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Irregular conformal block and conformal symmetry

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Arxiv:1207.4480 (JHEP 10 (2012) 138) with T. Nishinaka

Arxiv:1312.5535 (JHEP 04(2014) 106), Arxiv:1506.02421 with S. Choi

Arxiv:1411.4453 (PLB 742 (2015) 50) with S. Choi and H. Zhang

Arxiv:1504.07910, Arxiv:1506.03561, with H. Zhang

Virasoro primary and descendents

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Primary state: $L_0|\Delta >= \Delta|\Delta >, \qquad L_k|\Delta >= 0 \text{ when } k > 0$

$$|\Delta >= \lim_{z\to 0} V_\alpha(z)|0>$$

Descendent state:

$$L_0\Big(L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Delta >\Big)$$
$$= (n_1+n_2+\cdots n_k)\Big(L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Delta >\Big)$$

Irregular state

1. Simultaneous eigenstate of positive Virasoro generators;

$$[L_m, L_n] = (m-n)L_{m+n} \text{ for } m, n > 0.$$

2. Irregular module with rank n

$$L_k|I^{(n)}\rangle = 0 \quad \text{for} \quad 2n < k$$
$$L_k|I^{(n)}\rangle = \Lambda_k|I^{(n)}\rangle \quad \text{for} \quad n \le k \le 2n$$

3. Not the eigenstate of L_0 but a special combination of descendents. (Gaiotto state or Whittaker state)

4. Motivated by N=2 super Yang-Mills gauge theory, Argyres-Douglas limit

L_1 eigenstate

Gaiotto (2009)

$$L_1|\Delta,\Lambda^2\rangle = \Lambda^2|\Delta,\Lambda^2\rangle\,,\;\; L_2|\Delta,\Lambda^2\rangle = 0$$

$$|\Delta,\Lambda^2\rangle = \textstyle\sum_\ell \Lambda^{2n} u_n;$$

$$u_0 = |\Delta\rangle$$

$$u_1 = \frac{1}{2\Delta}L_{-1}|\Delta\rangle$$

$$u_2 = \frac{(c+8\Delta)L_{-1}^2 - 12\Delta L_{-2}}{4\Delta(2c\Delta + c + 16\Delta^2 - 10\Delta)}|\Delta\rangle$$

$$c = 1 + 6Q^2, \quad Q = b + 1/b$$

u_0 is not the minimal state.

L1 and L2 eigenstate

$$L_2|\Delta,\Lambda,\xi\rangle = -\Lambda^2|\Delta,\Lambda,\xi\rangle\,, \quad L_1|\Delta,\Lambda,\xi\rangle = -2\xi\Lambda|\Delta,\Lambda,\xi\rangle$$

$$|\Delta,\Lambda^2\rangle = \sum_{\ell} \Lambda^{\ell} w_n$$

$$w_0 = |\Delta\rangle$$

$$w_1 = -\frac{\xi}{\Delta} L_{-1}|\Delta\rangle$$

$$w_2 = \frac{(c\xi^2 + \Delta(3+8\xi^2))L_{-1}^2 - 2\Delta(1+2\Delta+6\xi^2)L_{-2}}{4\Delta(2\Delta + c + 16\Delta^2 - 10\Delta)}|\Delta\rangle$$

More details in Marshak et al (2009)

Irregular module of rank n

(Kanno, Maruyoshi, Shiba, Taki. 2013)

$$|G_{2n}\rangle = \sum_{\ell=0}^{\infty} \sum_{\ell_p} \Lambda^{\ell/n} \prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} b_i^{\ell_i} m^{\ell_n}$$

$$Q_\Delta^{-1}(2n^{\ell_{2n}}(2n-1)^{\ell_{2n-1}} \cdots 2^{\ell_2} 1^{\ell_1}; Y) L_{-Y} |\Delta\rangle \,,$$

Problem: The module is not determined completely by the eigenvalues when rank n>1.

$$b_i = ?$$

$$L_k|I^{(n)}\rangle = \Lambda_k |I^{(n)}\rangle \text{ for } n \le k \le 2n$$

$$\Lambda_n = \Lambda m, \quad \Lambda_k = \Lambda^{k/n} a_{2n-k} \text{ when } n < k \le 2n$$

Questions to be answered

1. Irregular module of rank n >1
2. Extension to W symmetry

How to construct the irregular module for rank n >1

1. Coherent state

Gaiotto &Teschner (2012)

Coherent coordinates of Heisenberg algebra

$$a_k|\mathrm{in}\rangle = c_k|\mathrm{in}\rangle \quad \text{for} \quad 1 \leq k \leq n$$

$$a_k|\mathrm{in}\rangle = 0 \qquad\qquad \text{for} \quad k > n$$

Eigenvalues of rank n: $L_k|\mathrm{in}\rangle = \Lambda_k|\mathrm{in}\rangle \quad \text{for} \quad n \leq k \leq 2n$

$$\Lambda_k = (k+1)Q\,c_k - \sum_{0\leq \ell \leq k} c_k c_{k-\ell}$$

Other positive mode representation is a differential operators

$$L_k|\mathrm{in}\rangle = (\Lambda_k + v_k)|\mathrm{in}\rangle \quad \text{for} \quad 0 \leq k \leq n-1$$

$$v_k = \sum_{0\leq \ell \leq m} c_{\ell+k} \frac{\partial}{\partial c_\ell}$$

2. Energy momentum tensor

Energy momentum tensor and Virasoro generator

$$T(z) = \sum_k \frac{L_k}{z^{k+2}}$$

Irregular module has the energy momentum tensor with poles of degree more than 2

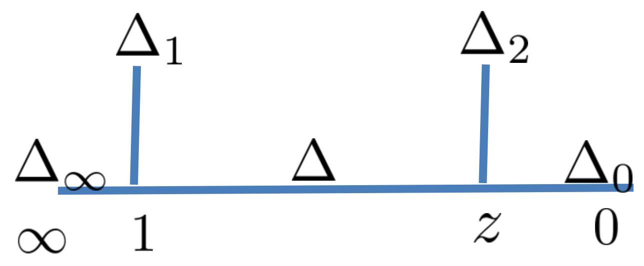
$$T(z)|I^{(n)}\rangle = \sum_{0\le k\le n} \frac{L_k}{z^{k+2}}|I^{(n)}\rangle$$

Seiberg-Witten curve:

$$< \mathrm{out}|T(z)|I^{(n)} >= \sum_{0\le k\le n} \frac{< \mathrm{out}|L_k|I^{(n)} >}{z^{k+2}}$$

3. Liouville conformal block

$$\langle V_{\Delta_\infty}(\infty) V_{\Delta_1}(1) V_{\Delta_2}(z) V_{\Delta_0}(0) \rangle$$

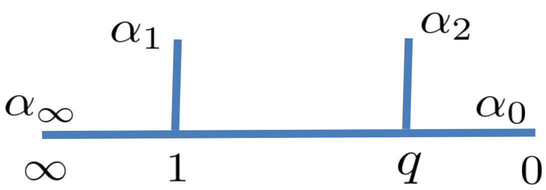


Liouville primary field: $V_\alpha(z) = e^{2\alpha\phi(z)}$

Conformal dimension = $\Delta = \alpha(Q - \alpha)$

Back ground charge= Q

Selberg integral of conformal block



$$q^{-2\alpha_2\alpha_0}(1-q)^{-2\alpha_1\alpha_2} \quad \times \int \left[\prod_{I=1}^{N} d\lambda_I\right] \prod_{I<J}(\lambda_I-\lambda_J)^{-2b^2}$$

$$\times \prod_I (\lambda_I)^{-2b\alpha_0}(\lambda_I-q)^{-2\alpha_2}(\lambda_I-1)^{-2\alpha_1}$$

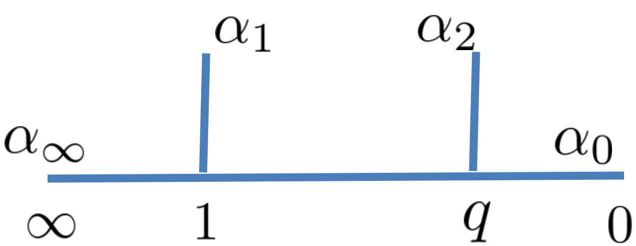
Screening operator: $\int dz\; e^{2b\phi(z)}$

Neutrality condition: $\alpha_0+\alpha_1+\alpha_2+\alpha_\infty+bN=Q$

Free field correlation: $\langle e^{2\alpha_1\phi(z)}e^{2\alpha_2\phi(w)}\rangle = (z-w)^{-2\alpha_1\alpha_2}\,,$

beta-deformed Penner-type matrix model

Dijkgraaf & Vafa (2009)



$$q^{-2\alpha_2\alpha_0}(1-q)^{-2\alpha_1\alpha_2}\;\; I_4(\alpha_i)$$

$$I_4 = \int \left[\prod_{I=1}^{N} d\lambda_I\right] \prod_{I<J} (\lambda_I - \lambda_J)^{-2b^2} \;\; \exp\left(-\frac{2b}{\hbar}\sum_I V(\lambda_I)\right)$$

$$\frac{V(\lambda_I)}{\hbar} = -\alpha_0 \log(\lambda_I) - \alpha_1 \log(\lambda_I - 1) - \alpha_2 \log(\lambda_I - q).$$

$$\beta = -b^2 \text{ or } b = i\sqrt{\beta}, \qquad\qquad -2b/\hbar = \sqrt{\beta}/g$$

4. Colliding limit

Eguchi & Maruyoshi (2010)
Gaiotto & Teschner (2012)

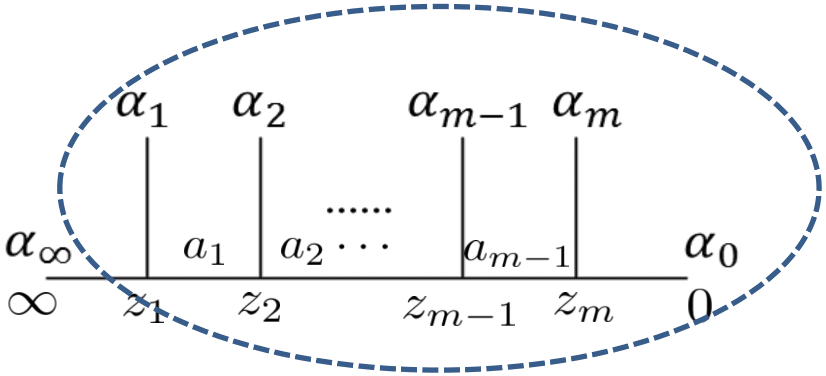
- Vertex operators are put at the same point
- The operators have infinite Liouville charge so that appropriate finiteness is maintained; $c_k = \sum_{i=1}^{m} \alpha_i z^k$

- Note that OPE of regular primary has the form,

$$V_{\alpha_1}(z) V_{\alpha_2}(o) \to z^{\Delta - \Delta_1 - \Delta_2} V_{\alpha}(0) + \text{ descendents}$$

- However, colliding limit produces more than primary state: They create coherent state.

5. Irregular matrix model for $< \Delta | I^{(m)} >$



$$\Longrightarrow Z_{(0,m)} \sim \langle \Delta | I^{(m)} \rangle$$

$$z_1, z_2, \cdots z_m \to 0$$

$$Z_{(0,m)} \equiv \left[\prod_{I=1}^{N} \int d\lambda_I\right] \Delta^{2\beta} \exp\left(-\frac{\sqrt{\beta}}{g} \sum_I V(\lambda_I)\right)$$

$$\frac{V(\lambda)}{\hbar} = -\sum_r \alpha_r \log(\lambda - z_r) \Longrightarrow -c_0 \log \lambda + \sum_{k>0} \frac{c_k}{k\lambda^k}$$

$$c_k = \sum_{i=1}^{m} \alpha_i z^k$$

6. Loop equation to find $< \Delta | I^{(m)} >$

$$\frac{f(z)}{4} = W(z)^2 + V'(z)W(z) + g\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) W'(z) + g^2 W(z,z)$$

$$W(z) = g\sqrt{\beta}\left\langle \sum_{I_1} \frac{1}{z-\lambda_{I_1}} \right\rangle_{\rm conn}$$

$$W(z,z) = \beta \left\langle \sum_{I_1} \frac{1}{z-\lambda_{I_1}} \sum_{I_2} \frac{1}{z-\lambda_{I_2}} \right\rangle_{\rm conn}$$

$$f(z) == \equiv -\frac{\hbar b}{2} \sum_{I=1}^{N} \left\langle \frac{V'(z)-V'(\lambda_I)}{z-\lambda_I} \right\rangle$$

$$f(z) = \sum_{k=0}^{m-1} \frac{d_k}{z^{2+k}} = \sum_{k=0}^{m-1} \frac{v_k\left(-\hbar^2 \log Z_N^{(m)}\right)}{z^{2+k}}$$

$$v_k = \sum_\ell c_{\ell+k} \frac{\partial}{\partial c_\ell}$$

$$\Longrightarrow v_k\left(-\hbar^2 \log Z_N^{(m)}\right) = d_k$$

(N & R. 2012, C&R 2013)

7. Loop equation and conformal symmetry

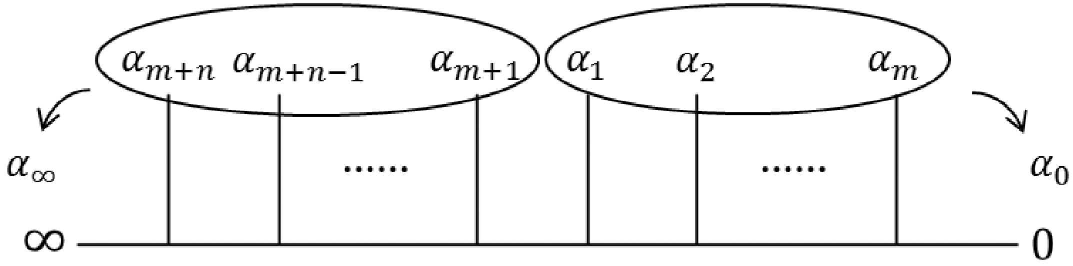
$$\frac{f(z)}{4} = W(z)^2 + V'(z)W(z) + g\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)W'(z) + g^2 W(z,z)$$

$$\Longrightarrow \xi_2 = x(z)^2 + \hbar Q x'(z) - \hbar^2 W(z,z)$$

$$x(z) = 2W(z) + V'(z)$$

$$\xi_2(z) = -\frac{1}{\hbar^2}(V'^2 + f + \hbar Q V'') = \frac{\langle \Delta | T(z) | I^{(n)} \rangle}{\langle \Delta | I^{(n)} \rangle}$$

7. Irregular matrix model $Z_{(n;m)}$



$$\frac{f(z)}{4} = \sum_{k=-(n-1)}^{m-1} \frac{d_k}{z^{2+k}}; \qquad \frac{V_{n,m}(z)}{\hbar} = -c_0 \log z + \sum_{k=-n}^{m} \frac{c_k}{k z^k}$$

$$d_k = v_k\Big(- \hbar^2 \log Z_{(n;m)}\Big) \text{ for } k \geq 0 : \qquad v_k = \sum_{\ell>0} c_{\ell+k} \frac{\partial}{\partial c_\ell}$$

$$d_{-k} = u_k\Big(- \hbar^2 \log Z_{(n;m)}\Big) + 2\hbar Q N c_{-k} \text{ for } k > 0 \ :$$

$$u_k = \sum_{\ell>0} c_{-\ell-k} \frac{\partial}{\partial c_{-\ell}}$$

8. Irregular conformal block as $\langle I^{(n)} | I^{(m)} \rangle$

$Z_{(n;m)} \sim \langle I^{(n)} | I^{(m)} \rangle$ with proper normalization

$$\langle I^{(n)} | I^{(m)} \rangle = \frac{e^{\zeta_{(m:n)}} Z_{(m:n)}(c_0; \{c_k\}, \{c_{-\ell}\})}{Z_{(0:n)}(c_0; \{c_k\}) Z_{(0:m)}(c_\infty; \{c_{-\ell}\})}$$

$$\zeta_{(m:n)} = \sum_k^{\min(m,n)} 2 c_k c_{-k}/k$$

Comparison with algebraic construction

$$|G_{2n}\rangle = \sum_{\ell=0}^{\infty} \sum_{\ell_p} \Lambda^{\ell/n} \prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} b_i^{\ell_i} m^{\ell_n}$$

$$Q_\Delta^{-1}(2n^{\ell_{2n}}(2n-1)^{\ell_{2n-1}} \cdots 2^{\ell_2} 1^{\ell_1}; Y) L_{-Y} |\Delta\rangle ,$$

$$b_k = v_k + \Lambda_k$$

How to extend to W3-symmetry

R & C, 1506.02421

$$L_k|I^{(n)}\rangle = \Lambda_k|I^{(n)}\rangle \text{ for } n \le k \le 2n$$

$$W_k|I^{(n)}\rangle = \Lambda_k|I^{(n)}\rangle \text{ for } 2n \le k \le 3n$$

- Toda field theory
- Conformal block and colliding limit
- Irregular matrix model
- Loop equation contains (A2 case)
  - Quadratic: energy momentum tensor
  - Cubic: W3 symmetry
- Evaluate the partition function using the symmetry

Summary

1. Virasoro irregular conformal block is studied using irregular matrix model.
2. Virasoro irregular module and their inner product is given in terms of irregular matrix model with appropriate potential
3. Loop equation contains the conformal symmetry.

4. One can extend the irregular matrix model from Toda field theory
5. Loop equation contains Virasoro and W3 symmetry.
6. Irregular module and their inner product are found using the symmetry alone.

Further outlook

- Classical limit of the irregular conformal block
  (Virasoro case: R& Z 1504.07910, 1506.03561)
- Extension to W-symmetry and relation with DDAHA
  (M, R & Z, 1405.3141)
- New physical application