Lie algebras with a set grading

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Lie Theory
and its applications in
physics

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Definition

Let L be an algebra and let $(G, +)$ be an abelian group. It is said that L is group-graded algebra if we have

$$L = \bigoplus_{g \in G} L_g$$

where any L_g is a linear subspace of L and where

$$[L_g, L_h] \subset L_{g+h}$$

for any $g, h \in G$.

**BY WORKING ON THE SUPPORT OF THE GRADING
IT IS POSSIBLE TO DESCRIBE
THE STRUCTURE OF THE WHOLE ALGEBRA
IN THE CASE OF GROUP-GRADED LIE ALGEBRAS**

OBJETIVE

$\Downarrow$

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EXTEND THE RESULTS ON GROUP-GRADED LIE ALGEBRAS TO LIE ALGEBRAS GRADED BY AN ARBITRARY SET

Definition

Let L be a Lie algebra and I an arbitrary (non-empty) set. It is said that L has a set grading, by means of I, if

$$L = \bigoplus_{i \in I} L_i$$

where any L_i is a linear subspace satisfying that for any $j \in I$ either $[L_i, L_j] = 0$ or $0 \neq [L_i, L_j] \subset L_k$ for some (unique) $k \in I$.

We call support of the grading to the set $\Sigma = \{i \in I : L_i \neq 0\}$.
We can write

$$L = \bigoplus_{i \in \Sigma} L_i$$

GENERAL PROCEDURE

Step 1. Define an equivalence relation $\sim$ on Σ (*connection relation*). Consider the set $\Sigma/\sim:=\{[i]:i\in\Sigma\}$.

Step 2. Associate an adequate ideal $I_{[i]}$ to any $[i]\in\Sigma/\sim$ satisfying $[I_{[i]},I_{[j]}]=0$ if $[i]\neq[j]$.

Step 3. State that $L=\bigoplus\limits_{[i]\in\Sigma/\sim} I_{[i]}$.

Step 4. Study when the simplicity of L can be characterized by the connectivity of Σ and show that the above decompositions is by means of the family of the minimal (simple) ideals of L.

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For each $i \in \Sigma$, a new variable $\bar{i} \notin \Sigma$ is introduced and we denote by

$$\overline{\Sigma} := \{\bar{i} : i \in \Sigma\}$$

the set consisting of all these new symbols.
We will also denote by $\mathcal{P}(A)$ the power set of a given set A.

Next, we consider the following operation,

$$\star : (\Sigma\dot{\cup}\overline{\Sigma}) \times (\Sigma\dot{\cup}\overline{\Sigma}) \to \mathcal{P}(\Sigma),$$

given by

- For $i, j \in \Sigma$,

$$i \star j = \begin{cases} \emptyset, & \text{if } [L_i, L_j] = 0; \\ \{k\}, & \text{if } 0 \neq [L_i, L_j] \subset L_k. \end{cases}$$

- For $i \in \Sigma$ and $\overline{j} \in \overline{\Sigma}$,

$$i \star \overline{j} = \overline{j} \star i = \{k \in \Sigma : 0 \neq [L_k, L_j] \subset L_i\}.$$

- For $\overline{i}, \overline{j} \in \overline{\Sigma}$,

$$\overline{i} \star \overline{j} = \emptyset.$$

In this moment we have to note that sometimes it is interesting to distinguish one element $\mathfrak{o}$ in the support of the grading, because the homogeneous space $L_{\mathfrak{o}}$ has, in a sense, a special behavior than the remaining elements in the set of homogeneous spaces $L_i, i \in \Sigma$.

Hence, let us now fix an element $\mathfrak{o}$ such that either $\mathfrak{o} \in \Sigma$ satisfying the property $\mathfrak{o} \star i \neq \{\mathfrak{o}\}$ for any $i \in \Sigma \setminus \{\mathfrak{o}\}$, or $\mathfrak{o} = \emptyset$.
Note that the possibility $\mathfrak{o} = \emptyset$ holds for the case in which it is not wished to distinguish any element in Σ.

Finally, we need to introduce the following map:

$$\phi : \mathcal{P}((\Sigma\dot{\cup}\overline{\Sigma}) \setminus \{\mathfrak{o}, \overline{\mathfrak{o}}\}) \times (\Sigma\dot{\cup}\overline{\Sigma}) \to \mathcal{P}((\Sigma\dot{\cup}\overline{\Sigma}) \setminus \{\mathfrak{o}, \overline{\mathfrak{o}}\}),$$

as

- $\phi(\emptyset, \Sigma\dot{\cup}\overline{\Sigma}) = \emptyset$,
- For any $\emptyset \neq \mathfrak{A} \in \mathcal{P}((\Sigma\dot{\cup}\overline{\Sigma}) \setminus \{\mathfrak{o}, \overline{\mathfrak{o}}\})$ and $a \in \Sigma\dot{\cup}\overline{\Sigma}$,

$$\phi(\mathfrak{A}, a) = ((\bigcup_{x \in \mathfrak{A}} (x \star a)) \setminus \{\mathfrak{o}\}) \cup \overline{((\bigcup_{x \in \mathfrak{A}} (x \star a)) \setminus \{\mathfrak{o}\})}.$$

Definition

Let i and j be two elements in $\Sigma \setminus \{0\}$. We say that i is *connected* to j if either $i = j$ or there exists a family

$$\{a_1, a_2, ..., a_{n-1}, a_n\} \subset \Sigma \dot{\cup} \overline{\Sigma}$$

satisfying the following conditions:

1. $a_1 \in \{i, \bar{i}\}$.

2. $\phi(\{a_1\}, a_2) \neq \emptyset$,
$\phi(\phi(\{a_1\}, a_2), a_3) \neq \emptyset$,
$\phi(\phi(\phi(\{a_1\}, a_2), a_3), a_4) \neq \emptyset$,
$\cdots\cdots\cdots$
$\phi(\phi(\cdots(\phi(\{a_1\}, a_2), \cdots), a_{n-2}), a_{n-1}) \neq \emptyset$.

3. $j \in \phi(\phi(\cdots(\phi(\{a_1\}, a_2), \cdots), a_{n-1}), a_n)$.

Proposition

The connection relation is an equivalence relation on $\Sigma \setminus \{\mathfrak{0}\}$.

Proposition

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From here, we can consider the set $(\Sigma \setminus \{\mathfrak{o}\})/\sim := \{[i] : i \in \Sigma \setminus \{\mathfrak{o}\}\}$.

Step 2. Associate an adequate ideal $I_{[i]}$ to any $[i] \in (\Sigma \setminus \{\mathfrak{o}\})/\sim$ satisfying $[I_{[i]}, I_{[j]}] = 0$ if $[i] \neq [j]$.

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- $L_{\mathfrak{o},[i]} \subset L_{\mathfrak{o}}$ as follows

$$L_{\mathfrak{o},[i]} := \Big(\sum_{j,k\in[i]} [L_j, L_k]\Big) \cap L_{\mathfrak{o}},$$

where $L_{\mathfrak{o}} := \{0\}$ whence $\mathfrak{o} = \emptyset$.
Next, we define

$$V_{[i]} := \bigoplus_{j\in[i]} L_j$$

Finally, we denote by $I_{[i]}$ the direct sum of the two subspaces above, that is,

$$I_{[i]} := L_{\mathfrak{o},[i]} \oplus V_{[i]}.$$

Theorem.

The following assertions hold.

1. *Proposition.*
   For any $i \in \Sigma \setminus \{\mathfrak{o}\}$, *the graded linear subspace* $I_{[i]}$ *is an ideal of* L.
2. *Corollary.*
   If L *is simple, then there exists a connection between any two elements of* $\Sigma \setminus \{\mathfrak{o}\}$.

Step 3. State that $L = \bigoplus_{[i]\in(\Sigma\setminus\{\mathfrak{o}\})/\sim} I_{[i]}$.

- **Theorem.**

  A set-graded Lie algebra L decomposes as

$$L = \mathfrak{u} \oplus \Big(\sum_{[i] \in (\Sigma \setminus \{\mathfrak{o}\})/\sim} I_{[i]} \Big),$$

where $\mathfrak{u}$ is a linear complement of $\sum_{[i] \in (\Sigma \setminus \{\mathfrak{o}\})/\sim} L_{\mathfrak{o},[i]}$ in $L_{\mathfrak{o}}$ and any $I_{[i]}$ is one of the ideals given above. Furthermore

$$[I_{[i]}, I_{[j]}] = 0$$

whenever $[i] \neq [j]$.

- **Corollary.**

  If $\mathfrak{o} = \emptyset$, then

$$L = \bigoplus_{[i]\in(\Sigma\setminus\{\mathfrak{o}\})/\sim} I_{[i]},$$

  where any $I_{[i]}$ is one of the ideals given above.

- **Corollary.**

  Suppose L is centerless and $L_{\mathfrak{o}}$ is tight, then the set-graded Lie algebra L decomposes as the direct sum of the ideals given above,

$$L = \bigoplus_{[i]\in(\Sigma\setminus\{\mathfrak{o}\})/\sim} I_{[i]}.$$

Step 4. Study when the simplicity of L can be characterized by the connectivity of $\Sigma \setminus \{\mathfrak{o}\}$ and show that the above decompositions is by means of the family of the minimal ideals of L.

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Definitions

- We say that L is of *maximal length* if $\dim L_i = 1$ for any $i \in \Sigma \setminus \{\mathfrak{o}\}$.
- We say that L is Σ-multiplicative if given $i, j \in \Sigma \setminus \{\mathfrak{o}\}$ such that and $i \in j \star a$ for some $a \in \Sigma \dot{\cup} \overline{\Sigma}$ then $L_i \subset [L_j, L_a + L_{\overline{a}}]$.

Theorem.

- *Let L be a centerless Σ-multiplicative Lie algebra with a set grading of maximal length and with L_0 tight. Then L is simple if and only if it has all of the elements in $\Sigma \setminus \{0\}$ connected.*

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- *Let L be a centerless Σ-multiplicative Lie algebra with a set grading of maximal length and with $L_{\mathfrak{o}}$ tight. Then L is simple if and only if it has all of the elements in $\Sigma \setminus \{\mathfrak{o}\}$ connected.*

Theorem.

- *Let L be a centerless Σ-multiplicative Lie algebra with a set grading of maximal length and with $L_{\mathfrak{o}}$ tight. Then L is the direct sum of the family of its minimal ideals,*

$$L = \bigoplus_{[i] \in (\Sigma \setminus \{\mathfrak{o}\})/\sim} I_{[i]},$$

each one being a simple Lie algebra with a set grading having all of the elements different to $\mathfrak{o}$ in its support connected.

THANK YOU FOR YOUR ATTENTION