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Higher genus amplitudes in SUSY double-well matrix model for 2D IIA superstring

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Mainly based on

- T. Kuroki and F. S., Nucl. Phys. B **867** (2013) 448, arXiv 1208.3263 ;
  JHEP **1403** (2014) 006, arXiv 1306.3561
- M. G. Endres, T. Kuroki, F. S. and H. Suzuki, Nucl. Phys. B **876** (2013) 758, arXiv 1308.3306
- S. M. Nishigaki and F. S., JHEP **1409** (2014) 104, arXiv 1405.1633
- T. Kuroki and F. S., in progress

1 Introduction

In this Talk,

◇ I would like to discuss correspondence between

A simple zero-dimensional SUSY double-well matrix model (MM)

and

2D type IIA superstring on a nontrivial RR background.

An interesting example of MMs for superstrings with target-space SUSY, in which various amplitudes are explicitly calculable.

e.g.) All-order results in the string perturbation, resurgence, ...

◇ Nonperturbative effect of the MM is computed in its double scaling limit.

SUSY is spontaneously broken due to instantons.

$\Downarrow$

In the type IIA theory,

SUSY is dynamically broken by a nonperturbative effect.

Contents

2 SUSY double-well MM

[Kuroki-F.S. 2009]

$$S_{\rm MM} = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2 - \mu^2) + \bar{\psi}(\phi\psi + \psi\phi)\right],$$

where

B, ϕ: $N \times N$ hermitian matrices (Bosonic),

ψ, $\bar{\psi}$: $N \times N$ Grassmann-odd matrices (Fermionic).

- SUSY:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0,$$
$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0.$$

$\Rightarrow Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$ (nilpotent)

- B, ψ, $\bar{\psi}$ integrated out

$$S_{\rm MM} \to N\mathrm{tr}\,\frac{1}{2}(\phi^2 - \mu^2)^2 - \ln\det(\phi \otimes 1\!\!1_N + 1\!\!1_N \otimes \phi)$$

$\uparrow$

Double-well scalar potential

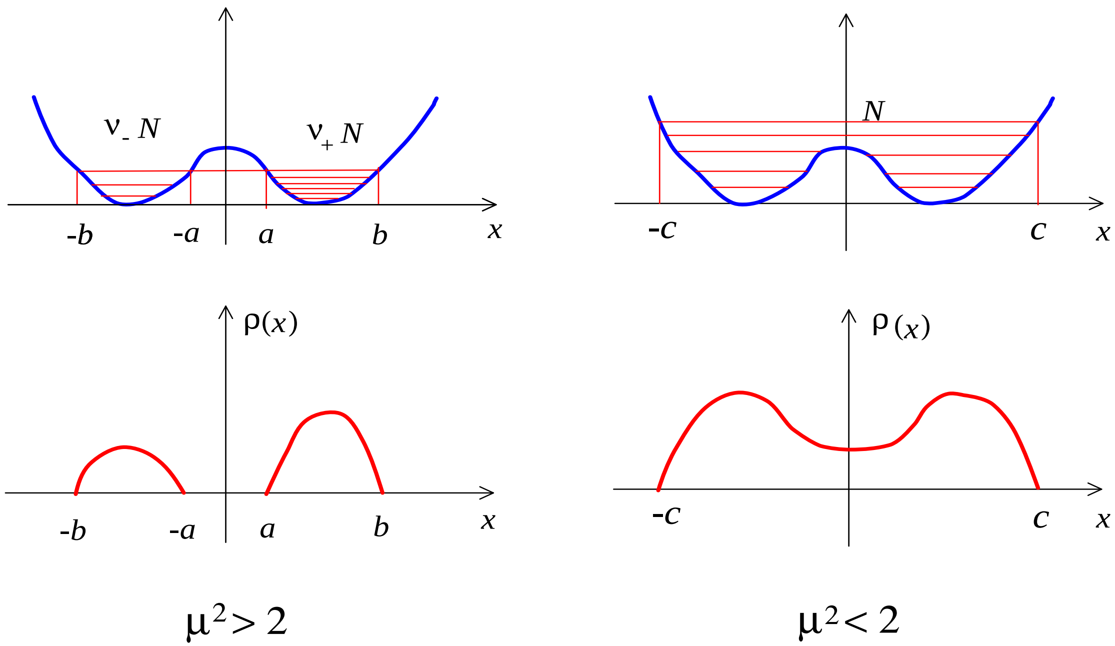

Figure 1: **(Left)**: "SUSY preserving" solution with $\langle \frac{1}{N}\mathrm{tr}\, B \rangle = 0$, **(Right)**: SUSY breaking solution with $\langle \frac{1}{N}\mathrm{tr}\, B \rangle \neq 0$, at the planar limit.

◇ Large-N saddle point solution for $\rho(x) \equiv \frac{1}{N}\mathrm{tr}\,\delta(x-\phi)$: Planar limit

[Kuroki-F.S. 2010]

3rd order phase transition between these two phases.

The 3rd derivative of the free energy w.r.t. μ^2 has a jump.

3 2D type IIA superstring [Kutasov-Seiberg 1990, Ita-Nieder-Oz 2005]

- (Target space) $= (x, \varphi) \sim$ Cylinder,
  where $x \in S^1$ with self-dual radius $(R = 1)$ and φ: Liouville.

- Holomorphic EM tensor (except ghost part) on string worldsheet:

$$T_m = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi_x \partial \psi_x - \frac{1}{2}(\partial \varphi)^2 + \frac{Q}{2}\partial^2 \varphi - \frac{1}{2}\psi_\ell \partial \psi_\ell$$

with $Q = 2$.

- Target-space SUSY is nilpotent.

$$q_+(z) = e^{-\frac{1}{2}\phi - \frac{i}{2}H - ix}(z), \qquad Q_+ = \oint \frac{dz}{2\pi i}\, q_+(z),$$
$$\bar{q}_-(\bar{z}) = e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H} + i\bar{x}}(\bar{z}), \qquad \bar{Q}_- = \oint \frac{d\bar{z}}{2\pi i}\, \bar{q}_-(\bar{z}),$$

where $\psi_\ell \pm i\psi_x = \sqrt{2}e^{\mp iH}$.

$\Rightarrow Q_+^2 = \bar{Q}_-^2 = \{Q_+, \bar{Q}_-\} = 0$. ($\leftarrow$ Same as the matrix model!)

• Vertex operators (holomorphic sector):

$$\text{NS sector } (-1)\text{-picture}: \quad T_k(z) = e^{-\phi + ikx + p_\ell \varphi}(z)$$

$$\text{R sector } (-\tfrac{1}{2})\text{-picture}: \quad V_{k,\,\epsilon}(z) = e^{-\frac{1}{2}\phi + \frac{i}{2}\epsilon H + ikx + p_\ell \varphi}(z)$$

with $\epsilon = \pm 1$.

Locality with supercurrents, mutual locality, superconformal inv., level matching
$\Rightarrow$ physical on-shell vertex operators with $p_\ell = 1 - |k|$ and $k = \epsilon |k|$

Winding background: [Ita-Nieder-Oz 2005]

$$(\text{NS}, \text{NS}): \quad T_k(z)\,\bar{T}_{-k}(\bar{z}) \quad (k \in \mathbf{Z} + \frac{1}{2})$$

massless scalar

winding

$$(\text{R}+, \text{R}-): \quad V_{k,+1}(z)\,\bar{V}_{-k,-1}(\bar{z}) \quad (k = \frac{1}{2}, \frac{3}{2}, \cdots)$$

$$(\text{R}-, \text{R}+): \quad V_{-k,-1}(z)\,\bar{V}_{k,+1}(\bar{z}) \quad (k = 0, 1, 2, \cdots)$$

RR 2-form field strength

winding

$$(\text{NS}, \text{R}-): \quad T_{-k}(z)\,\bar{V}_{-k,-1}(\bar{z}) \quad (k = \frac{1}{2}, \frac{3}{2}, \cdots)$$

fermion$(-)$

momentum

$$(\text{R}+, \text{NS}): \quad V_{k,+1}(z)\,\bar{T}_k(\bar{z}) \quad (k = \frac{1}{2}, \frac{3}{2}, \cdots)$$

fermion$(+)$

momentum

4 Correspondence between the MM and the IIA theory

Observation:

Under the identification of supercharges between the MM and the type IIA theory:

$$(Q, \bar{Q}) \Leftrightarrow (Q_+, \bar{Q}_-).$$

$\Rightarrow$ SUSY transformation properties lead to

$$\Phi_1 = \frac{1}{N}\mathrm{tr}\,\phi \Leftrightarrow \int d^2z\, V_{\frac{1}{2},+1}(z)\, \bar{V}_{-\frac{1}{2},-1}(\bar{z}) \qquad (\mathrm{R}+,\ \mathrm{R}-),$$

$$\Psi_1 = \frac{1}{N}\mathrm{tr}\,\psi \Leftrightarrow \int d^2z\, T_{-\frac{1}{2}}(z)\, \bar{V}_{-\frac{1}{2},-1}(\bar{z}) \qquad (\mathrm{NS},\ \mathrm{R}-),$$

$$\bar{\Psi}_1 = \frac{1}{N}\mathrm{tr}\,\bar{\psi} \Leftrightarrow \int d^2z\, V_{\frac{1}{2},+1}(z)\, \bar{T}_{\frac{1}{2}}(\bar{z}) \qquad (\mathrm{R}+,\ \mathrm{NS}),$$

$$\frac{1}{N}\mathrm{tr}(-iB) \Leftrightarrow \int d^2z\, T_{-\frac{1}{2}}(z)\, \bar{T}_{\frac{1}{2}}(\bar{z}) \qquad (\mathrm{NS},\ \mathrm{NS}).$$

Quartet w.r.t. $(Q, \bar{Q})$ $\Leftrightarrow$ Quartet w.r.t. $(Q_+, \bar{Q}_-)$

Furthermore, it is natural to extend it to higher $k(= 1, 2, \cdots)$ as

$$\Phi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + \cdots \Leftrightarrow \int d^2z\, V_{k+\frac{1}{2},+1}(z)\,\bar{V}_{-k-\frac{1}{2},-1}(\bar{z}),$$

$$\Psi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\psi^{2k+1} + \cdots \Leftrightarrow \int d^2z\, T_{-k-\frac{1}{2}}(z)\,\bar{V}_{-k-\frac{1}{2},-1}(\bar{z}),$$

$$\bar{\Psi}_{2k+1} = \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2k+1} + \cdots \Leftrightarrow \int d^2z\, V_{k+\frac{1}{2},+1}(z)\,\bar{T}_{k+\frac{1}{2}}(\bar{z}),$$

(Single trace operators in the MM) $\Leftrightarrow$ (Integrated vertex operators in IIA)

(Powers of matrices) $\Leftrightarrow$ (Windings or Momenta)

Note:

- RR 2-form field strength in (R$-$, R$+$) is a singlet under the target-space SUSYs $\boldsymbol{Q}_{+}$, $\bar{\boldsymbol{Q}}_{-}$, and appears to have no MM counterpart.
- Expectation values of operators measuring the RR charge (e.g. $\langle \boldsymbol{\Phi}_{2k+1} \rangle_0$) are nonvanishing in the MM.

$\Rightarrow$ The MM is considered to correspond to IIA on a background of the RR 2-form. $\boldsymbol{\nu}_{+} - \boldsymbol{\nu}_{-} \Leftrightarrow$ (RR flux)

We can explicitly check the correspondence by computing various amplitudes in the MM and the IIA theory. [Kuroki-F.S. 2014]

5 Nonperturbative SUSY breaking in the MM

◇ SUSY double-well MM

$$S_{\rm MM} = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2-\mu^2) + \bar{\psi}(\phi\psi+\psi\phi)\right].$$

After integrating out matrices other than ϕ, the partition function is expressed in terms of eigenvalues λ_i $(i=1,\cdots,N)$ as

$$\begin{aligned} Z_{\rm MM} &= \tilde{C}_N \int \Big(\prod_{i=1}^N d\lambda_i\Big) \triangle(\lambda)^2 \prod_{i,j=1}^N (\lambda_i+\lambda_j)\, e^{-N\sum_{i=1}^N \frac{1}{2}(\lambda_i^2-\mu^2)^2} \\ &= \sum_{\nu_-N=0}^N \frac{N!}{(\nu_+N)!(\nu_-N)!} Z_{(\nu_+,\nu_-)}, \end{aligned}$$

where the partition function in the (ν_+,ν_-) sector is defined by the integration region

$$\int_0^\infty \prod_{i=1}^{\nu_+ N} d\lambda_i \qquad \int_{-\infty}^0 \prod_{j=\nu_+ N+1}^{N} d\lambda_j.$$

By $\lambda_j \to -\lambda_j$ $(j = \nu_+ N + 1, \cdots, N)$, it is easy to see

$$Z_{(\nu_+,\nu_-)} = (-1)^{\nu_- N}\, Z_{(1,0)}.$$

Thus, the total partition function vanishes:

$$Z_{\mathrm{MM}} = \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!}\, Z_{(\nu_+,\nu_-)} = (1 + (-1))^N\, Z_{(1,0)} = 0.$$

$\Rightarrow$ Expectation values normalized by Z_{MM} become ill-defined.

Let us regularize it as

$$Z_\alpha \equiv \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} e^{-i\alpha\nu_- N} Z_{(\nu_+,\nu_-)} = (1 - e^{-i\alpha})^N Z_{(1,0)}.$$

◇ Order parameter of spontaneous SUSY breaking:

$$\left\langle \frac{1}{N}\mathrm{tr}(iB) \right\rangle_\alpha = \frac{1}{N^2}\frac{1}{Z_\alpha}\frac{\partial}{\partial(\mu^2)} Z_\alpha = \frac{1}{N^2}\frac{1}{Z_{(1,0)}}\frac{\partial}{\partial(\mu^2)} Z_{(1,0)}$$

is independent of α and well-defined in the limit $\alpha \to 0$.

Problem reduces to computing $Z_{(1,0)}$.

After the variable change $x_i = \mu^2 - \lambda_i^2$ (Nicolai mapping),

$$Z_{(1,0)} = \tilde{C}_N \int_{-\infty}^{\mu^2} \left(\prod_{i=1}^{N} dx_i \right) \triangle(x)^2 \, e^{-N\sum_{i=1}^{N} \frac{1}{2} x_i^2}.$$

◇ Techniques in the random matrix theory [Tracy-Widom 1994] give a closed form for the partition function in the double scaling limit
(the soft-edge scaling limit)

$$N \to \infty, \qquad \mu^2 \to 2 \qquad \text{with} \qquad s = N^{2/3}(\mu^2 - 2) \quad \text{fixed}$$

as

$$F = -\ln Z_{(1,0)} = \int_s^\infty (x - s)q(x)^2 dx,$$

where $q(x)$ is a solution to the Painléve II differential equation

$$q''(x) = xq(x) + 2q(x)^3$$

with $q(x) \sim \mathrm{Ai}(x)$ as $x \to +\infty$.

- The solution is unique. [Hastings-McLeod 1980]
- $g_{st} \sim 1/N \sim s^{-3/2}$

$\Rightarrow$ $s \gg 1$: weakly coupled, $0 < s \ll 1$: strongly coupled.

5.1 Weak coupling expansion

◇ The partition function is given by the Fredholm determinant of the Airy kernel: [Tracy-Widom 1994]

$$Z_{(1,0)} = \mathrm{Det}(1 - \hat{K}_{\mathrm{Ai}}|_{[s,\infty)}).$$

By using the form of the Airy kernel

$$K_{\mathrm{Ai}}(s,t) \equiv \frac{\mathrm{Ai}(s)\mathrm{Ai}'(t) - \mathrm{Ai}'(s)\mathrm{Ai}(t)}{s-t},$$

the free energy expressed as an instanton sum

$$F = -\ln Z_{(1,0)} = \sum_{k=1}^{\infty} F_{k-\mathrm{inst.}}$$

is expanded as

$$F_{k-\mathrm{inst.}} = \frac{1}{k}\int_s^{\infty} dt_1 \ldots dt_k \, K_{\mathrm{Ai}}(t_1,t_2)\, K_{\mathrm{Ai}}(t_2,t_3) \cdots K_{\mathrm{Ai}}(t_k,t_1)$$
$$\sim \frac{1}{k}\left(\frac{1}{16\pi s^{3/2}} e^{-\frac{4}{3}s^{3/2}}\right)^k \left[1 + a_1^{(k)} s^{-3/2} + a_2^{(k)} s^{-3} + \cdots\right].$$

- $N^{4/3} \cdot \left\langle \frac{1}{N}\mathrm{tr}(iB) \right\rangle^{(1,0)} = -\frac{dF}{ds} \neq 0$

  $\Rightarrow$ SUSY is spontaneously broken due to instantons.

- The Airy-kernel expression of $F_{k-\mathrm{inst.}}$ contains all perturbative contributions around the k-instanton configuration.

5.2 Strong coupling expansion

◇ The Taylor series expansion of $F = \int_s^\infty (x - s) q(x)^2 dx$ around $s = 0$ is

$$F = 0.0311059853 - 0.0690913807 s + 0.0673670913 s^2 - 0.0361399144 s^3 + \cdots$$

This gives strong coupling expansion of the IIA superstring theory.

- The strongly coupled limit is regular!

- The expression of F is smoothly continued to the $s < 0$ region.

  The 3rd order phase transition in the planar limit becomes smooth crossover in the double scaling limit.
  Singular behavior at the "string tree level" is smeared by quantum effects.

  Similar to the unitary one-matrix model.

6 Higher genus amplitudes in the MM

We calculate the one-point function $\langle \Phi_{2k+1} \rangle \simeq \left\langle \frac{1}{N} \mathrm{tr}\, \phi^{2k+1} \right\rangle$ to the all orders in the string perturbation theory.

Not protected by SUSY

- $$\left\langle \frac{1}{N} \mathrm{tr}\, \phi^{2k+1} \right\rangle^{(\nu_+,\nu_-)} = (\nu_+ - \nu_-) \left\langle \frac{1}{N} \mathrm{tr}\, \phi^{2k+1} \right\rangle^{(1,0)}$$
$$= (\nu_+ - \nu_-) \oint_{[a,b]} \frac{dz}{2\pi i} z^{2k+1} \cdot 2z \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{z^2 - \phi^2} \right\rangle^{(1,0)}$$

- From the resolvent in the Gaussian MM, we have

$$\left\langle \frac{1}{N} \mathrm{tr} \frac{1}{z^2 - \phi^2} \right\rangle^{(1,0)} = -\langle R_G(\mu^2 - z^2) \rangle$$
$$= -(\text{planar part}) - \sum_{h=1}^{\infty} \frac{1}{N^{2h}} \sum_{r=2h}^{3h-1} C_{h,r} \left((a^2 - z^2)(b^2 - z^2)\right)^{-r-1/2}.$$

We end up with

$$N^{\frac{2}{3}(k+2)} \left\langle \frac{1}{N} \mathrm{tr}\, \phi^{2k+1} \right\rangle^{(1,0)} \Bigg|_{\mathrm{pert.}}$$

$$= \frac{1}{2\pi^{3/2}} \Gamma\left(k+\frac{3}{2}\right) \sum_{h=0}^{\left[\frac{k+2}{3}\right]} \left(-\frac{1}{12}\right)^h \frac{s^{k-3h+2}}{h!(k-3h+2)!} \ln s$$

$$+ \frac{(-1)^{k+1}}{2\pi^{3/2}} \Gamma\left(k+\frac{3}{2}\right) \sum_{h=\left[\frac{k+2}{3}\right]+1}^{\infty} \frac{(3h-k-3)!}{h!} \frac{s^{k+2-3h}}{12^h}.$$

↑

Divergent series (not Borel summable)

Borel resummation

(= Insert "$1 = \frac{1}{(2h)!} \int_0^\infty dz\, z^{2h} e^{-z}$"

and change the order of the sum and the integral)

leads to

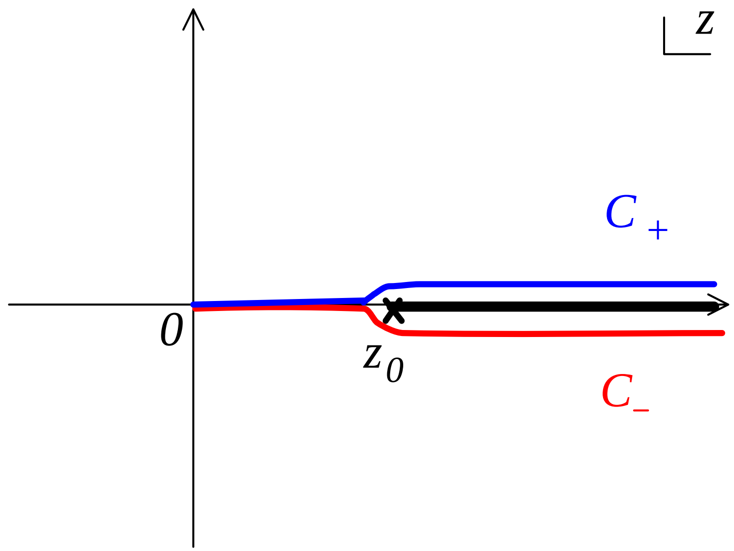

Figure 2: Singular point $z = z_0$ and integration contours in the Borel plane.

$$\text{(2nd line)} \simeq \frac{1}{4\pi} \frac{s^{k+2}}{(k+\frac{3}{2})(k+\frac{5}{2})} \int_0^{\infty} dz \left(1 - \frac{z^2}{z_0^2}\right)^{k+5/2} e^{-z}$$

with $z_0 \equiv \frac{4}{3} s^{3/2}$.

◇ The branch point singularity $z = z_0$ is on the integration contour $\mathbf{R}_+$.
⇒ The integral is ambiguous. How to avoid the singularity (C_+ or C_-)?

- The ambiguity gives a exponentially small imaginary part:

(2nd line with C_+) − (2nd line with C_-)

$$= \frac{i}{2\pi}\frac{(-1)^k}{3^{k+5/2}}\frac{s^{k+2}}{(k+\frac{3}{2})(k+\frac{5}{2})}\int_{z_0}^{\infty} dz \left(\frac{z^2}{z_0^2}-1\right)^{k+5/2} e^{-z}$$

↑

The order of $e^{-z_0} = e^{-\frac{4}{3}s^{3/2}}$: instanton contribution!

- Resurgence:
  The ambiguity from the perturbation series should cancel with the ambiguity from instanton contributions.
  In total, the expression is well-defined.

  We are computing fluctuations around the instanton contribution to the one-point function for further check of the resurgence.

7 Summary and discussions

◇ We computed correlation functions in the double-well SUSY MM, and discussed its correspondence to 2D type IIA superstring theory on (R−,R+) background by computing amplitudes in both sides.

- Case of $(\nu_+ - \nu_-)$ not small?
  Related to black-hole (cigar) target space? cf. [Hori-Kapustin 2001]

- General operators
  $\mathrm{tr}(\phi^k \psi^\ell \phi^m \psi^n \cdots) \Leftrightarrow$ (polynomial of $\partial x,\ \partial\varphi, \cdots)e^{ikx+p_\ell\varphi+\cdots}$
  are suggested by SUSY transformation properties.

- MMs for higher-dimensional noncritical superstrings $(D = 4, 6, 8, (10))$?
  $D = \quad 2 \quad + \quad (D-2)$ [Kutasov-Seiberg 1990]
  $\nearrow$ (under 2), $\nwarrow$ (under $(D-2)$)
  (x, φ): Nilpotent SUSY $\qquad \mathbf{R}^{D-2}$: Usual SUSY generating translations

◇ The full nonperturbative expression of the free energy of the MM and the all-order perturbative result for $\left\langle \frac{1}{N}\mathrm{tr}\,\phi^{2k+1}\right\rangle$ obtained.

- Strong coupling expansion

  $\Rightarrow$ existence of the S-dual theory (noncritical M theory)?

- D-brane computation in the type IIA side.

- Check of resurgence in the MM for 2D IIA superstring

Thank you very much for your attention!