Four Types of Orthogonal Polynomials of Affine Weyl Groups

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Outline

Motivation

Aim

generalization of multidimensional trigonometric transforms, Chebyshev-like polynomials and related Fourier methods to affine Weyl groups

- mathematics
  - cubature formulas, orthogonal polynomials, numerical solutions of differential equations, Fourier methods
- physics
  - variable transform between two integrable systems, quantum walks, fluid simulations, waves scattering, conformal field theory
- scientific computing and applications
  - signal processing, data compression, image analysis and reconstruction, jpeg, MPEG-4, data hiding

Outline

Lie algebras

- the simple complex Lie algebra of rank n
- the set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$, $\operatorname{span}_{\mathbb{R}} \Delta = \mathbb{R}^n$, scalar product $\langle\, ,\, \rangle$
- with roots of the same length A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, E_8
- with two different lengths of the roots, B_n $(n \geq 3)$, C_n $(n \geq 2)$, F_4, G_2,

$$\Delta = \Delta_s \cup \Delta_l$$

- the highest root $\xi \equiv -\alpha_0 = m_1\alpha_1 + \cdots + m_n\alpha_n$
- $m_j \dots$ the **marks** of G
- the Cartan matrix C

$$C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \quad i, j \in \{1, \dots, n\}$$

- and its determinant $c = \det C$

Root and weight lattices

- the root lattice Q of G

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$$

- the $\mathbb{Z}$-dual lattice to Q, with $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$

$$P^\vee = \{\omega^\vee \in \mathbb{R}^n \,|\, \langle \omega^\vee, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta\} = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee$$

- the dual root lattice

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee\,, \quad \text{where} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

- the $\mathbb{Z}-$dual lattice to $Q^\vee$, with $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$

$$P = \{\omega \in \mathbb{R}^n \,|\, \langle \omega, \alpha^\vee \rangle \in \mathbb{Z}, \forall \alpha^\vee \in \Delta^\vee\} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$$

Weyl group and affine Weyl group

- the Weyl group W is generated by n reflections r_α, $\alpha \in \Delta$

$$r_{\alpha_i} a \equiv r_i a \equiv r_i^\vee a = a - \frac{2\langle a,\, \alpha_i\rangle}{\langle \alpha_i,\, \alpha_i\rangle}\alpha_i\,, \qquad a \in \mathbb{R}^n$$

- the affine reflections

$$r_0 a = r_\xi a + \frac{2\xi}{\langle \xi,\, \xi\rangle}\,, \qquad r_\xi a = a - \frac{2\langle a,\, \xi\rangle}{\langle \xi,\, \xi\rangle}\xi$$

$$r_0^\vee a = r_\eta a + \frac{2\eta}{\langle \eta,\, \eta\rangle}\,, \qquad r_\eta a = a - \frac{2\langle a,\, \eta\rangle}{\langle \eta,\, \eta\rangle}\eta$$

Affine Weyl group

The affine Weyl group

$$W^{\mathrm{aff}} = Q^\vee \rtimes W$$

- W^{aff} is generated by $n+1$ reflections

$$R = \{r_0, r_1, \dots, r_n\}$$

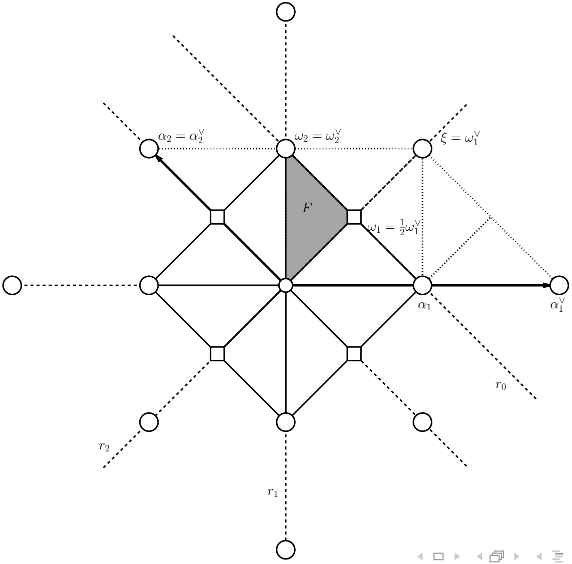
- the retraction homomorphism $\psi : W^{\mathrm{aff}} \to W$ and the mapping $\tau : W^{\mathrm{aff}} \to Q^\vee$

$$\psi(w^{\mathrm{aff}}) = w,$$
$$\tau(w^{\mathrm{aff}}) = q^\vee.$$

- fundamental domain F of W^{aff} contains exactly one point from each W^{aff} orbit

$$F = \left\{ y_1\omega_1^\vee + \dots + y_n\omega_n^\vee \,\middle|\, y_0 + y_1 m_1 + \dots + y_n m_n = 1 \right\}.$$

The fundamental domain F of C_2



Dual Affine Weyl group

The dual affine Weyl group

$$\widehat{W}^{\mathrm{aff}} = Q \rtimes W$$

- $\widehat{W}^{\mathrm{aff}}$ is generated by $n+1$ reflections
$$R^\vee = \{r_0^\vee, r_1^\vee, \dots, r_n^\vee\}$$
- the dual retraction homomorphism $\widehat{\psi} : \widehat{W}^{\mathrm{aff}} \to W$ and the mapping $\widehat{\tau} : W^{\mathrm{aff}} \to Q$
$$\widehat{\psi}(w^{\mathrm{aff}}) = w,$$
$$\widehat{\tau}(w^{\mathrm{aff}}) = q.$$
- the dual fundamental domain $F^\vee$ of $\widehat{W}^{\mathrm{aff}}$ contains exactly one point from each $\widehat{W}^{\mathrm{aff}}$ orbit
$$F^\vee = \left\{ z_1\omega_1 + \dots + z_n\omega_n \,\middle|\, z_0 + z_1 m_1^\vee + \dots + z_n m_n^\vee = 1 \right\}.$$

Outline

Sign homomorphisms

- an abstract presentation of W

$$r_i^2 = 1, \quad (r_i r_j)^{m_{ij}} = 1, \quad i, j = 1, \dots, n$$

- m_{ij} are elements of the Coxeter matrix.
- 'sign' homomorphisms $\sigma : W \to \{\pm 1\}$

$$\sigma(r_i)^2 = 1, \quad (\sigma(r_i)\sigma(r_j))^{m_{ij}} = 1, \quad i, j = 1, \dots, n$$

- the four sign homomorphisms $\mathfrak{1}$, σ^e, σ^s, σ^l:

$$\mathfrak{1}(r_\alpha) = 1$$
$$\sigma^e(r_\alpha) = -1$$
$$\sigma^s(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_l \\ -1, & \alpha \in \Delta_s \end{cases}$$
$$\sigma^l(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_s \\ -1, & \alpha \in \Delta_l \end{cases}$$

Fundamental domains F^σ

- subset R^σ of generators R of W^{aff}

$$R^\sigma = \left\{ r \in R \,\middle|\, \sigma \circ \psi(r) = -1 \right\}$$

- a subset of boundary of F:

$$H^\sigma = \{a \in F \,|\, (\exists r \in R^\sigma)(ra = a)\}\,.$$

- fundamental domain F^σ

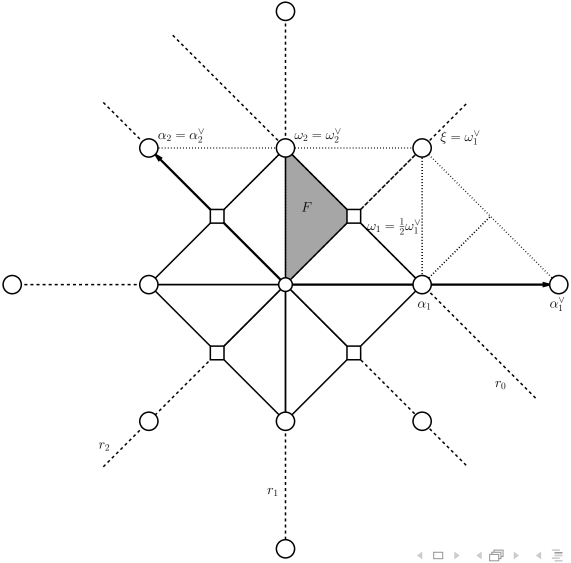
$$F^\sigma = F \setminus H^\sigma.$$

- the symbols $y_i^\sigma \in \mathbb{R}$, $i = 0, \dots, n$

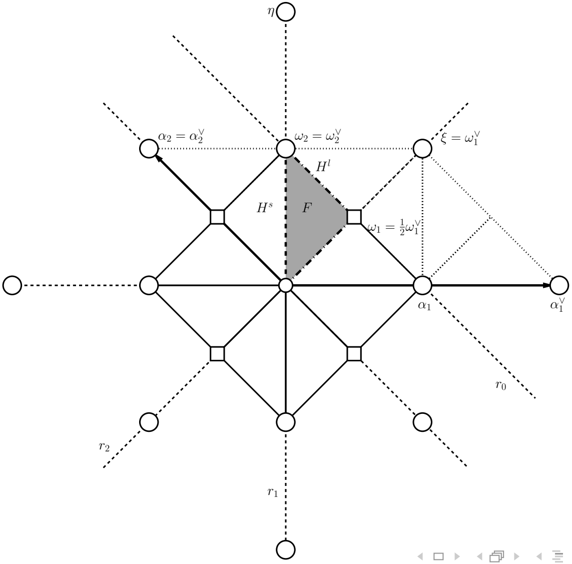
$$y_i^\sigma \in \begin{cases} \mathbb{R}^{>0}, & r_i \in R^\sigma \\ \mathbb{R}^{\geq 0}, & r_i \in R \setminus R^\sigma. \end{cases}$$

$$F^\sigma = \left\{ y_1^\sigma \omega_1^\vee + \dots + y_n^\sigma \omega_n^\vee \,\middle|\, y_0^\sigma + y_1^\sigma m_1 + \dots + y_n^\sigma m_n = 1 \right\}.$$

The fundamental domain F of C_2



The fundamental domains F^{σ^s} and F^{σ^l} of C_2



Dual fundamental domains $F^{\sigma\vee}$

- $R^{\sigma\vee}$ of generators $R^\vee$ of $\widehat{W}^{\mathrm{aff}}$

$$R^{\sigma\vee} = \left\{ r \in R^\vee \,\middle|\, \sigma \circ \widehat{\psi}\,(r) = -1 \right\}$$

- a subset of boundary of $F^\vee$:

$$H^{\sigma\vee} = \{ a \in F^\vee \mid (\exists r \in R^{\sigma\vee})(ra = a) \} \,.$$

- fundamental domain $F^{\sigma\vee}$

$$F^{\sigma\vee} = F^\vee \setminus H^{\sigma\vee}.$$

- the symbols $z_i^\sigma \in \mathbb{R}$, $i = 0, \dots, n$

$$z_i^\sigma \in \begin{cases} \mathbb{R}^{>0}, & r_i \in R^{\sigma\vee} \\ \mathbb{R}^{\geq 0}, & r_i \in R^\vee \setminus R^{\sigma\vee}. \end{cases}$$

$$F^{\sigma\vee} = \left\{ z_1^\sigma \omega_1 + \cdots + z_n^\sigma \omega_n \,\middle|\, z_0^\sigma + z_1^\sigma m_1^\vee + \cdots + z_n^\sigma m_n^\vee = 1 \right\}.$$

Outline

$C-$, $S-$, S^s- and S^l-functions

- for $\sigma \in \{\mathbb{1}, \sigma^e, \sigma^s, \sigma^l\}$, $b \in P$ are the complex functions $\varphi_b^\sigma : \mathbb{R}^n \to \mathbb{C}$

$$\varphi_b^\sigma(a) = \sum_{w \in W} \sigma(w)\, e^{2\pi i \langle wb,\, a\rangle}, \quad a \in \mathbb{R}^n$$

- $\sigma = \sigma^e \dots$ $S-$functions (known from the Weyl character formula)
- $\sigma = \mathbb{1} \dots$ $C-$functions
- $\sigma = \sigma^s \dots$ S^s-functions
- $\sigma = \sigma^l \dots$ S^l-functions

Proposition

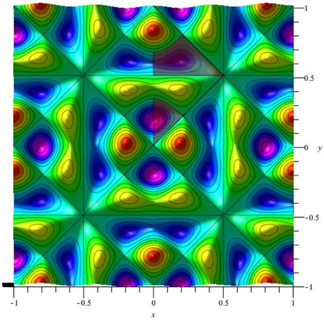
Let $b \in P$. Then for any $w^{\mathrm{aff}} \in W^{\mathrm{aff}}$ and $a \in \mathbb{R}^n$ it holds that

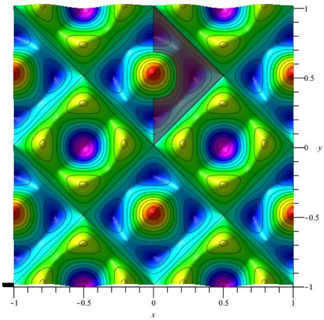
$$\varphi_b^\sigma(w^{\mathrm{aff}} a) = \sigma \circ \psi(w^{\mathrm{aff}})\, \varphi_b^\sigma(a).$$

Moreover the functions φ_b^σ are zero on the boundary H^σ, i.e.

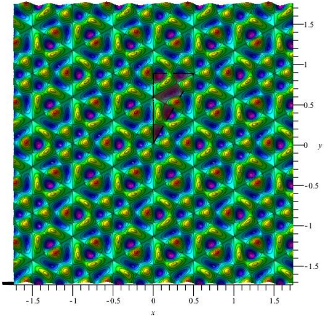
$$\varphi_b^\sigma(a') = 0, \quad a' \in H^\sigma.$$

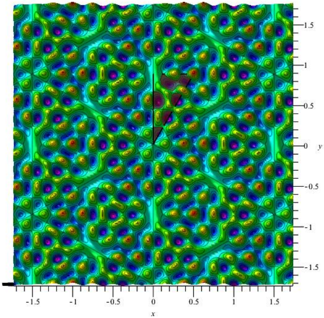
S^l–functions $\varphi^l_{(1,2)}(x,y)$ and $\varphi^l_{(2,1)}(x,y)$ of C_2



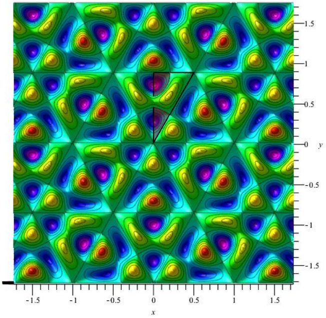


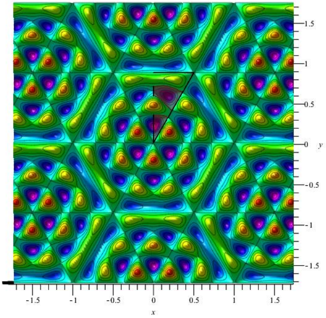
S^s–functions $\varphi^s_{(2,3)}(x,y)$ and $\varphi^s_{(3,2)}(x,y)$ of G_2





S^l−functions $\varphi^l_{(1,2)}(x,y)$ and $\varphi^l_{(2,1)}(x,y)$ of G_2





Characters χ_λ^σ

- the four vectors $\varrho^\sigma \in \{\varrho^{\mathbf{1}}, \varrho^{\sigma^e}, \varrho^{\sigma^s}, \varrho^{\sigma^l}\}$:

$$\varrho^{\mathbf{1}} = 0$$

$$\varrho^{\sigma^e} = \sum_{\alpha_i \in \Delta} \omega_i$$

$$\varrho^{\sigma^s} = \sum_{\alpha_i \in \Delta_s} \omega_i$$

$$\varrho^{\sigma^l} = \sum_{\alpha_i \in \Delta_l} \omega_i$$

The four characters χ_λ^σ

$$\chi_\lambda^\sigma(x) = \frac{\varphi^\sigma_{\lambda+\varrho^\sigma}(x)}{\varphi^\sigma_{\varrho^\sigma}(x)}, \quad \lambda \in P^+$$

$C-$, $S-$, S^s- and S^l-functions

- we can consider the functions φ^σ_b, $b \in P$ on the domain F^σ only

Proposition

Let $a \in \frac{1}{M}P^\vee$ with $M \in \mathbb{N}$. Then for any $w^{\mathrm{aff}} \in \widehat{W}^{\mathrm{aff}}$ and $b \in \mathbb{R}^n$ it holds that

$$\varphi^\sigma_{Mw^{\mathrm{aff}}(\frac{b}{M})}(a) = \sigma \circ \widehat{\psi}(w^{\mathrm{aff}})\, \varphi^\sigma_b(a). \tag{1}$$

Moreover the functions φ^σ_b are identically zero on the boundary $MH^{\sigma\vee}$, i.e.

$$\varphi^\sigma_b \equiv 0, \quad b \in MH^{\sigma\vee}. \tag{2}$$

- we can consider the functions $\varphi^\sigma_b(a)$, $a \in \frac{1}{M}P^\vee$ with the labels b from $MF^{\sigma\vee}$ only

Outline

Discretization of orbit functions

- the set of points

$$F^\sigma_M = \frac{1}{M} P^\vee \cap F^\sigma$$

- the set of labels of orbit functions

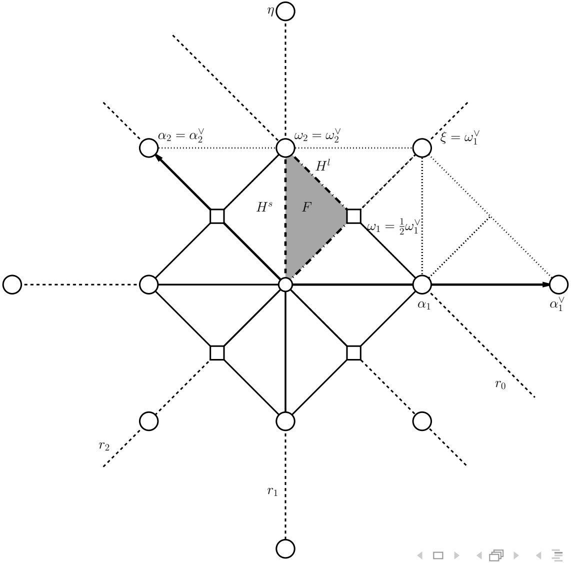
$$\Lambda^\sigma_M = P \cap M F^{\sigma\vee}.$$

Theorem

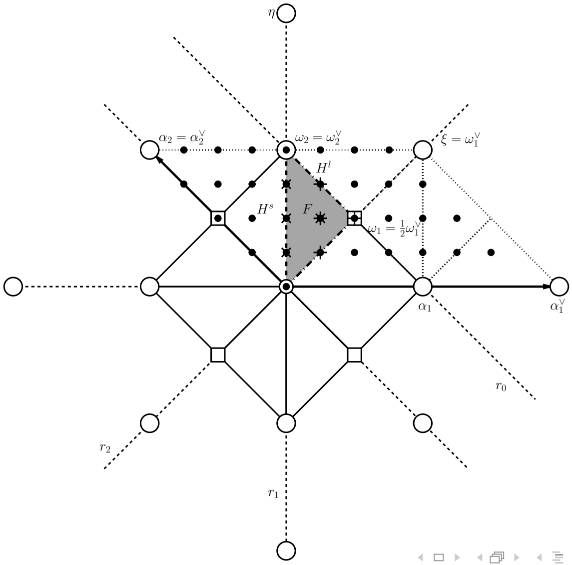
It holds that

$$|\Lambda^\sigma_M| = |F^\sigma_M|.$$

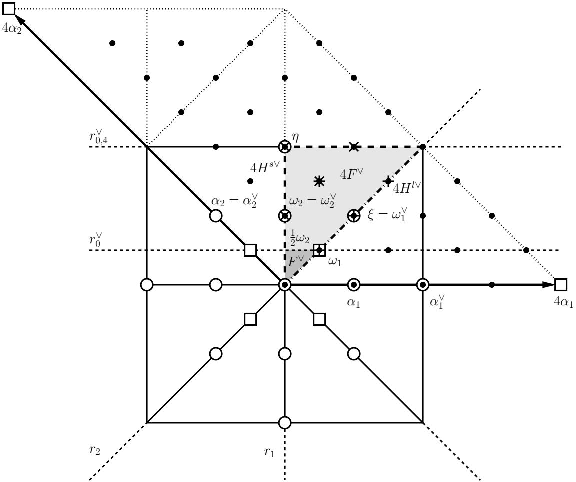
Grids $F_4^{\sigma^s}$ and $F_4^{\sigma^l}$ of C_2



Grids $F_4^{\sigma^s}$ and $F_4^{\sigma^l}$ of C_2



Grids $\Lambda_4^{\sigma^s}$ and $\Lambda_4^{\sigma^l}$ of C_2



Orthogonality of orbit functions

Theorem

For any $b, b' \in \varrho^\sigma + P^+$ it holds that

$$\int_{F^\sigma} \varphi_b^\sigma(a)\overline{\varphi_{b'}^\sigma(a)}\, da = |W|\,|F|\,|\mathrm{Stab}_W(b)|\,\delta_{b,b'},$$

Theorem

For any $b, b' \in \Lambda_M^\sigma$ it holds that

$$\sum_{a\in F_M^\sigma} |\mathrm{Stab}_{W^{\mathrm{aff}}}(a)|^{-1}\varphi_b^\sigma(a)\overline{\varphi_{b'}^\sigma(a)} = cM^n\left|\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}\left(\frac{b}{M}\right)\right|\delta_{b,b'},$$

Outline

Polynomials $\mathbb{T}_\lambda^\sigma$

- the mapping $X:\mathbb{R}^n \to \mathbb{C}^n$

$$X(x) = (\chi^{\mathbf{1}}_{\omega_1}(x),\dots,\chi^{\mathbf{1}}_{\omega_n}(x))$$

The four families polynomials $\mathbb{T}_\lambda^\sigma(y_1,\dots,y_n)$

$$\mathbb{T}_\lambda^\sigma(X(x)) = \chi_\lambda^\sigma(x), \quad \lambda\in P^+$$

The weight polynomial $K(y_1,\dots,y_n)$

$$K(X(x)) = |\varphi^{\mathbf{1}}_{\varrho^{\mathbf{1}}}(x)|^2$$

The four weight polynomials $J^\sigma(y_1,\dots,y_n)$

$$J^\sigma(X(x)) = |\chi^\sigma_{\varrho^\sigma}(x)|^2$$

Polynomials $\mathbb{T}_\lambda^\sigma$

- the mapping $X : \mathbb{R}^n \to \mathbb{C}^n$

$$X(x) = (\chi^{\mathbf{1}}_{\omega_1}(x), \dots, \chi^{\mathbf{1}}_{\omega_n}(x))$$

- the restriction of X to F_M^σ

$$X_M^\sigma \equiv X \restriction_{F_M^\sigma}$$

The set Ω^σ

$$\Omega^\sigma = X(F^\sigma)$$

The point set $\Omega_M^\sigma \subset \Omega^\sigma$

$$\Omega_M^\sigma \equiv X_M^\sigma(F_M^\sigma)$$

Orthogonality of polynomials $\mathbb{T}^\sigma_\lambda$

Theorem

For any $\lambda, \lambda' \in P^+$ it holds that

$$\int_{\Omega^\sigma} \frac{J^\sigma(y)}{\sqrt{K(y)}} \mathbb{T}^\sigma_\lambda(y) \overline{\mathbb{T}^\sigma_{\lambda'}(y)} \, dy = (2\pi)^n |\mathrm{Stab}_W(\lambda + \varrho^\sigma)| \, \delta_{\lambda,\lambda'}.$$

Theorem

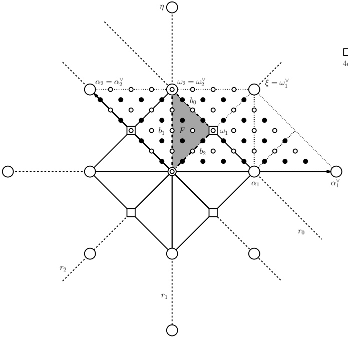
For any $\lambda, \lambda' \in \Lambda^\sigma_M - \varrho^\sigma$ it holds that

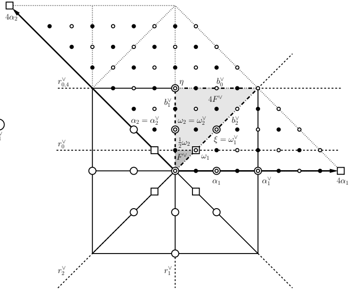
$$\sum_{y \in \Omega^\sigma_M} \frac{J^\sigma(y)}{|\mathrm{Stab}_{W^{\mathrm{aff}}}(X_{\frac{1}{M}}{}^{-1} y)|} \mathbb{T}^\sigma_\lambda(y) \overline{\mathbb{T}^\sigma_{\lambda'}(y)} = cM^n \left| \mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}} \left(\frac{\lambda + \varrho^\sigma}{M} \right) \right| \delta_{\lambda,\lambda'}.$$

Outline

Summary

- discrete and continuous orthogonality of φ_λ^σ and $\mathbb{T}_\lambda^\sigma$ explicitly
- polynomial interpolation methods, cubature formulas
- possible generalizations: the shifted transforms[1]
- recurrence formulas and generating functions for $\mathbb{T}_\lambda^\sigma$

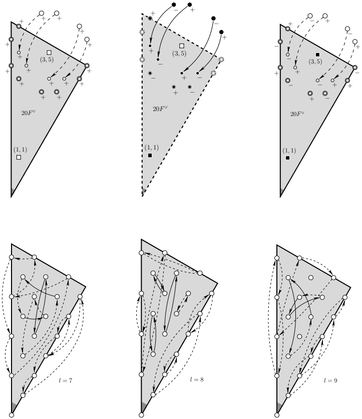




[1] T. Czyżycki, J. Hrivnák, *Generalized discrete orbit function transforms of affine Weyl groups*, J. Math. Phys. **55**, (2014) 113508

Summary

- modified multiplication and Galois symmetry in conformal field theory [2]



[2] J. Hrivnák, M. Walton, *Discretized Weyl-orbit functions: modified multiplication and Galois symmetry*, J. Phys. A: Math. Theor. **48** (2015) 175205

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