A star product for the volume form Nambu-Poisson structure on a Kähler manifold.

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Deformations of Lie algebras

Deformations of algebras were first studied by Gerstenhaber[Gerstenhaber:64], the Lie algebra case in particular was done by Nijenhuis and Richardson[nijenhuis:67].

Definition

A deformation of a Lie algebra $\mathfrak{g}$ with Lie bracket $\{f, g\}$ is a family of Lie brackets depending on a parameter t, defined by

$$\{f, g\}_t := \{f, g\} + \sum_{n=1}^{\infty} t^n C_n(f, g),$$

where the C_n are bilinear, skewsymetric $\mathfrak{g}$-valued maps, $C_n : \wedge^2 \mathfrak{g} \to \mathfrak{g}$. Furthermore we require that the deformed bracket $\{f, g\}_t$ satifies the Jacobi identity for every t. this gives another set of conditions that must be satisfied by the C_n.

The algebraic structure of Hamiltonian mechanics

The setting for Hamiltonian mechanics is a symplectic manifold (M, ω). In physics terminology the manifold M is called the phase space.

Example

1. For a free particle moving in n-dimensions the phase space is $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$.
2. For a simple pendulum, it's phase space is T^*S^1.

The (Poisson) algebra of obseravables

The the set of smooth functions on M, $C^\infty(M)$ together with the Poisson bracket,

$$\{f, g\} = \omega(df, dg)$$

is known as the classical algebra of observables. $C^\infty(M)$ is also a commutative and associative algebra under the operation of pointwise multiplication of functions.

The algebraic structure of quantum mechanics, Hilbert space formulations

In (Hilbert space formulations of) quantum mechanics the phase space manifold is replaced by a Hilbert space $\mathcal{H}$

The quantum algebra of observables

The set of operators on the Hilbert space $\mathcal{H}$, $\mathrm{End}(\mathcal{H})$ with the Poisson bracket given by the commutator of operators,

$$[A, B] := AB - BA$$

is known as the quantum algebra of observables. $\mathrm{End}(\mathcal{H})$ is also a non-commutative algebra and associative alegbra under the composition of operators.

Algebraic structures of classical and quantum mechanics, a comparison

	Hamiltonian mechanics	Quantum mechanics
Phase space	(M, ω)	$\mathcal{H}$
Observables	$C^{\infty}(M)$	$End(\mathcal{H})$
Assoc alg	$\cdot$	$\circ$
Poisson alg	$\{\cdot, \cdot\}$	$[\cdot, \cdot]$

deformation quantization

In deformation quantization we begin with a Poisson algebra of the type $(C^{\infty}(M), \{\cdot, \cdot\})$ and we try to deform into a Possion algebra of the type $(\mathcal{H}, [\cdot, \cdot])$.

Remark (Berezin-Toeplitz operator quantization)

This is in contrast to operator quantization procedure, where one assigns non-commutative operators to functions. One example of this is the Berezin-Toeplitz operator quantization, where one assigns a Toeplitz operator T_f to each function f.

Preliminaries

Let (M, ω) be a compact connected n-dimensional Kähler manifold $(n \geq 1)$. Denote by $\{.,.\}$ the Poisson bracket on M coming from ω. Assume that the Kähler form ω is integral. Let L be a hermitian holomorphic line bundle on M such that the curvature of the hermitian connection is equal to $-2\pi i\omega$.

Star products

Definition (star product)

Let $A = C^\infty(M)[[t]]$ be the algebra of formal power series in the variable t over the algebra $C^\infty(M)$. A product $\star$ on A is called a **(formal) star product** if it is an associative $\mathbb{C}[[t]]$-linear product such that

1. $A/tA \cong C^\infty(M)$, i.e. $f \star g \mod t = fg$,
2. $\frac{1}{t}(f \star g - g \star f) \mod t = -i\{f, g\}$,

where $f, g \in C^\infty(M)$. We can also write

$$f \star g = \sum_{j=0}^{\infty} C_j(f, g) t^j, \tag{1}$$

with $C_j(f, g) \in C^\infty(M)$. The C_j should be $\mathbb{C}$-bilinear in f and g.

Star products

The condition 1. can be reformulated as

$$C_0(f, g) = fg, \tag{2}$$

The billinearity of the C_j guarentees that the star product will be bilinear. The condition 1. says that the star product is in fact a deformation of the associative algebra $(C^\infty(M), \bullet)$, where $f \bullet g$ is the usual pointwise multiplication of functions [Gerstenhaber:64].

Star products

Every star product defines a skew symmetric bracket of functions by the formula,

$$[f, g]_Q := \frac{1}{t}(f \star g - g \star f). \tag{3}$$

Condition 2. can be reformulated as

$$C_1(f, g) - C_1(g, f) = -i\{f, g\} \tag{4}$$

Condition 2. is equivalent to the correspondance principal (of Quantum Mechanics) for the quantum bracket defined by 3. That is, 2. says that $[f, g]_Q$ is a deformation of the Poisson algebra $(C^\infty(M), \{,\})$ [Gerstenhaber:64].

In the article [Sclich:00], Schlichenmaier gives a proof of the following theorem.

Theorem (Berezin-Toeplitz star product)

There exists a unique (formal) star product on $C^\infty(M)$

$$f \star g \equiv \sum_{j=0}^{\infty} \nu^j C_j(f,g), \tag{5}$$

in such a way that for f,g$\in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all m

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} (\frac{1}{m})^j T_{C_j(f,g)}^{(m)}|| = K_N(f,g)(\frac{1}{m})^N \tag{6}$$

Theorem [Bordemann, Meinrenken, Schlichenmaier]

Theorem ([BMS] Th. 4.1, 4.2, [S2] Th. 3.3)

For $f, g \in C^\infty(M)$, as $k \to \infty$,

(i)

$$||ik[T_f^{(k)}, T_g^{(k)}] - T_{\{f,g\}}^{(k)}|| = O(\frac{1}{k}),$$

(ii) *there is a constant $C = C(f) > 0$ such that*

$$|f|_\infty - \frac{C}{k} \leq ||T_f^{(k)}|| \leq |f|_\infty.$$

	Hamiltonian mechanics	Quantum mechanics
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$$C^\infty(M) \xrightarrow{T} End(\mathcal{H})$$

m-plectic structures

Definition (m-plectic structure)

An $(m+1)$-form Ω on a smooth manifold M is called an **m-plectic form** if it is closed ($d\Omega = 0$) and non-degenerate ($v \in T_x M, v \lrcorner \Omega_x = 0 \Rightarrow v = 0$). If Ω is a multiplectic form on M, (M,Ω) is called a **multisymplectic** or **m-plectic**, manifold.

Example

The canonical example of an m-plectic manifold is $\bigwedge^m T^*M$, where M is any smooth manifold, this generalizes the canonical symplectic (1-plectic) structure on T^*M.

Nambu-Poisson structures

Definition (Nambu-Poisson bracket)

Let M be a smooth manifold. A multilinear map

$$\{., ..., .\} : (C^\infty(M))^{\otimes j} \to C^\infty(M)$$

is called a **Nambu-Poisson bracket** or **(generalized) Nambu bracket of order** j if it satisfies the following properties:

- (skew-symmetry) $\{f_1, ..., f_j\} = \epsilon(\sigma)\{f_{\sigma(1)}, ..., f_{\sigma(j)}\}$ for any $f_1, ..., f_j \in C^\infty(M)$ and for any $\sigma \in S_j$,
- (Leibniz rule) $\{f_1, ..., f_{j-1}, g_1 g_2\} = \{f_1, ..., f_{j-1}, g_1\}g_2 + g_1\{f_1, ..., f_{j-1}, g_2\}$ for any $f_1, ..., f_{j-1}, g_1, g_2 \in C^\infty(M)$,
- (Fundamental Identity)

$$\{f_1, ..., f_{j-1}, , \{g_1, ..., g_j\}\} = \sum_{i=1}^{j} \{g_1, ..., \{f_1, ..., f_{j-1}, g_i\}, ..., g_j\},$$

for all $f_1, ..., f_{j-1}, g_1, ..., g_j \in C^\infty(M)$.

Volume form NP-structure

Example (Volume form Nambu-Poisson structure)

Let (M, ω) be a Kähler manifold. Every volume form is closed and non-degenerate so the volume form $\Omega = \frac{\omega^n}{n!}$ is a $(2n-1)$-plectic form. The bracket $\{., ..., .\} : \bigwedge^{2n} C^\infty(M) \to C^\infty(M)$ defined by

$$df_1 \wedge ... \wedge df_{2n} = \{f_1, ..., f_{2n}\}\Omega$$

is a Nambu-Poisson bracket [G, Cor. 1 p. 106] .

Deforming the Nambu-Poisson structure

Definition (Generalized star product)

Consider the (2n-1)-plectic manifold (M, Ω) obtained from the symplectic manifold (M, ω), where $\Omega = \frac{1}{n!}\omega^n$. Define a star product (the terminology is justified by the next lemma) of 2n functions in $C^\infty(M)$ by the formula

$$\star(f_1, f_2, \ldots, f_{2n-1}, f_{2n}) = \sum_{j=0}^{\infty} t^j D_j(f_1, f_2, \ldots, f_{2n-1}, f_{2n}) \tag{7}$$

where

$$D_j(f_1, f_2, \ldots, f_{2n-1}, f_{2n}) := C_j(f_1, f_2) \ldots C_j(f_{2n-1}, f_{2n}) \tag{8}$$

Proposition

Proposition

The 2n-ary product defined on the previous slide gives simultaneously a deformation of the pointwise product of functions and a deformation of the Nambu-Poisson structure.

1. $D_0(f_1, f_2, \dots, f_{2n-1}, f_{2n}) = f_1 f_2 \dots f_{2n-1} f_{2n}$
2. $\sum_{\sigma \in S_{2n}} \epsilon(\sigma) D_1(f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(2n-1)}, f_{\sigma(2n)}) = n!(-i)^n \{f_1, f_2, \dots, f_{2n-1}, f_{2n}\}$

Resolution in terms of Poisson brackets

Lemma

For $f_1, ..., f_{2n} \in C^\infty(M)$, *where* $2n = dim_{\mathbb{R}}(M)$

$$\{f_1, ..., f_{2n}\} = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \epsilon(\sigma) \prod_{j=1}^{n} \{f_{\sigma(2j-1)}, f_{\sigma(2j)}\} \qquad (9)$$

In particular, for $n = 2$

$$\{f_1, f_2, f_3, f_4\} = \{f_1, f_2\}\{f_3, f_4\} - \{f_1, f_3\}\{f_2, f_4\} + \{f_1, f_4\}\{f_2, f_3\}.$$

Definition

Definition

For $j \leq n$ define,

$$\{f_1, \ldots, f_{2j}\} = \frac{1}{2^j j!} \sum_{\sigma \in S_{2j}} \epsilon(\sigma) \prod_{i=1}^{j} \{f_{\sigma(2i-1)}, f_{\sigma(2i)}\} \tag{10}$$

Lemma

$$\begin{aligned}\{f_1, \ldots, f_{2j}\} = \{f_1, f_2, \}\{f_3, \ldots, f_{2n}\} \\ + \sum_{i=3}^{2n-1} (-1)^i \{f_1, f_j\}\{f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{2n}\} \\ + \{f_1, f_{2n}\}\{f_2, \ldots, f_{2n-1}\}\end{aligned} \tag{11}$$

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