Cubature formulas of multivariate polynomials arising from symmetric orbit functions

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Outline

Simple Lie groups

- simple roots: $\alpha_1, \dots, \alpha_n$;
- simple dual roots: $\alpha_1^\vee, \dots, \alpha_n^\vee$, where $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$;
- marks m_i of the highest root $\xi = m_1\alpha_1 + \dots + m_n\alpha_n$;
- Weyl group W generated by reflections r_i, $i = 1, \dots, n$, $r_i(x) = x - \frac{2\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i$ for $x \in \mathbb{R}^n$;
- affine Weyl group W^{aff} generated by r_i and r_0, $r_0(x) = r_\xi(x) + \frac{2\xi}{\langle \xi, \xi \rangle}$, $r_\xi(x) = x - \frac{2\langle x, \xi \rangle}{\langle \xi, \xi \rangle}\xi$ for $x \in \mathbb{R}^n$;
- fundamental domain F of W^{aff} given as convex hull of the points $\left\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\right\}$ with $\langle \omega_i^\vee, \alpha_j \rangle = \delta_{ij}$.

Four types of lattices in $\mathbb{R}^n$

- root lattice: $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n\,;$
- $\mathbb{Z}$-dual lattice to Q:

$$P^\vee = \left\{\omega^\vee \in \mathbb{R}^n \mid \langle \omega^\vee, \alpha_i \rangle \in \mathbb{Z}\right\} = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee$$

($\omega_i^\vee$ are called dual weights, $\langle \omega_i^\vee, \alpha_j \rangle = \delta_{ij}$);
- dual root lattice:

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee\,, \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}\,;$$

- $\mathbb{Z}$-dual lattice to $Q^\vee$:

$$P = \left\{\omega \in \mathbb{R}^n \mid \langle \omega, \alpha_i^\vee \rangle \in \mathbb{Z}\right\} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$$

(ω_i are called weights, $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$).

C, S, S^s- and S^l-functions

For σ a "sign" homomorphism on W, we define

$$\varphi^\sigma_\lambda(x) = \sum_{w\in W} \sigma(w) e^{2\pi i \langle w(\lambda), x\rangle} \,.$$

$$\begin{array}{lll} C\text{-functions:} & \sigma = 1\,, & \varphi^\sigma = \Phi\,; \\ S\text{-functions:} & \sigma = \det\,, & \varphi^\sigma = \varphi\,; \\ S^s\text{-functions:} & \sigma = \sigma^s\,, & \varphi^\sigma = \varphi^s\,; \\ S^l\text{-functions:} & \sigma = \sigma^l\,, & \varphi^\sigma = \varphi^l\,. \end{array}$$

σ^s : all "short" signs equal to -1, all "long" signs equal to 1;
σ^l : all "long" signs equal to -1, all "short" signs equal to 1.

Numerical integration

- A cubature formula is an approximation of a weighted integral of any function f of several variables by a linear combination of function values at points called nodes,

$$\int_{\Omega} f(y)\omega(y)\,dy \approx \sum_{j=1}^{N} \omega_j f(y^j)\,,$$

  where $y = (y_1, \dots, y_n)$.
- One criteria to specify cubature formulas is based on their behaviour for specific sets of functions. **We require that cubature formulas are exact equalities for polynomial functions up to a certain degree.**

Gaussian cubature formulas

Optimal cubature formulas with the lowest bounds for the number of nodal points required are known as Gaussian. The following statements characterising Gaussian cubature formulas hold.

- The number of nodes of a Gaussian cubature formula of degree $2n-1$ is equal to the dimension of polynomials of degree at most $n-1$.
- A Gaussian cubature formula of degree $2n-1$ exists if and only if the number of common distinct zeros of the corresponding orthogonal polynomials of degree n is equal to the dimension of the polynomials of degree at most $n-1$.

Chebyshev polynomials

The Chebyshev polynomials of the first kind are the orthogonal polynomials given by

$$T_n(x) = \cos n\theta\,, \quad x = \cos\theta\,, \quad 0 \le \theta \le \pi$$

or by three-terms recurrence relation

$$T_0(x) = 1\,, \quad T_1(x) = x\,, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)\,.$$

They satisfy the orthogonality relation:

$$\int_{-1}^{1} T_m(x)\,T_k(x)\frac{dx}{\sqrt{1-x^2}} = \pi c_m \delta_{mk}$$

with $c_m = 1$ if $m = 0$ otherwise $c_m = \frac{1}{2}$.

Gauss-Chebyshev quadrature formula

The following formula holds for any polynomial of degree at most $2n-1$

$$\int_{-1}^{1} \frac{p(x)}{\sqrt{1-x^2}} dx = \sum_{i=1}^{n} \frac{\pi}{n} \, p(\xi_i^{(n)}) \,,$$

where $\xi_i^{(n)}$ are the points that satisfy

$$T_n(x) = \cos n\theta = 0 \,,$$

i.e.

$$\xi_i^{(n)} = \cos \frac{\pi(2i-1)}{2n} \,, \quad i = 1, \ldots, n.$$

Polynomials arising from C-functions

We consider C-functions defined as the Weyl group orbit sums,

$$C_\lambda(x) \equiv \sum_{\nu \in W\lambda} e^{2\pi i \langle \nu, x \rangle}, \lambda \in P^+, x \in \mathbb{R}^n.$$

Let $\nu = \nu_1\omega_1 + \cdots + \nu_n\omega_n$, any function C_λ can be expressed as a polynomial in $Z_i = C_{\omega_i}$,

$$C_\lambda = \sum_{\nu \preceq \lambda,\, \nu \in P^+} d_\nu Z_1^{\nu_1} Z_2^{\nu_2} \ldots Z_n^{\nu_n}, \quad d_\nu \in \mathbb{C}, \quad d_\lambda = 1.$$

The functions Z_j are real-valued except for the following cases.

$$A_{2k}(k \geq 1): \quad Z_j = \overline{Z_{2k-j+1}},\ j = 1, \ldots, k,$$
$$A_{2k+1}(k \geq 1): \quad Z_j = \overline{Z_{2k-j+2}},\ j = 1, \ldots, k,$$
$$D_{2k+1}(k \geq 2): \quad Z_{2k} = \overline{Z_{2k+1}}, \qquad E_6: \quad Z_2 = \overline{Z_4}, Z_1 = \overline{Z_5}.$$

Polynomial variables

- Analogously to Chebyshev polynomials, we connect the abstract polynomial variables $y_1, \dots, y_n$ with the real-valued functions.
- For the real-valued Z_j, we define $y_j = Z_j$, otherwise

$$
\begin{aligned}
A_{2k}: &\quad y_j = \Re(Z_j),\ y_{2k-j+1} = \Im(Z_j),\ j = 1, \dots, k,\\
A_{2k+1}: &\quad y_j = \Re(Z_j),\ y_{2k-j+2} = \Im(Z_j),\ j = 1, \dots, k,\\
D_{2k+1}: &\quad y_{2k} = \Re(Z_{2k}),\ y_{2k+1} = \Im(Z_{2k}),\\
E_6: &\quad y_1 = \Re(Z_1),\ y_2 = \Re(Z_2),\ y_4 = \Im(Z_2),\ y_5 = \Im(Z_1).
\end{aligned}
$$

- Any function C_λ, $\lambda \in P^+$, can be rewritten as a polynomial in y_j.

m-degree

- We say that a monomial $y^\lambda \equiv y_1^{\lambda_1} \dots y_n^{\lambda_n}$ has *m*-degree
$$\mathrm{deg}_m y^\lambda = m_1^\vee \lambda_1 + \dots + m_n^\vee \lambda_n.$$
The *m*-degree of any polynomial $p \in \mathbb{C}[y]$ is defined as the largest *m*-degree of a monomial occurring in $p(y)$.
- The *C*-function C_λ, $\lambda = \lambda_1\omega_1 + \dots + \lambda_n\omega_n \in P^+$, expressed as a polynomial in y has *m*-degree equal to
$$|\lambda|_m = m_1^\vee \lambda_1 + \dots + m_n^\vee \lambda_n\,.$$
- The subspace $\Pi_M \subset \mathbb{C}[y]$ is formed by the polynomials of *m*-degree at most M, i.e.
$$\Pi_M \equiv \{p \in \mathbb{C}[y] \mid \mathrm{deg}_m\, p \leq M\}\ .$$

Integration region and set of nodes

The variables $y_1, \dots, y_n$ induce a map $\Xi : \mathbb{R}^n \to \mathbb{R}^n$:

$$\Xi(x) = (y_1(x), \dots, y_n(x)), \quad x \in \mathbb{R}^n.$$

- The image $\Omega \subset \mathbb{R}^n$ of the fundamental domain F under the map Ξ forms the integration domain on which the cubature rule is formulated, i.e.

$$\Omega \equiv \Xi(F).$$

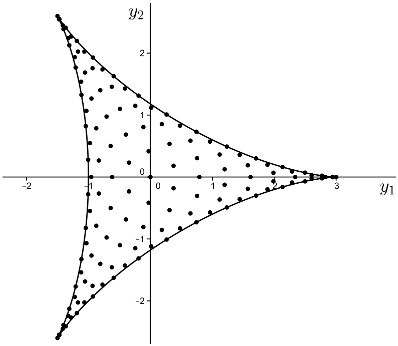
- The image $\Omega_M \subset \mathbb{R}^n$ of the set of points

$$F_M \equiv \frac{1}{M} P^\vee / Q^\vee \cap F$$

under the map Ξ forms the set of nodes for the cubature rule, i.e.

$$\Omega_M \equiv \Xi(F_M).$$

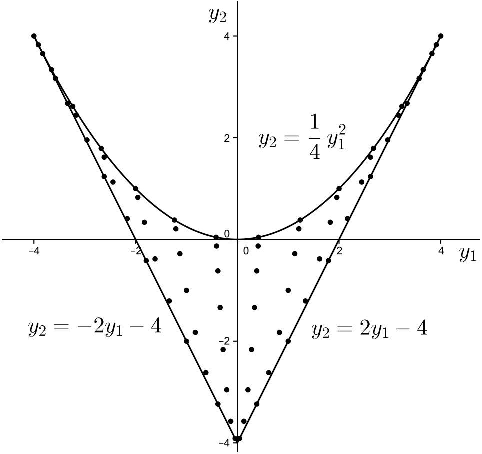
Lie algebra A_2: Ω and Ω_{15}



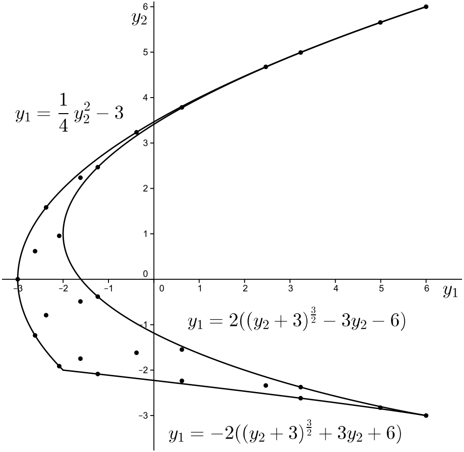
The boundary of Ω is defined by the equation

$$K(y_1, y_2) = -(y_1^2 + y_2^2 + 9)^2 + 8(y_1^3 - 3y_1 y_2^2) + 108 = 0.$$

Lie algebra C_2: Ω and Ω_{15}



Lie algebra G_2: Ω and Ω_{15}



Weight functions

We define

- the strictly positive function in the interior of Ω by

$$K(y_1,\dots,y_n)\equiv|S_\varrho|^2,$$

  where $\varrho=\sum_i\omega_i$, and
- the discrete function

$$\tilde{\varepsilon}(y)\equiv\varepsilon(\Xi_M^{-1}y),\quad y\in\Omega_M,$$

  where $\varepsilon(x)=|Wx|$ and $\Xi_M\equiv\Xi\restriction_{F_M}$.

The functions K and $\tilde{\varepsilon}$ are well-defined since $S_\varrho\overline{S_\varrho}$ is W-invariant and Ξ_M is injective.

Weight functions for Lie algebras of rank 2

A_2: $K(y_1, y_2) = -(y_1^2 + y_2^2 + 9)^2 + 8(y_1^3 - 3y_1y_2^2) + 108.$

C_2: $K(y_1, y_2) = (y_1^2 - 4y_2)((y_2 + 4)^2 - 4y_1^2).$

G_2: $K(y_1, y_2) = (y_2^2 - 4y_1 - 12)(y_1^2 - 4y_2^3 + 12y_1y_2 + 24y_1 + 36y_2 + 36).$

Cubature formulas

Theorem

For any $M \in \mathbb{N}$ and any $p \in \Pi_{2M-1}$ it holds that

$$\int_{\Omega} p(y) K^{-\frac{1}{2}}(y)\, dy = \frac{\kappa}{c|W|} \left(\frac{2\pi}{M}\right)^n \sum_{y \in \Omega_M} \tilde{\varepsilon}(y) p(y)\,,$$

where

$$\kappa = \begin{cases} 2^{-\lfloor \frac{n}{2} \rfloor} & \text{for } A_n \\ \frac{1}{2} & \text{for } D_{2k+1} \\ \frac{1}{4} & \text{for } E_6 \\ 1 & \text{otherwise.} \end{cases}$$

Example of numerical integration

The cubature formula is the exact equality for any polynomial function of m-degree up to $2M-1$. It can be used in numerical integration to approximate a weighted integral of any function by finite summing.
As our test function, we choose the function

$$f(y_1, y_2) = K^{\frac{1}{2}}(y_1, y_2)\,.$$

M	10	20	50	100	exact value
A_2	6.0751	6.2314	6.2749	6.2811	$2\pi \doteq 6.2832$
C_2	10.056	10.5133	10.6421	10.6605	$32/3 = 10.666\overline{6}$
G_2	7.4789	8.2561	8.4885	8.5221	$128/15 = 8.533\overline{3}$

Symmetric cosine functions

The symmetric C-functions arising from B_n and C_n coincide, up to a constant, with the symmetric cosine functions given by

$$\cos_k^+(x) = \sum_{\sigma \in S_n} \cos\left(\pi k_{\sigma(1)} x_1\right) \cos\left(\pi k_{\sigma(2)} x_2\right) \cdots \cos\left(\pi k_{\sigma(n)} x_n\right)$$

with $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ satisfying $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$. They can be expressed as polynomials in variables

$$y_1 = \cos^+_{(1,0,\dots,0)},\ y_2 = \cos^+_{(1,1,0,\dots,0)},\ \dots,\ y_n = \cos^+_{(1,1,\dots,1)}.$$

Gaussian cubature formula

Theorem

For any $N \in \mathbb{N}$ and any polynomial f of degree at most $2N-1$, the following cubature formulas are exact.

$$\int_{\mathfrak{F}(\tilde{S}_n^{\mathrm{aff}})} f(y)\omega^{I,+}(y)\,dy = \left(\frac{1}{N}\right)^n \sum_{y \in \mathfrak{F}_N^{\mathrm{II},+}} \mathcal{H}_y^{-1} f(y).$$

This formula is Gaussian since number of nodes in $\mathfrak{F}_N^{\mathrm{II},+}$ is equal to the dimension of polynomials of degree at most $N-1$. Moreover, the nodes $\mathfrak{F}_N^{\mathrm{II},+}$ are common zeros of the set of orthogonal polynomials $\cos_k^+$ with $k_1 = N$ of degree N.

Polynomial approximation

We define a weighted Hilbert space $\mathcal{L}^2_K(\Omega)$ as a space of cosets of measurable complex-valued functions f such that $\int_\Omega |f|^2 K^{-\frac{1}{2}} < \infty$ with an inner product defined by

$$(f,g)_K = \frac{1}{\kappa(2\pi)^n} \int_\Omega f(y)\overline{g(y)} K^{-\frac{1}{2}}(y)\, dy.$$

The set of C-functions C_λ, $\lambda \in P^+$, expressed as polynomials in y forms a Hilbert basis of $\mathcal{L}^2_K(\Omega)$, i.e. any $f \in \mathcal{L}^2_K(\Omega)$ can be expanded in terms of C_λ,

$$f = \sum_{\lambda \in P^+} a_\lambda C_\lambda\,, \quad a_\lambda = h_\lambda (f, C_\lambda)_K$$

Optimality

Theorem

For any $f \in \mathcal{L}^2_K(\Omega)$ the polynomial

$$u_M[f] = \sum_{|\lambda|_m \leq M} a_\lambda C_\lambda\,, \quad a_\lambda = h_\lambda\,(f, C_\lambda)_K.$$

is the best approximation of f, relative to the $\mathcal{L}^2_K(\Omega)$-norm, by any polynomial from Π_M.

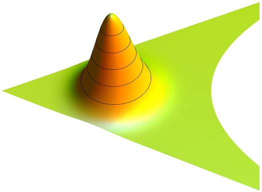
Rather than the optimal polynomial approximation one may consider for practical applications its weakened version:

$$v_M[f] = \sum_{\lambda \in P^+_M} a_\lambda p_\lambda\,, \quad a_\lambda = \frac{h_\lambda}{c|W|M^n} \sum_{y \in \Omega_M} \tilde{\varepsilon}(y) f(y) \overline{p_\lambda(y)}.$$

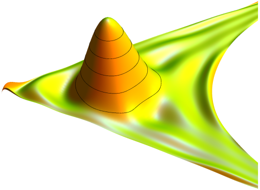
Example of cubature polynomial approximation

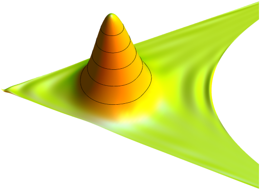
As a specific example of a continuous model function in the case C_2, we consider

$$f(y_1, y_2) = e^{-(y_1^2+(y_2+1.8)^2)/(2\times 0.35^2)}.$$



Lie algebra C_2: Approximations $v_{20}[f]$ and $v_{30}[f]$





M	10	20	30
$\int_\Omega \lvert f - v_M[f]\rvert^2 K^{-\frac{1}{2}} dy_1\, dy_2$	0.0636842	0.0035217	0.0000636

Concluding remarks

- studying Clenshaw-Curtis method for deriving cubature formulas;
- studying common zeros of C-functions;
- determining of Lebesgue constant;
- construction of cubature formulas of higher efficiency;
- possibility of an extension of the current cubature formulas to Macdonald polynomials.

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