variable coefficients: Exact solutions and their interactions, *Chin. Phys. B* **22** (2013), 050203, 8 pp.

It was found in [Xu2013] that equations

$$u_t + auu_{xxx} + bu_x u_{xx} + cu^2 u_x + fuu_x + gu_{xxxxx} + hu_{xxx} + mu + nu_x + kxu_x = 0$$

with $k = 0$ are Painlevé integrable in the following three cases

I. $b = a, \quad c = \mu_1 a e^{\int m\,\mathrm{d}t}, \quad f = 2\mu_2 a,$
$g = \frac{a}{5\mu_1} e^{-\int m\,\mathrm{d}t}, \quad h = \frac{\mu_2}{\mu_1} a e^{-\int m\,\mathrm{d}t};$

II. $b = 2a, \quad c = \mu_1 a e^{\int m\,\mathrm{d}t}, \quad f = 2\mu_1 h e^{\int m\,\mathrm{d}t}, \quad g = \frac{3a}{10\mu_1} e^{-\int m\,\mathrm{d}t};$

III. $b = \frac{5}{2} a, \quad c = \mu_1 a e^{\int m\,\mathrm{d}t}, \quad f = 2\mu_2 a,$
$g = \frac{a}{5\mu_1} e^{-\int m\,\mathrm{d}t}, \quad h = \frac{\mu_2}{\mu_1} a e^{-\int m\,\mathrm{d}t}.$

In [Yu2010,Yu2012] only one integrable case was derived

$$a = b = 15g\nu e^{\int (m-2k)\,\mathrm{d}t}, \quad c = 45g\nu^2 e^{2\int (m-2k)\,\mathrm{d}t}, \quad f = h = 0.$$

We show that all the mentioned cases of integrable equations are reduced by point transformations to well-known fifth-order integrable evolution equations. To achieve this goal, we present a complete description of admissible transformations between equations from this class.

An admissible transformation can be interpreted as a triple consisting of two fixed equations from a class and a point transformation that links these equations. The set of admissible transformations considered with the standard operation of composition of transformations is also called the equivalence groupoid.

An equivalence transformation applied to any equation from the class always maps it to another equation from the same class whereas an admissible transformation may exist only for a specific pair of equations from the class under consideration.

Theorem

The usual point equivalence group G of the class

$$u_t = F(t)u_n + G(t, x, u_0, u_1, \dots, u_{n-1}),$$
$$F \neq 0, \quad G_{u_i u_{n-1}} = 0, \ i = 1, \dots, n-1,$$

consists of the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad \tilde{u} = U^1(t,x)u + U^0(t,x),$$
$$\tilde{F} = \frac{(X^1)^n}{T_t} F,$$
$$\tilde{G} = \frac{1}{T_t}\Bigg[U^1 G - \left(\sum_{k=0}^{n-1} \binom{n}{k} U^1_{(n-k)} u_{(k)} + U^0_{(n)}\right) F$$
$$+ U^1_t u + U^0_t - \frac{X^1_t x + X^0_t}{X^1}\left((U^1 u)_x + U^0_x\right)\Bigg],$$

where $T_t X^1 U^1 \neq 0$. This class is normalized with respect to the group G.

Theorem

The generalized extended equivalence group $G^{\sim}$ of the class

$$u_t+auu_{xxx}+bu_xu_{xx}+cu^2u_x+fuu_x+gu_{xxxxx}+hu_{xxx}+mu+nu_x+kxu_x=0$$

consists of the transformations

$$\tilde t=\alpha(t),\quad \tilde x=\beta(t)x+\gamma(t),\quad \tilde u=\varphi(t)\left(u+\sigma e^{-\int m\,\mathrm{d}t}\right),$$

$$\tilde a=\frac{\beta^3}{\alpha_t\varphi}a,\quad \tilde b=\frac{\beta^3}{\alpha_t\varphi}b,\quad \tilde c=\frac{\beta}{\alpha_t\varphi^2}c,\quad \tilde f=\frac{\beta}{\alpha_t\varphi}\left(f-2\sigma ce^{-\int m\,\mathrm{d}t}\right),$$

$$\tilde g=\frac{\beta^5}{\alpha_t}g,\quad \tilde h=\frac{\beta^3}{\alpha_t}\left(h-\sigma ae^{-\int m\,\mathrm{d}t}\right),\quad \tilde m=\frac{1}{\alpha_t}\left(m-\frac{\varphi_t}{\varphi}\right),$$

$$\tilde n=\frac{\beta}{\alpha_t}\left(n+\left(\frac{\gamma}{\beta}\right)_t-k\frac{\gamma}{\beta}+\sigma^2ce^{-2\int m\,\mathrm{d}t}-\sigma fe^{-\int m\,\mathrm{d}t}\right),\quad \tilde k=\frac{1}{\alpha_t}\left(k+\frac{\beta_t}{\beta}\right).$$

where α, β, γ, and φ run through the set of smooth functions of t with $\alpha_t\beta\varphi\neq0$, and σ is an arbitrary constant.

Corollary

Any equation from the class

$$u_t + a(t)uu_{xxx} + b(t)u_x u_{xx} + c(t)u^2 u_x + f(t)uu_x + g(t)u_{xxxxx} + h(t)u_{xxx} + m(t)u + n(t)u_x + k(t)xu_x = 0$$

can be reduced by the point transformation

$$\tilde{t} = \int g e^{-5\int k\,\mathrm{d}t}\,\mathrm{d}t, \quad \tilde{x} = e^{-\int k\,\mathrm{d}t}x - \int n e^{-\int k\,\mathrm{d}t}\,\mathrm{d}t, \quad \tilde{u} = e^{\int m\,\mathrm{d}t}u$$

to an equation from the same class with $g = 1$ *and* $m = n = k = 0$.

Consider the class of constant-coefficient fifth-order KdV equations of the form

$$u_t + Auu_{xxx} + Bu_x u_{xx} + Cu^2 u_x + u_{xxxxx} = 0,$$

where A, B and C are nonzero constants. Up to scale transformations, there exist three inequivalent triples (A, B, C) such that the latter equations are integrable. These are the triples $(10, 20, 30)$, $(15, 15, 45)$ and $(10, 25, 20)$, which respectively give

- Lax's fifth-order KdV equation

$$u_t + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x + u_{xxxxx} = 0;$$

- the Sawada–Kotera equation (equiv. to the Caudrey–Dodd–Gibbon Eq.)

$$u_t + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x + u_{xxxxx} = 0;$$

- the Kaup–Kupershmidt equation

$$u_t + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x + u_{xxxxx} = 0.$$

Corollary

The usual equivalence group $G^{\sim}_{\rm const}$ *of the class* (14) *consists of the transformations*

$$\tilde t=\beta^5t+\delta,\quad \tilde x=\beta x+\gamma,\quad \tilde u=\frac{u}{\beta^2\lambda},\quad \tilde A=\lambda A,\quad \tilde B=\lambda B,\quad \tilde C=\lambda^2C.$$

Here β, γ, δ, *and* λ *are arbitrary constants with* $\beta\lambda\neq0$.

In view of this assertion it is obvious, e.g., that the Caudrey–Dodd–Gibbon equation in which $(\tilde A,\tilde B,\tilde C)=(30,30,180)$ is similar to the Sawada–Kotera equation with $(A,B,C)=(15,15,45)$. The similarity between the equations is realized by the scale transformation $\tilde t=t$, $\tilde x=x$, $\tilde u=\frac12u$.

Theorem

An equation from the class

$$u_t + auu_{xxx} + bu_x u_{xx} + cu^2 u_x + fuu_x + gu_{xxxxx} + hu_{xxx} + mu + nu_x + kxu_x = 0$$

is similar to a constant-coefficient equation of the form

$$u_t + Auu_{xxx} + Bu_x u_{xx} + Cu^2 u_x + u_{xxxxx} = 0$$

with $ABC \neq 0$ *if and only if its coefficients satisfy the conditions*

$$\left(\frac{b}{a}\right)_t = \left(\frac{b^2}{cg}\right)_t = \left(\frac{f}{c} e^{\int m\,\mathrm{d}t}\right)_t = 0, \quad \left(\frac{b}{g}\right)_t = \frac{b}{g}(m - 2k), \quad af = 2ch.$$

The Lax pairs of the Sawada–Kotera equation
$u_t + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x + u_{xxxxx} = 0$ are

$$L = {\partial_x}^3 + 3u\partial_x,$$
$$P = 9{\partial_x}^5 + 45u{\partial_x}^3 + 45u_x{\partial_x}^2 + 15(2u_{xx} + 3u^2)\partial_x$$

and

$$L = {\partial_x}^3 + 3u\partial_x + 3u_x,$$
$$P = 9{\partial_x}^5 + 45u{\partial_x}^3 + 90u_x{\partial_x}^2 + 15(5u_{xx} + 3u^2)\partial_x + 30(u_{xxx} + 3uu_x).$$

We carry out the transformation

$$\tilde{t} = \int g e^{-5\int k\,\mathrm{d}t}\,\mathrm{d}t, \quad \tilde{x} = e^{-\int k\,\mathrm{d}t}x - \int n e^{-\int k\,\mathrm{d}t}\,\mathrm{d}t, \quad \tilde{u} = \nu e^{\int m\,\mathrm{d}t}u$$

in the associated spectral problems, $L\psi = \lambda\psi$, $\psi_t = P\psi$.

We derive the corresponding Lax pairs for the variable-coefficient equation

$$u_t+15g\Upsilon uu_{xxx}+15g\Upsilon u_xu_{xx}+45g\Upsilon^2u^2u_x+gu_{xxxxx}+mu+nu_x+kxu_x=0$$

$$L=e^{3\int k\,\mathrm{d}t}({\partial_x}^3+3\Upsilon u\partial_x),$$
$$P=9g{\partial_x}^5+45g\Upsilon u{\partial_x}^3+45g\Upsilon u_x{\partial_x}^2+(30g\Upsilon u_{xx}+45g\Upsilon^2u^2-kx-n)\partial_x$$

and

$$L=e^{3\int k\,\mathrm{d}t}({\partial_x}^3+3\Upsilon u\partial_x+3\Upsilon u_x),$$
$$P=9g{\partial_x}^5+45g\Upsilon u{\partial_x}^3+90g\Upsilon u_x{\partial_x}^2+(75g\Upsilon u_{xx}+45g\Upsilon^2u^2-kx-n)\partial_x$$
$$+30g\Upsilon u_{xxx}+90g\Upsilon^2uu_x.$$

Here we use the notation $\Upsilon=\nu e^{\int(m-2k)\,\mathrm{d}t}$ with a nonzero constant ν.

The Kaup–Kupershmidt equation
$u_t + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x + u_{xxxxx} = 0$ admits the Lax pair

$$L = {\partial_x}^3 + 2u\partial_x + u_x,$$
$$P = 9{\partial_x}^5 + 30u{\partial_x}^3 + 45u_x{\partial_x}^2 + 5(7u_{xx} + 4u^2)\partial_x + 10(u_{xxx} + 2uu_x).$$

Using the equivalence transformation we derive the corresponding Lax pair for the equation

$$u_t+10g\Upsilon uu_{xxx}+25g\Upsilon u_xu_{xx}+20g\Upsilon^2u^2u_x+gu_{xxxxx}+mu+nu_x+kxu_x = 0,$$

that is of the form

$$L = e^{3\int k\,\mathrm{d}t}({\partial_x}^3 + 2\Upsilon u\partial_x + \Upsilon u_x),$$
$$P = 9g{\partial_x}^5 + 30g\Upsilon u{\partial_x}^3 + 45g\Upsilon u_x{\partial_x}^2 + (35g\Upsilon u_{xx} + 20g\Upsilon^2u^2 - kx - n)\partial_x$$
$$+ 10g\Upsilon u_{xxx} + 20g\Upsilon^2uu_x.$$

Equivalence transformations well fit into the study of integrability of variable-coefficient PDEs, where they can be used for several targets:

- to look for inessential arbitrary elements of the class of variable-coefficient PDEs under consideration and to gauge these elements to chosen simple values from the very beginning;
- to establish the similarity of integrable variable-coefficient PDEs, which are separated by another method (e.g., the Painlevé test) from the class under consideration, to well-known (usually, constant-coefficient) integrable equations; or, more generally, to select canonical representatives in the obtained list of integrable equations;
- to check listed integrable cases using the established similarity to previously known integrable equations;
- to derive all objects related to a singled out integrable equation and their properties from those of a similar well-studied integrable equation; such objects include, but are not exhausted by, local symmetries, cosymmetries, conservation laws, recursion operators, Bäcklund transformations, exact solutions, the Painlevé expansion, bilinear representations and Lax pairs.

Thank you for your attention!