# Sphere Partition Functions and the Zamolodchikov Metric

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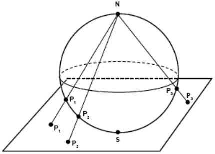
It is important to understand which observables exist in Quantum Field Theory and what is their interpretation.

Recently, there has been interest in placing conformal field theories on $\mathbb{S}^d$ via the stereographic map.




Since this is an angle-preserving transformation, there is a canonical way to implement this compactification of conformal field theories

$$ds^2 = dx^i dx^i \longrightarrow \frac{1}{(x^2 + r^2)^2} dx^i dx^i \ .$$

Now the theory is free of infrared divergences because space is compact. The UV divergences are the same as in flat space. One can therefore try to compute

$$Z_{S^d} \equiv \int [DX] e^{-S[X;g_{ij}]} , \qquad g_{ij} = \delta_{ij} \frac{1}{(x^2 + r^2)^2}$$

If the theory has various coupling constants $\lambda_i$ then we can compute

$$Z_{S^d}(\lambda_i)$$

Is this well defined? The partition function has various power divergences with the UV cutoff $\Lambda_{UV}$ of the sort

$$\log Z_{S^d} = \Lambda_{UV}^d r^d + \Lambda_{UV}^{d-2} r^{d-2} + ...$$

which correspond to the counter-terms

$$\Lambda_{UV}^d \int d^d x \sqrt{g} + \Lambda_{UV}^{d-2} \int d^d x \sqrt{g} R + \cdots$$

But there may be also terms that cannot be removed by changing the scheme.

In odd $d$, we can have a finite piece

$$\log Z_{S^d} = -F(\lambda_i)$$

and in even $d$ we can have both a log and a finite piece

$$\log Z_{S^d} = a(\lambda_i)\log(r\Lambda_{UV}) - F(\lambda_i)\ .$$

In odd $d$, the finite piece

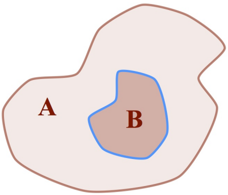
$$\log Z_{S^d} = -F(\lambda_i)$$

is physical! Interesting observable in QFT. Can prove that for exactly marginal couplings $F(\lambda_i) = F$

- Measures the total number of degrees of freedom and decreases along renormalization group flows

$$F_{UV} > F_{IR} \ .$$

- Can be mapped to the Vacuum Entanglement Entropy



In even $d$,

$$\log Z_{S^d} = a(\lambda_i)\log(r\Lambda_{UV}) - F(\lambda_i)$$

and we can show that $a(\lambda_i) = a$ which counts degrees of freedom and decreases along renormalization group flows

$$a_{UV} > a_{IR} \ .$$

The term $F(\lambda_i)$ is unphysical. It can be removed by the counter-term

$$\int d^d x F(\lambda_i) E_d$$

with $E_d$ the Gauss-Bonnet term (which exists only in even dimensions).

Interestingly, if we add supersymmetry, in some situations the space of allowed counter-terms is sufficiently reduced to allow for a *physical* finite part in d=2,4!

Consider a $(2,2)$ supersymmetric theory in $\mathbb{R}^2$. We have four supercharges $Q_+, Q_-, \tilde{Q}_+, \tilde{Q}_-$. Suppose the theory is superconformal. Then we have additional four supercharges $S_+, S_-, \tilde{S}_+, \tilde{S}_-$. In addition we have $U(1)_V \times U(1)_A$ $R$-symmetry.

$(2,2)$ superconformal field theories often have exactly marginal operators, that are either chiral primaries ($\bar{D}_\pm \Phi = 0$) or twisted chiral primaries ($\bar{D}_+ Y = D_- Y = 0$). So we can deform the action by

$$\delta S = \int d^2\theta \lambda^i \Phi_i + \int d^2\tilde{\theta}\tilde{\lambda}^A Y_A + c.c.$$

If our SCFT is a sigma model with Calabi-Yau target space, then the chiral and twisted chiral exactly marginal operators describe the complex structure and Kähler deformations of the Calabi-Yau.

So such $(2,2)$ superconformal field theories are part of some space of conformal theories – the conformal manifold

$$\mathcal{M}_{chiral} \times \mathcal{M}_{t.chiral}$$

such that both $\mathcal{M}_{chiral}$, and $\mathcal{M}_{t.chiral}$ are Kähler spaces with Kähler potential

$$K = K_{chiral} + K_{t.chiral}$$

This space is interesting for various reasons

- $\mathcal{M}_{chiral} \times \mathcal{M}_{t.chiral}$ is the space of massless fields in space-time if we interpret the $(2,2)$ SCFT as a string worldsheet. $K$ is the Kähler potential of these massless space-time fields.
- The spaces $\mathcal{M}_{chiral}$, $\mathcal{M}_{t.chiral}$ are interchanged by mirror symmetry.
- Duality symmetries should preserve $\mathcal{M}_{chiral} \times \mathcal{M}_{t.chiral}$.
- The geometry on the space of CY deformations is interesting to mathematicians.

These theories can be placed on $\mathbb{S}^2$ while preserving all the supercharges:

$$\nabla_m \epsilon = \gamma_m \eta \ , \qquad \nabla_m \tilde{\epsilon} = \gamma_m \tilde{\eta} \ ,$$

which has eight solutions, corresponding to the 4 supercharges and 4 conformal supercharges.

$$SU(1,1|1) \times SU(1,1|1) \ .$$

The bosonic generators of $SU(2)$ lead to isometries of $\mathbb{S}^2$ and there are also 3 conformal isometries.

Since the partition functions are UV divergent, we need to discuss massive subalgebras, namely, subalgebras that only include the isometries. There are two maximal massive subalgebras. Our UV regulator would be invariant under one of these massive subalgebras.

The two Massive Subalgebras correspond to

$$SU(2|1)_A \subset SU(1,1|1) \times SU(1,1|1) \ ,$$

$$SU(2|1)_B \subset SU(1,1|1) \times SU(1,1|1) \ .$$

For example, in $SU(2|1)_A$ we retain only $SU(2) \times U(1)_V$ and four supercharges out of the original eight.

Corresponding to these two massive subalgebras, we can define two partition functions, $Z_{\mathbb{S}^2;A}$ and $Z_{\mathbb{S}^2;B}$. The claim is that

$$Z_{\mathbb{S}^2;A} = r^{\frac{c}{3}} e^{-K_{t.chiral}}$$

$$Z_{\mathbb{S}^2;B} = r^{\frac{c}{3}} e^{-K_{chiral}}$$

Notice that this is unlike the non-SUSY case, in which the finite part of $\mathbb{S}^2$ partition functions is non-universal.

Using supersymmetric localization, one can actually explicitly compute $Z_{\mathbb{S}^2;A}, Z_{\mathbb{S}^2;B}$ in essentially every SCFT that can be obtained as the infrared of some asymptotically free theory. Therefore one can give explicit forms for $K_{t.chiral}$ and $K_{chiral}$ in a large class of theories.

For example, in a sigma model with only twisted chiral fields,

$$Z_{\mathbb{S}^2;B} = r^{c/3} \int dY_0 e^{-i\tilde{W}(Y_0)+c.c.}$$

[Benini-Cremonesi; Doroud-Gomis-Le Floch-Lee...]

We now explain why the surprising claims

$$Z_{\mathbb{S}^2;A} = e^{-K_{t.chiral}}$$
$$Z_{\mathbb{S}^2;B} = e^{-K_{chiral}}$$

are correct.

The partition functions become physical in these theories because a general Einstein-Hilbert counter-term cannot be supersymmetrized. For instance, in $SU(2|1)_A$ one finds

$$\int d^2\tilde{\theta}\mathcal{F}(\tilde{\lambda}^A)\mathcal{R} + c.c. = \mathcal{F}(\tilde{\lambda}^A)R + c.c. + \cdots$$

with $\mathcal{F}(\tilde{\lambda}^A)$ a holomorphic function of the twisted chirals.

The partition functions therefore have a smaller scheme dependence than in the nonSUSY case; only a holomorphic function of the twisted chirals is allowed in $SU(2|1)_A$. This also explains why $Z_{\mathbb{S}^2;A} = e^{-K_{t.chiral}}$ is consistent with Kähler transformations

$$K_{t.chiral}\left(\tilde{\lambda}^A, \bar{\tilde{\lambda}}^{\bar{A}}\right) \to K_{t.chiral}\left(\tilde{\lambda}^A, \bar{\tilde{\lambda}}^{\bar{A}}\right) + \mathcal{F}\left(\tilde{\lambda}^A\right) + \bar{\mathcal{F}}\left(\bar{\tilde{\lambda}}^{\bar{A}}\right) \ .$$

Consider some general marginal operators in $d = 2$ CFTs: $\delta S = \int d^2x \lambda^i O_i(x)$. The two-point function

$$\langle O_i(x) O_j(0) \rangle = \frac{G_{ij}(\lambda)}{x^4} .$$

$G_{ij}(\lambda^i)$ is the (Zamolodchikov) metric on the conformal manifold $\{\lambda^i\}$. In momentum space

$$\langle O_i(x) O_j(0) \rangle = G_{ij}(\lambda) p^2 \log\left(\frac{\mu^2}{p^2}\right) .$$

This logarithm in a CFT signifies a trace anomaly.

$$\langle O_i(x) O_j(0) \rangle = G_{ij}(\lambda) p^2 \log\left(\frac{\mu^2}{p^2}\right) \ .$$

The complete trace anomaly is thus given by

$$T^\mu_\mu = \frac{-c}{24\pi} R + G_{ij}(\lambda) \partial^\mu \lambda^i \partial_\mu \lambda^j \ .$$

These two trace anomalies are fundamentally different: the usual central charge trace anomaly never manifests itself with logarithms in correlation functions, while the new trace anomaly does come from a logarithm. The former type is called type A and the latter type B.

Such trace anomalies also appeared in the formalism of [Osborn].

Let us be more precise. In general, we could have improved the energy momentum tensor by adding

$$T_{\mu\nu} \to T_{\mu\nu} + \left(\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2\right) F(\lambda^i) \ ,$$

with an arbitrary function $F(\lambda^i)$. So the formula for the trace would be

$$T^\mu_\mu = \frac{-c}{24\pi} R + G_{ij}(\lambda)\partial^\mu \lambda^i \partial_\mu \lambda^j + \Box F(\lambda^i) \ .$$

While the first two pieces are fixed, the third piece is arbitrary.

In supersymmetric theories there are additional constraints.

- The metric $G_{ij}$ is Kähler and factorizes between chirals and twisted chirals

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K_{chiral} \ , \qquad G_{A\bar{B}} = \partial_A \partial_{\bar{B}} K_{t.chiral}$$

- $T_\mu^\mu$ appears as the SUSY variation of the $R$-current, so this allows us to fix the improvement ambiguity almost completely. (For the details of how the multiplet looks like see [Dumitrescu-Seiberg])

Implementing the constraints of supersymmetry we find that the trace of the energy-momentum tensor in $SU(2|1)_A$ is

$$T^\mu_\mu = \frac{-c}{24\pi} R + G_{ij}(\lambda)\partial^\mu \lambda^i \partial_\mu \lambda^j + \Box K(\tilde{\lambda}^A, \bar{\tilde{\lambda}}^{\bar{A}}) \ .$$

There is a leftover ambiguity that corresponds to Kähler transformations that we have not written down explicitly.

We see that $SU(2|1)_A$ fixes the improvement in terms of the Kähler potential for twisted chiral fields.

Under Weyl transformations, the partition function transforms by

$$\delta_\sigma \log(Z) = \int d^2x \sqrt{g} \sigma \langle T^\mu_\mu \rangle$$

and we thus see that we have the term

$$\delta_\sigma \log(Z_{\mathbb{S}^2;A}) = \int d^2x \sqrt{g} \Box \sigma K(\tilde{\lambda}^A, \bar{\tilde{\lambda}}^{\bar{A}})$$

This can be integrated for large conformal transformations to take the form $\log Z_{\mathbb{S}^2;A} \supset \int d^2x \sqrt{g} R K(\tilde{\lambda}^A, \bar{\tilde{\lambda}}^{\bar{A}})$.
By the Gauss-Bonnet theorem $\int_{\mathbb{S}^2} R = 1$ and thus we derive

$$Z_{\mathbb{S}^2;A} = e^{-K(\tilde{\lambda}^A, \bar{\tilde{\lambda}}^{\bar{A}})} \ .$$

# Final Comments

- An analogous derivation can be done for $Z_{\mathbb{S}^2;B}$.
- A similar new anomaly associated to the conformal manifold exists in $d = 4$ theories and it leads in $\mathcal{N} = 2$ theories to a similar conclusion $Z = e^{-K}$. This can be useful for the geometry of $AdS_5$ vacua, the relation to 2d Liouville theories, exact computations of extremal correlators etc. (see e.g. [Baggio-Niarchos-Papadodimas])
- Some outstanding open questions includes various other superalgebras that can be realized on $S^2$. For example, we can thread the $S^2$ with monopole flux. What does this compute? see e.g. [Closset-Cremonesi-Park]
- What are the global properties of the conformal manifold?