Automorphisms of multiloop Lie algebras

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Multiloop Lie algebras

- k an algebraically closed field of characteristic 0
- $\xi_m \in k$, $m \geq 1$, primitive m-th roots of unity
- L a finite-dimensionsional split simple (Chevalley) Lie algebra over k of type $A_l - G_2$
- $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $n \geq 1$
- $\tilde{R} = k[x_1^{\pm \frac{1}{m}}, \ldots, x_n^{\pm \frac{1}{m}}]$, $m \geq 1$

The ring extension $\tilde{R}/R$ is Galois, $\mathrm{Gal}(\tilde{R}/R) \cong (\mathbb{Z}/m\mathbb{Z})^n$.
Fix n pairwise commuting automorphisms of L of order m:

$$\sigma = (\sigma_1, \ldots, \sigma_n) \in \Big(\mathrm{Aut}_k(L)\Big)^n.$$

This determines a $\mathbb{Z}^n$-grading on L:

$$L_{i_1 \ldots i_n} = \{x \in L \mid \sigma_j(x) = \xi_m^{i_j} x,\ 1 \leq j \leq n\}.$$

Definition (Allison, Berman, Moody, Pianzola, 2009)

The **multiloop Lie algebra** $\mathcal{L}(L, \sigma)$ is the $\mathbb{Z}^n$-graded k-Lie subalgebra

$$\mathcal{L}(L,\sigma) = \bigoplus_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} L_{i_1\ldots i_n} \otimes x_1^{\frac{i_1}{m}} \ldots x_n^{\frac{i_n}{m}}$$

of the k-Lie algebra $L \otimes_k k[x_1^{\pm\frac{1}{m}}, \ldots, x_n^{\pm\frac{1}{m}}]$.

- As a k-Lie algebra, $\mathcal{L}(L, \sigma)$ is infinite-dimensional.
- As an R-Lie algebra, $\mathcal{L}(L, \sigma)$ is an $\tilde{R}/R$-twisted form of the R-Lie algebra $L \otimes_k R$, i.e.

$$\mathcal{L}(L,\sigma) \otimes_R \tilde{R} \cong (L \otimes_k R) \otimes_R \tilde{R}.$$

Lie tori

- Δ a finite irreducible root system $A_l - G_2$ or BC_l; $0 \in \Delta$
- set $\Delta^\times = \Delta \setminus \{0\}$, $Q = \mathbb{Z}\,\Delta$, and $\Delta^\times_{ind} = \{\alpha \in \Delta^\times \mid \frac{1}{2}\alpha \notin \Delta\}$

Definition (Yoshii, 2004)

A **Lie torus of type Δ and nullity** $n \geq 1$ is a $Q \times \mathbb{Z}^n$-graded Lie algebra $\mathcal{L} = \bigoplus_{(\alpha,\lambda)\in Q\times\mathbb{Z}^n} \mathcal{L}^\lambda_\alpha$ over k satisfying

1. $\Delta^\times_{ind} \times 0 \subseteq \mathrm{supp}(\mathcal{L}) \subseteq \Delta \times \mathbb{Z}^n$.
2. $\mathbb{Z}^n$ is generated by the $\mathbb{Z}^n$-components of $\mathrm{supp}(\mathcal{L})$.
3. For all $(\alpha,\lambda) \in \Delta^\times \times \mathbb{Z}^n$, one has

$$\mathcal{L}^\lambda_\alpha = k \cdot e^\lambda_\alpha, \quad \mathcal{L}^{-\lambda}_{-\alpha} = k \cdot f^\lambda_\alpha, \quad \text{and} \quad [[e^\lambda_\alpha, f^\lambda_\alpha], x] = \langle \beta, \alpha^\vee \rangle x$$

   for all $(\beta,\mu) \in \Delta \times \mathbb{Z}^n$, $x \in \mathcal{L}^\mu_\beta$.

4. $\mathcal{L}$ is generated as a k-Lie algebra by $\mathcal{L}^\lambda_\alpha$, $(\alpha,\lambda) \in \Delta^\times \times \mathbb{Z}^n$.

- [ABFP, 2009] If a centerless Lie torus $\mathcal{L}$ is finitely generated over its centroid (**fgc**), then the centroid is k-isomorphic to

$$k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = R,$$

  and $\mathcal{L}$ is R-isomorphic to a multiloop Lie algebra $\mathcal{L}(L, \sigma)$.
- Not all multiloop Lie algebras are Lie tori: by definition, Lie tori are **isotropic**.
- [Neher, 2004]
  - All Lie tori are either fgc, or quantum of type A_n.
  - Lie tori are centerless cores of EALA's;
    for $n = 1$ EALA $=$ affine Kac–Moody Lie algebra;
    centerless core $=$ derived subgroup/center.

Theorem (Chernousov, Gille, Pianzola, 2011)

Let $\mathcal{L}$ be a fgc centerless Lie torus over k of nullity $n \geq 1$. Then the type Δ is an invariant of $\mathcal{L}$.

Automorphisms

Let $\mathcal{L}$ be a fgc centerless Lie torus over k with the centroid $R \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.
There is a short exact sequence

$$1 \to \mathrm{Aut}_R(\mathcal{L}) \to \mathrm{Aut}_k(\mathcal{L}) \to \mathrm{Aut}_k(R).$$

In this sequence

- $\mathrm{Aut}_k(R) \cong (k^\times)^n \rtimes \mathrm{GL}_n(\mathbb{Z})$
- $1 \to G \to \mathrm{Aut}_R(\mathcal{L}) \to \mathrm{Out}_R(\mathcal{L}) \to 1$,
  where
- $\mathrm{Out}_R(\mathcal{L}) = 1$ if L has type B_l, C_l, E_7, E_8, F_4, G_2, and is an $\tilde{R}/R$-twisted form of $\mathbb{Z}/l\,\mathbb{Z}$ or $(\mathbb{Z}/2\,\mathbb{Z})^2$ in other cases.
- $G = \mathrm{Aut}_R(\mathcal{L})^\circ$ is an adjoint simple group (scheme) over R, an $\tilde{R}/R$-twisted form of $\mathrm{Aut}_R(L \otimes_k R)^\circ$.

Definition (Tits, ∼1964)

Let H be a simple algebraic group, K an arbitrary field. The **Whitehead group of** H is

$$W(K, H) = H(K)/H(K)^+,$$

where $H(K)^+ = \langle g \in H(K) \mid g \text{ is unipotent} \rangle$.

Theorem (St., 2014)

Let $\mathcal{L}$ be a fgc centerless Lie torus of type Δ with centroid $R \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Set $G = \mathrm{Aut}_R(\mathcal{L})^\circ$ and

$$E_{exp}(\mathcal{L}) = \Big\langle \exp(\mathrm{ad}_x),\ x \in \mathcal{L}_\alpha^\lambda,\ (\alpha, \lambda) \in \Delta^\times \times \Lambda \Big\rangle \le G.$$

If $\mathrm{rank}(\Delta) \ge 2$, *then*

$$G/E_{exp}(\mathcal{L}) \cong W\Big(k((x_1)) \ldots ((x_n)), G\Big).$$

How to compute $W(F,G)$, where $F = k((x_1))\dots((x_n))$?

Let G^{sc} be the simply connected cover of G, and $C = \mathrm{Cent}(G^{sc})$. We have an exact sequence

$$1 \to C(F) \to W(F, G^{sc}) \to \mathbf{W(F,G)} \to$$
$$\to H^1(F, C(\bar{F})) \to H^1(F, G^{sc}(\bar{F})).$$

Proposition

1. *If L has type F_4, G_2, or L has type E_8 and* $\mathrm{rank}(\Delta) \geq 4$, *then* $W(F,G) = 1$.
2. *If $n = 1$, then $W(F,G) \cong H^1(F, C(\bar{F}))$ computed explicitly in all cases. If L is of type B_l, C_l, E_7, or inner type D_{2l+1}, A_l, E_6, then $H^1(F, C(\bar{F})) \cong \mathbb{Z}/d\,\mathbb{Z}$ for a suitable $d \geq 2$.*
3. *If $n = 2$, then $W(F,G) \cong H^1(F, C(\bar{F})) \cong (\mathbb{Z}/d\,\mathbb{Z})^2$ for a suitable d if L has type B_l, C_l; inner type E_6; E_7 with* $\mathrm{rank}(\Delta) \geq 3$.

$$1 \to C(F) \to W(F, G^{sc}) \to \mathbf{W}(\mathbf{F}, \mathbf{G}) \to$$
$$\to H^1(F, C(\bar{F})) \to H^1(F, G^{sc}(\bar{F}))$$

Kneser–Tits problem: compute $W(K, G^{sc})$ for any field K any any simply connected simple algebraic K-group G^{sc}.

Theorem (Wang, 1950)

If L is of type A_l and $l+1$ is square-free, then $W(F, G^{sc}) = 1$.

Theorem (Chernousov, Platonov, 1998)

One has $W(F, G^{sc}) = 1$ whenever

1. *L is of type B_l, C_l $(l \geq 2)$, D_4, E_6, F_4, G_2,*
2. *L is of type A_l $(l \geq 2)$, D_l $(l \geq 5)$, E_7, E_8, and* $\operatorname{rank}(\Delta) \geq \lfloor \frac{l}{2} \rfloor$,
3. *L is of type D_l $(l \geq 5)$, and Δ is of type B_m.*

$$1 \to C(F) \to W(F, G^{sc}) \to \mathbf{W(F, G)} \to$$
$$\to H^1(F, C(\bar{F})) \to H^1(F, G^{sc}(\bar{F}))$$

The group $H^1(F, C(\bar{F}))$ is computed in all cases.

- If L has type E_8, F_4, G_2, then $C(\bar{F}) = 1$ and $H^1(F, C(\bar{F})) = 1$.
- If L is of type B_l, C_l, E_7, or inner type D_{2l+1}, A_l, E_6, then $H^1(F, C(\bar{F})) \cong (\mathbb{Z}/d\,\mathbb{Z})^n$ for a suitable $d \geq 2$.
- Slightly more complicated for D_{2l} and outer A_l, D_{2l+1}, E_6.

The group $H^1(F, G^{sc}(\bar{F}))$ is

- trivial if $n \leq 2$;
- tricky in general; some case-by-case progress via "cohomological invariants" (Serre, Rost, Merkurjev, Chernousov, Garibaldi, Gille etc., 1990–present)

Comment

An *extended affine Lie algebra*, is a pair (E, H) consisting of a Lie algebra E over F and subalgebra H satisfying the following axioms (EA1) – (EA6).

(EA1) *E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$.*

(EA2) *H is nontrivial finite-dimensional toral and self-centralizing subalgebra of E.*

H induces a decomposition of E via the adjoint representation:

$$\begin{aligned} E &= \bigoplus_{\alpha\in H^*} E_\alpha, \\ E_\alpha &= \{e \in E : [h, e] = \alpha(h)e \text{ for all } h \in H\}. \end{aligned} \tag{4.1}$$

We can now define

$$\begin{aligned} R &= \{\alpha \in H^* : E_\alpha \neq 0\} \quad (\textit{set of roots of } (E, H)), \\ R^0 &= \{\alpha \in R : (\alpha \mid \alpha) = 0\} \quad (\textit{null roots}), \\ R^{\mathrm{an}} &= \{\alpha \in R : (\alpha \mid \alpha) \neq 0\} \quad (\textit{anisotropic roots}). \end{aligned} \tag{4.2}$$

We define the *core of* (E, H) as the subalgebra E_c of E generated by all anisotropic root spaces:

$$E_c = \langle \bigcup_{\alpha \in R^{\mathrm{an}}} E_\alpha \rangle_{\mathrm{subalg}}$$

(EA3) *For every* $\alpha \in R^{\mathrm{an}}$ *and* $x_\alpha \in E_\alpha$, *the operator* $\mathrm{ad} x_\alpha$ *is locally nilpotent on* E.

(EA4) R^{an} *is connected* in the sense that for any decomposition $R^{\mathrm{an}} = R_1 \cup R_2$ with $(R_1 \mid R_2) = 0$ we have $R_1 = \emptyset$ or $R_2 = \emptyset$.

(EA5) *The centralizer of the core* E_c *of* E *is contained in* E_c: $\{e \in E : [e, E_c] = 0\} \subset E_c$.

(EA6) *The subgroup* $\mathcal{L} = \mathbb{Z} R^0 \subset H^*$ *is isomorphic to* $\mathcal{L} \cong \mathbb{Z}^n$ *for some* $n \geq 0$.