Plactic and Pre-Plactic Algebra

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June 17, 2015

Lie Theory XI
and
its Applications in Physics ,
Varna 2015

Outline

- Plactic Monoid $\mathfrak{Plac}$

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- Parastatistics algebra and its Fock spaces PS

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- Deformed Parastatistics space PS_q and $\mathfrak{Plac}$

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- Parastatistics algebra and its Fock spaces PS
- Deformed Parastatistics space PS_q and $\mathfrak{Plac}$
- (Quantum) Pre-Plactic Algebra (D. Krob, J.-Y. Thibon)

Robinson-Schensted correspondence

Algorithm

$$\mathfrak{S}_r \leftrightarrow (STab_\lambda, STab_\lambda)$$

$$\sigma = \begin{pmatrix} 1 & \dots & r \\ \sigma(1) & \dots & \sigma(r) \end{pmatrix} \leftrightarrow (Q(\sigma), P(\sigma))$$

$STab_\lambda$: Standard Young Tableaux of shape $\lambda \vdash r$

$$V^{\otimes r} \leftrightarrow (STab_\lambda, Tab_\lambda(V))$$

Word $w := w_1 \dots w_r$ with letters from the basis of $V = \oplus_{i=1}^{\dim V} \mathbb{C} e_i$

$$w = \begin{pmatrix} 1 & \dots & r \\ w_1 & \dots & w_r \end{pmatrix} \leftrightarrow (Q(\sigma), P(w))$$

$Tab_\lambda(V)$: SemiStandard Young Tableaux of shape $\lambda \vdash r$ with entries from the basis of V

Plactic Monoid(Lascoux and Schutzenberger)

$T(V)$ is the free monoid with letters from the basis of V.

Definition (Plactic algebra)

The Plactic algebra $\mathfrak{Plac}(V)$ is the quotient of the free monoid

$$\mathfrak{Plac}(V) := T(V)/\equiv$$

by the Knuth congruence

$$\begin{aligned} xzy &\equiv zxy \qquad && x \leq y < z \\ yxz &\equiv yzx \qquad && x < y \leq z \end{aligned}$$

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Theorem (Knuth)

The classes of the Plactic monoid $\mathfrak{Plac}$ are the classes of the Robinson-Schensted P-equivalence.

Robinson-Schensted-Knuth correspondence

$$\mathrm{Hom}(V^{\otimes r}) \cong V^{*\otimes r} \otimes V^{\otimes r} \leftrightarrow (Tab_\lambda(V^*), Tab_\lambda(V))$$

Words of biletters from the basis of $\mathrm{Hom}(V) = \oplus_{i,j=1}^{\dim V} \mathbb{C} e_j^i$

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v_1 & \dots & v_r \\ w_1 & \dots & w_r \end{pmatrix} \leftrightarrow (Q(v), P(w))$$

Creation-Annihilation Parastatistics Algebra of H.S.Green (1953)

$$\begin{aligned} [[a_i^\dagger, a_j], a_k^\dagger] &= 2\delta_{jk} a_i^\dagger \qquad & [[a_i^\dagger, a_j], a_k] &= -2\delta_{ik} a_j \\ [[a_i^\dagger, a_j^\dagger], a_k^\dagger] &= 0 \qquad & [[a_i, a_j], a_k] &= 0 \end{aligned} \tag{1}$$

Physical basis for $\mathfrak{so}_{2n+1}$

$$\mathfrak{so}_{2n+1} = \mathfrak{n}^* \rtimes \mathfrak{gl}_n \ltimes \mathfrak{n}$$

Lie algebra $\mathfrak{g}$ of type B_n, written in the oscillator basis

$$\mathfrak{g} \cong \mathfrak{so}_{2n+1} \qquad \begin{cases} a_i^\dagger = E_{e_i} \\ a_i = E_{-e_i} \end{cases}$$

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$\{e_i\}_{i=1}^n$ orthonormal basis in the root space, $(e_i, e_j) = \delta_{ij}$

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Parastatistics Fock space $PS(V)$ is the UEA for $\mathfrak{n} = V \oplus \wedge^2 V$

$$U\mathfrak{n} = T(V)/([[V, V]_\otimes, V]_\otimes) \qquad = PS(V)$$

where $(\mathfrak{I})$ stands for a two-sided ideal

Theorem

Let $S^\lambda(V)$ be the Schur module associated with Young diagram λ. The algebra $PS(V)$ is a $GL(V)$-model, i.e., every irreducible polynomial $GL(V)$-representations appears once and exactly once

$$PS(V) = \bigoplus_\lambda S^\lambda(V)$$

The Young tableaux label the basis of $PS(V)$.

Parastatistics Algebra $PS(V)$ and Young Tableaux

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Proof: The Cauchy formula is an identity of characters

$$\prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_\lambda s_\lambda(x)$$

where $s_\lambda(x) = chS^\lambda(V)$ stands for the Schur polynomial of dim V variables.

Quantum Universal Enveloping Algebra $U_q(\mathfrak{so}_{2n+1})$ (Drinfeld)
Deformed ParaStatistics Fock space (Palev)

$$PS_q(V) := U_q\mathfrak{n} = T(V)/(\mathfrak{L}(V))$$

the ideal $(\mathfrak{L})$ is generated by $U_q(\mathfrak{gl}_n)$-module $\mathfrak{L}(V)$ of type $(2,1)$

$$\begin{array}{rcll}
\mathfrak{L}^{i_1,i_2}_{i_3} & = & [[a^+_{i_1}, a^+_{i_3}], a^+_{i_2}]_{q^2} + q[[a^+_{i_1}, a^+_{i_2}], a^+_{i_3}] & \text{with} \quad i_1 < i_2 < i_3 \\
\mathfrak{L}^{i_1,i_1}_{i_2} & = & [[a^+_{i_1}, a^+_{i_2}], a^+_{i_2}]_q & \text{with} \quad i_1 < i_2 \\
\mathfrak{L}^{i_1,i_3}_{i_2} & = & [a^+_{i_2}, [a^+_{i_1}, a^+_{i_3}]]_{q^2} + q[a^+_{i_1}, [a^+_{i_2}, a^+_{i_3}]] & \text{with} \quad i_1 < i_2 < i_3 \\
\mathfrak{L}^{i_1,i_2}_{i_2} & = & [a^+_{i_1}, [a^+_{i_1}, a^+_{i_2}]]_q & \text{with} \quad i_1 < i_2
\end{array}$$

The Lie module $\mathfrak{L}(V)$ contains the Serre relations of $U_q(\mathfrak{so}_{2n+1})$

The crystal basis $q = 0$ of the module $\mathfrak{L}$ yields the ideal generating the Knuth relations

Tropicalization: $PS(V)$ versus $\mathfrak{Plac}(V)$

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The Fock space $PS_q(V) = U_q\mathfrak{n}$ is a version of the *Plactic algebra*. The elements in $\mathfrak{L}$ are primitive with respect to the Hopf structure of $U_q(\mathfrak{so}_{2n+1})$. Thus $PS_q(V)$ is a braded Lie algebra.
(related to Poirier-Reutenauer Hopf algebra)

Quantum Group $GL_q(n)$

Let $\mathcal{F}[GL(n)]$ be the commutative algebra of the regular functions on the group $GL(n)$. It is a Hopf algebra with a coproduct

$$\begin{array}{rcc} \Delta : \mathcal{F}[GL(n)] & \to & \mathcal{F}[GL(n)] \otimes \mathcal{F}[GL(n)] \\ \Delta x^i_j & \mapsto & \sum_s x^i_s \otimes x^s_j \end{array}$$

$\mathcal{F}[GL(n)]$ is in duality with the algebra of the vector fields $U(\mathfrak{gl}_n)$.
The deformation $\mathcal{F}_q[GL(n)]$ of the coordinate ring $\mathcal{F}[GL(n)]$

$$\begin{array}{lll} x^j_k x^i_k = q x^i_k x^j_k & j > i & \\ x^i_k x^i_l = q x^i_l x^i_k & k > l & \\ x^j_k x^i_l = x^i_l x^j_k & j > i & k < l \\ x^j_k x^i_l = x^i_l x^j_k + (q - q^{-1}) x^i_k x^j_l & j > i & k > l \end{array}$$

Hopf algebra duality $\mathcal{F}_q[GL(n)]^* \cong U_q(\mathfrak{gl}_n)$
The quantum group $GL_q(n)$ is the spectrum of $\mathcal{F}_q[GL(n)]$.

The quantum group $GL_q(2)$

The deformed relations

$$\begin{array}{lll} ba = qab & ca = qac & bc = cb \\ db = qbd & dc = qcd & da - ad = (q - q^{-1})bc \end{array}$$

define a Hopf coideal for

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$Det_q = da - qbc = ad - q^{-1}cb$$

$$Det_q = \frac{-q^{-1}(da - q^2 ad)}{q - q^{-1}} = \frac{-q^{-1}[d, a]_{q^2}}{q - q^{-1}} = \frac{1}{(1 - q^2)}[d, a]_{q^2}$$

Non-commutative characters of $GL_q(V)$-comodules

M a $GL_q(V)$-comodule with basis $m_J \subset V^{\otimes r}$, $J = \{j_1, \dots, j_r\}$

$$\begin{array}{rcl} \delta : M & \to & GL_q(n) \otimes M \\ \delta(m_I) & \mapsto & \sum_J X_I^J \otimes m_J \end{array}$$

The element $\sum_I X_I^I$ is independent of the choice of m_I
The quantum determinant $Det_q \in T_q(n)$, $n = \dim V$

$$Det_q = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x^1_{\sigma(1)} \dots x^n_{\sigma(n)} = \frac{1}{(1-q^2)^{n-1}} [x^n_n, [\dots, [x^2_2, x^1_1]_{q^2}]_{q^2}$$

The character of the r-th exterior power Λ^r_q is the polynomial

$$\chi(\Lambda^r_q) = \Lambda_r(q, \Delta) := \sum_{1 \le i_1 < \dots < i_r \le n} \frac{1}{(1-q^2)^{r-1}} [x^{i_r}_{i_r}, [\dots, [x^{i_2}_{i_2}, x^{i_1}_{i_1}]_{q^2}]_{q^2}$$

with entries in $\Delta = \{x^1_1, \dots, x^n_n\}$, $\Lambda_r(q, \Delta)$ span $T_q(n)$-subalgebra

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$$\begin{array}{rrcll}
\mathfrak{L}_j^{i,i}: & x_j^j x_i^i x_i^i - (1+q^2) x_i^i x_j^j x_i^i + q^2 x_i^i x_i^i x_j^j & = & 0 & i<j \\
\mathfrak{L}_j^{i,j}: & x_j^j x_j^j x_i^i - (1+q^2) x_j^j x_i^i x_j^j + q^2 x_i^i x_j^j x_j^j & = & 0 & i<j \\
\mathfrak{L}_j^{i,k} - \mathfrak{L}_k^{i,j}: & [x_j^j,[x_i^i,x_k^k]] & = & 0 & i<j<k
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$$

Definition (D. Krob, J.-Y. Thibon)

The Quantum pre-Plactic $\mathfrak{P}_q(W)$ is cubic algebra with relations

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Conjecture: The algebra $T_q(n)$ has no other independent relations

$$T_q(n) = \mathfrak{P}_q(W) \qquad \dim W = n$$

Quantum Eulerian Idempotent

Eulerian idempotent $e(q)$ in the Hecke algebra $\mathcal{H}_3(q)$

$$
\begin{aligned}
e(q) \quad := \quad & \frac{1}{[3]}\left(T_{123} - \frac{1}{2}(T_{231} + T_{213} + T_{132} + T_{312}) + T_{321}\right) \\
+ \quad & \frac{q - q^{-1}}{2[3]}\,(T_{213} - T_{312} - T_{231} + T_{132})\,. \qquad (2)
\end{aligned}
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The elements $\mathfrak{L}_3^{12}$ and $\mathfrak{L}_2^{13}$ (such that $2e(q) = \mathfrak{L}_3^{12} + \mathfrak{L}_2^{13}$)

$$q(T_{213} - T_{231}) + T_{123} - T_{132} - T_{231} + T_{321} + q^{-1}(T_{312} - T_{132})\;,$$
$$q(T_{132} - T_{312}) + T_{123} - T_{213} - T_{312} + T_{321} + q^{-1}(T_{231} - T_{213})\;.$$

span the $\mathcal{H}_3(q)$-module $\mathfrak{L} := e(q)\mathcal{H}_3(q) = \mathbb{C}(q)\mathfrak{L}_3^{12} \oplus \mathbb{C}(q)\mathfrak{L}_2^{13}$

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span the $\mathcal{H}_3(q)$-module $\mathfrak{L} := e(q)\mathcal{H}_3(q) = \mathbb{C}(q)\mathfrak{L}_3^{12} \oplus \mathbb{C}(q)\mathfrak{L}_2^{13}$
The Schur functor sends $\mathfrak{L}$ to a $U_q(\mathfrak{gl}(V)$-module $\mathfrak{L}(V)$

$$\mathfrak{L}_q(V) = \mathfrak{L}_q \otimes_{\mathcal{H}_3(q)} V^{\otimes 3}$$

Malvenuto-Reutenauer algebra $\mathfrak{MR} = \bigcup_{n\geq 0} \mathbb{C}[\mathfrak{S}_n]$
$\alpha \in \mathfrak{S}_r$ and $\beta \in \mathfrak{S}_p$ (Standardization $st(\alpha)$)

$$\alpha * \beta = \sum_{st(u)=\alpha,\, st(v)=\beta} uv \quad \in \mathfrak{S}_{r+p} \tag{3}$$

$$\Delta(\alpha) = \sum_{i=0}^{r} \alpha_{\{1,\dots,i\}} \otimes st(\alpha_{\{i+1,\dots,r\}}) \tag{4}$$

$$\begin{array}{rccc} * : & \mathbb{C}[\mathfrak{S}_r] \times \mathbb{C}[\mathfrak{S}_p] & \longrightarrow & \mathbb{C}[\mathfrak{S}_{r+p}] \uparrow^{\mathfrak{S}_{r+p}}_{\mathfrak{S}_r \times \mathfrak{S}_p}, \\ \Delta : & \mathbb{C}[\mathfrak{S}_r] & \longrightarrow & \bigoplus_{i=0}^{r} \mathbb{C}[\mathfrak{S}_i] \otimes \mathbb{C}[\mathfrak{S}_{r-i}] \downarrow^{\mathfrak{S}_r}_{\mathfrak{S}_i \times \mathfrak{S}_{r-i}} . \end{array}$$

Malvenuto-Reutenauer algebra

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Multilinearization of the Hopf tensor algebra $(T(V), \otimes, \Delta)$.

$$(v_1, \ldots, v_p) \otimes (v_{p+1}, \ldots, v_{p+q}) = (v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q})$$

$$\Delta(v_1, \ldots, v_n) = \sum_{q=n-p} \sum_{\sigma \in Sh_{p,q}} (v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \otimes (v_{\sigma(p+1)}, \ldots, v_{\sigma(n)})$$

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Lemma (J.-L. Loday and T.P.)

Define $PS_q = \mathfrak{MR}_q/(\mathfrak{L}_q)$ with elements in STab.
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$$\pi(T_{[2,[1,3]]}) = 0 = \pi(\mathfrak{L}_3^{12} - \mathfrak{L}_2^{13})$$

pre-Plactic $\mathfrak{P}_q$

Let us define a mapping $\begin{array}{lcl} \pi : \mathfrak{MR}_q & \to & \mathfrak{MR}_q \\ \pi : T_\sigma & \mapsto & T_{\sigma^{-1}} T_\sigma \end{array}$

$$\pi(T_{[2,[1,3]]}) = 0 = \pi(\mathfrak{L}_3^{12} - \mathfrak{L}_2^{13})$$

Definition

The Quantum pre-Plactic algebra $\mathfrak{P}_q$ is the quotient

$$\mathfrak{P}_q = \mathfrak{MR}_q/(T_{[2,[1,3]]})$$

the pre-Plactic Algebra is $\mathfrak{P}_{q=0}$

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Definition

The Quantum pre-Plactic algebra $\mathfrak{P}_q$ is the quotient

$$\mathfrak{P}_q = \mathfrak{MR}_q / (T_{[2,[1,3]]})$$

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Theorem (??!)

The Quantum pre-Plactic algebra $\mathfrak{P}_q$ is the quotient

$$\mathfrak{P}_q \stackrel{?}{=} \mathfrak{MR}_q / \ker \pi$$

that is $\ker \pi = (T_{[2,[1,3]]})$

Dimensions of $\mathfrak{P}_q$

Quantum pre-Plactic algebra $\mathfrak{P}_q = \oplus_{n\geq 0}\mathfrak{P}_q(n)$
The dimensions d_n of the degrees of $\mathfrak{P}_q$

$$d_n = \dim \mathfrak{P}_q(n) = a_{n+1}$$

are related to the numbers of the series $a_0, a_1, a_2, a_3, \ldots$

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, \ldots$$

from the Sloane's encyclopedia
Surprisingly the number of primitive elements $Prim\mathfrak{P}_q(n)$

$$\dim Prim(\mathfrak{P}_q(n)) = a_{n-1}$$

Recurency relation for a_n (one starts with $a_0 = 1$ and $a_1 = 1$

$$\exp \sum_{n\geq 1} a_{n-1}\frac{x^n}{n!} = \sum_{n\geq 0} a_{n+1}\frac{x^n}{n!}$$

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