A HHL 3-point correlation function in the η-deformed $AdS_5 \times S^5$

Changrim Ahn, Plamen Bozhilov

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1 Introduction

The AdS/CFT duality [1] between string theories on curved spacetimes with Anti-de Sitter subspaces and conformal field theories in different dimensions has been actively investigated in the last years. A lot of impressive progresses have been made in this field of research based mainly on the integrability structures discovered on both sides of the correspondence (for recent review on the AdS/CFT duality, see [2]). For the most studied case of the $\mathcal{N} = 4$ super Yang-Mills theory, the anomalous dimensions of gauge-invariant single-trace operators match non-perturbatively with the string energies in the curved $AdS_5 \times S^5$ background. Integrability provides tools to solve the finite-volume spectral problem exactly.

After these successes, one direction of interesting developments is to generalize the duality to larger theories which include the original AdS/CFT as a special case and the other is to go beyond the spectral problem by computing general correlation functions, in particular, the three-point functions, or the structure constants.

An interesting development for the former direction is to study the string theory on the η-deformed $AdS_5 \times S^5$ background [3]. The bosonic part of the superstring sigma model Lagrangian on this η-deformed background and perturbative worldsheet S-matrix were obtained in [4]. The TBA for spectrum and explicit dispersion relation for giant magnon [5] have been derived in [6]. Finite-size effect on the giant magnon spectrum has been computed in [7]. For three-point correlation functions, quite a lot of interesting results on both strong and weak coupling regions were accumulated although non-perturbative results are much more difficult than the spectral problem.

In this letter, we compute the three-point correlation function of two giant magnon heavy operators with finite-size J_1 and a single dilaton light operator of the string theory with the η-deformed $AdS_5 \times S^5$ background [3].

2 Exact semiclassical structure constant

According to [8], the three-point functions of two "heavy" operators and a "light" operator can be approximated by a supergravity vertex operator evaluated at the "heavy" classical string configuration:

$$\langle V_H(x_1)V_H(x_2)V_L(x_3)\rangle = V_L(x_3)_{\text{classical}}.$$

For $|x_1| = |x_2| = 1$, $x_3 = 0$, the correlation function reduces to

$$\langle V_H(x_1)V_H(x_2)V_L(0)\rangle = \frac{C_{123}}{|x_1 - x_2|^{2\Delta_H}}.$$

Then, the normalized structure constant

$$\mathcal{C}_3 = \frac{C_{123}}{C_{12}}$$

can be found from

$$\mathcal{C}_3 = c_\Delta V_L(0)_{\text{classical}}, \tag{2.1}$$

where c_Δ is the normalized constant of the "light" vertex operator. Actually, we are going to compute the normalized structure constant (2.1). For the case under consideration, the "light" state is represented by the dilaton with zero momentum.

According to [9], C_3 for the infinite-size giant magnons and dilaton with zero momentum in the undeformed $AdS_5 \times S^5$ is given by

$$C_3 = c^d_\Delta \int_{-\infty}^{+\infty} \frac{d\tau_e}{\cosh^4(\kappa\tau_e)} \int_{-\infty}^{+\infty} d\sigma \left(\kappa^2 + \partial X_K \bar{\partial} X_K\right) \quad (2.2)$$
$$= \frac{4c^d_\Delta}{3\kappa} \int_{-\infty}^{+\infty} d\sigma \left(\kappa^2 + \partial X_K \bar{\partial} X_K\right),$$

where $t = \kappa\tau_e$ is the Euclidean AdS time and the term $\partial X_K \bar{\partial} X_K$ is proportional to the string Lagrangian on S^2 computed on the giant magnon solution living in the $R_t \times S^2$ subspace.

Since here we are interested in *finite-size* giant magnons, we have to replace

$$\int_{-\infty}^{+\infty} d\sigma \to \int_{-L}^{+L} d\sigma = 2\int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\theta'},$$

where L gives the size of the giant magnon and θ is the non-isometric angle on the two-sphere [11].

Going to the η-deformed $AdS_5 \times S^5$ case, we have to compute the term $\partial X_K \bar{\partial} X_K$ for this background, which is proportional to the string Lagrangian on S^2_η for *finite-size* giant magnons:

$$L_{S^2_\eta} = -\frac{T}{2} \partial X_K \bar{\partial} X_K,$$

where $X_K = (\phi_1, \theta)$ are the isometric and non-isometric string coordinates on S^2_η correspondingly.

Working in conformal gauge and applying the ansatz

$$\phi_1(\tau,\sigma) = \tau + F_1(\xi), \quad \theta(\tau,\sigma) = \theta(\xi),$$
$$\xi = \alpha\sigma + \beta\tau, \quad \alpha, \beta - constants,$$

one finds

$$L_{S^2_\eta} = -\frac{T}{2}\Bigg\{(\alpha^2-\beta^2)\frac{\theta'^2}{1+\tilde{\eta}^2(1-\chi)} \tag{2.3}$$
$$+(1-\chi)\Big[(\alpha^2-\beta^2)(F_1')^2-2\beta F_1'-1\Big]\Bigg\},$$

where $\tilde{\eta}$ is related to the deformation parameter η according to [4]

$$\tilde{\eta} = \frac{2\eta}{1-\eta^2}, \tag{2.4}$$

and the new variable χ is defined by

$$\chi = \cos^2\theta.$$

The prime here and below is the derivative $d/d\xi$. The string tension T for the η deformed case is related to the coupling constant g by

$$T = g\sqrt{1+\tilde{\eta}^2}. \tag{2.5}$$

The first integrals of the equations of motion F_1' and θ' can be written as

$$F_1' = \frac{\beta}{\alpha^2-\beta^2}\left(-\frac{\kappa^2}{1-\chi}+1\right), \tag{2.6}$$

$$\theta'^2 = \frac{1+\tilde{\eta}^2(1-\chi)}{(\alpha^2-\beta^2)^2}\left[(\alpha^2+\beta^2)\kappa^2-\frac{\beta^2\kappa^4}{1-\chi}-\alpha^2(1-\chi)\right]. \tag{2.7}$$

Inserting (2.6), (2.7) in (2.3), we obtain:

$$L_{S^2_\eta} = -\frac{T}{2}\frac{\beta^2\kappa^2+\alpha^2(\kappa^2-2(1-\chi))}{\alpha^2-\beta^2}. \tag{2.8}$$

Now we introduce the new parameters

$$v = -\frac{\beta}{\alpha}, \quad W = \kappa^2,$$

which leads to

$$L_{S^2_\eta} = -\frac{T}{2}\frac{(1+v^2)W - 2(1-\chi)}{1-v^2}. \tag{2.9}$$

Therefore, for the case at hand, the normalized structure constant takes the form

$$C_3^{\tilde{\eta}} = \frac{8c^d_\Delta}{3\sqrt{W}}\int_{\chi_m}^{\chi_p}\frac{d\chi}{\chi'}\left[W + \frac{(1+v^2)W - 2(1-\chi)}{1-v^2}\right], \tag{2.10}$$

where

$$\chi_m = \chi_{min}, \quad \chi_p = \chi_{max}.$$

One can rewrite Eq.(2.7) as

$$\chi' = \frac{2\tilde{\eta}}{1-v^2}\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}, \tag{2.11}$$

where [7]

$$\chi_m = 1 - W, \quad \chi_p = 1 - v^2 W, \quad \chi_\eta = 1 + \frac{1}{\tilde{\eta}^2}. \tag{2.12}$$

Using this, we can express all the results in terms of $\chi_p, \ \chi_m$ by eliminating $v, \ W$.

Replacing (2.11) into (2.10) and using (2.12), we obtain

$$C_3^{\tilde{\eta}} = \frac{8c^d_\Delta}{3\tilde{\eta}\sqrt{1-\chi_m}}\int_{\chi_m}^{\chi_p}\sqrt{\frac{\chi - \chi_m}{(\chi_\eta - \chi)(\chi_p - \chi)\chi}}\, d\chi. \tag{2.13}$$

The above integral can be easily expressed by $\mathbf{K}$ and $\mathbf{\Pi}$ - the complete elliptic integrals of first and third kind, respectively, as follows:

$$\begin{aligned} C_3^{\tilde{\eta}} &= \frac{16c_\Delta^d}{3\tilde{\eta}} \frac{\chi_m}{\sqrt{\chi_p(1-\chi_m)(\chi_\eta-\chi_m)}} \times \\ & \left[\mathbf{\Pi}\left(1-\frac{\chi_m}{\chi_p}, 1-\epsilon\right) - \mathbf{K}\left(1-\epsilon\right)\right], \end{aligned} \tag{2.14}$$

where we introduced the short notation ϵ by

$$\epsilon = \frac{\chi_m(\chi_\eta-\chi_p)}{\chi_p(\chi_\eta-\chi_m)}. \tag{2.15}$$

Eq.(2.14) is our main result, which is an *exact* semiclassical result for the normalized structure constant $C_3^{\tilde{\eta}}$ valid for any value of $\tilde{\eta}$ and the string angular momentum J_1. Here, χ_p and χ_m are determined by J_1 and the world-sheet momentum p from the following equations:

$$\begin{aligned} J_1 &= \frac{2T}{\tilde{\eta}} \frac{1}{\sqrt{\chi_p(\chi_\eta-\chi_m)}} \times \\ & \left[\chi_p \mathbf{K}\left(1-\epsilon\right) - \chi_m \mathbf{\Pi}\left(1-\frac{\chi_m}{\chi_p}, 1-\epsilon\right)\right], \end{aligned} \tag{2.16}$$

$$\begin{aligned} p &= \frac{2\chi_m}{\tilde{\eta}} \sqrt{\frac{1-\chi_p}{\chi_p(1-\chi_m)(\chi_\eta-\chi_m)}} \times \\ & \left[\mathbf{K}\left(1-\epsilon\right) - \mathbf{\Pi}\left(\frac{\chi_p-\chi_m}{\chi_p(1-\chi_m)}, 1-\epsilon\right)\right]. \end{aligned} \tag{2.17}$$

The world-sheet energy of the giant magnon is given by

$$E = \frac{2T}{\tilde{\eta}} \frac{\chi_p-\chi_m}{\sqrt{\chi_p(1-\chi_m)(\chi_\eta-\chi_m)}} \mathbf{K}\left(1-\epsilon\right).. \tag{2.18}$$

One nontrivial check of the above result is that the g derivative of the conformal dimension $\Delta = E - J_1$ should be proportional to the normalized structure constant $C_3^{\tilde{\eta}}$ since the g derivative of the two-point function inserts the dilaton (Lagrangian) operator into the two-point function of the heavy operators [10]. This can be expressed by

$$C_3^{\tilde{\eta}} = \frac{8c_\Delta^d}{3\sqrt{1+\tilde{\eta}^2}}\frac{\partial\Delta}{\partial g}. \tag{2.19}$$

To check that if Eqs.(2.14), (2.16), and (2.18) do satisfy Eq.(2.19), we use the fact that

$$\frac{\partial J_1}{\partial g} = \frac{\partial p}{\partial g} = 0 \tag{2.20}$$

as noticed in [11] for the case of undeformed giant magnons. From these, we can obtain the expressions for $\partial\chi_p/\partial g$ and $\partial\chi_m/\partial g$ which can be inserted into $\partial\Delta/\partial g$. The η-deformed case involves much more complicated expressions which can be dealt with Mathematica. It can be shown that Eq.(2.14) do satisfy the consistency condition (2.19) exactly.

In the limit $\tilde{\eta}\to 0$ with $\tilde{\eta}^2\chi_\eta \to 1$, Eq.(2.14) becomes

$$C_3^0 = \frac{16c_\Delta^d}{3}\sqrt{\frac{\chi_p}{1-\chi_m}}\left[\mathbf{E}\left(1-\epsilon\right) - \epsilon\mathbf{K}\left(1-\epsilon\right)\right], \quad \epsilon = \frac{\chi_m}{\chi_p}, \tag{2.21}$$

where we used the identity $(1-a)\mathbf{\Pi}(a,a) = \mathbf{E}(a)$. This is the structure constant of the undeformed theory derived in [11].

3 Leading finite-size effect on $C_3^{\tilde{\eta}}$

It is straightforward to compute the leading finite-size effect on $C_3^{\tilde{\eta}}$ for $J_1 \gg g$ by taking the limit $\epsilon\to 0$ in (2.14).

First, we expand the parameters χ_p, W and v for small ϵ as follows:

$$\begin{aligned}
\chi_p &= \chi_{p0} + (\chi_{p1} + \chi_{p2}\log\epsilon)\epsilon, \\
W &= 1 + W_1\epsilon, \\
v &= v_0 + (v_1 + v_2\log\epsilon)\epsilon.
\end{aligned} \tag{3.1}$$

Inserting into Eq.(2.14), we obtain

$$\begin{aligned}
C^d_{\tilde\eta} \approx & \frac{16c^d_\Delta}{3\tilde\eta^2\sqrt{\left(1+\frac{1}{\tilde\eta^2}\right)\chi_{p0}}}\Bigg\{\sqrt{(1+\tilde\eta^2)\chi_{p0}}\ \mathrm{arctanh}\frac{\tilde\eta\sqrt{\chi_{p0}}}{\sqrt{1+\tilde\eta^2}} \\
& -\Bigg[\frac{W_1}{2}\sqrt{(1+\tilde\eta^2)\chi_{p0}}\ \mathrm{arctanh}\frac{\tilde\eta\sqrt{\chi_{p0}}}{\sqrt{1+\tilde\eta^2}} + \frac{\tilde\eta}{4\left(1+\tilde\eta^2(1-\chi_{p0})\right)}\times \\
& \Big((1+\tilde\eta^2)(\chi_{p0}-2\chi_{p1}) \\
& -4\left((1+\tilde\eta^2)\chi_{p0} + 2W_1\left(1+\tilde\eta^2(1-\chi_{p0})\right)\right)\log 2\Big)\Bigg]\epsilon \\
& -\frac{\tilde\eta}{4\left(1+\tilde\eta^2(1-\chi_{p0})\right)}\Big(\Big((1+\tilde\eta^2)(\chi_{p0}-2\chi_{p2}) \\
& + 2W_1\left(1+\tilde\eta^2(1-\chi_{p0})\right)\Big)\Big)\epsilon\log\epsilon\Bigg\}.
\end{aligned} \tag{3.2}$$

In view of Eqs.(2.12) and (2.15), we can express all the auxiliary parameters in terms of v (or its coefficients v_0, v_1, and v_2):

$$\begin{aligned}
\chi_{p0} = 1 - v_0^2, \quad \chi_{p1} = 1 - v_0^2 - 2v_0v_1 - \frac{(1-v_0^2)^2}{1+\tilde\eta^2 v_0^2}, \\
\chi_{p2} = -2v_0v_2, \quad W_1 = -\frac{(1+\tilde\eta^2)(1-v_0^2)}{1+\tilde\eta^2 v_0^2}.
\end{aligned} \tag{3.3}$$

This leads to

$$C_3^{\tilde{\eta}} \approx \frac{16c_\Delta^d}{3\tilde{\eta}}\Bigg\{\text{arctanh}\frac{\tilde{\eta}\sqrt{1-v_0^2}}{\sqrt{1+\tilde{\eta}^2}} + \frac{1}{4\sqrt{(1+\tilde{\eta}^2)(1-v_0^2)}(1+\tilde{\eta}^2 v_0^2)^2} \times$$
$$\Bigg[(1+\tilde{\eta}^2)\left((1-v_0^2)\left(1+\tilde{\eta}^2 v_0^2\right)\right.$$
$$\left(2\sqrt{(1+\tilde{\eta}^2)\left((1-v_0^2\right)}\text{arctanh}\frac{\tilde{\eta}\sqrt{1-v_0^2}}{\sqrt{1+\tilde{\eta}^2}} - \tilde{\eta}\log 16\right)$$
$$\left.-\tilde{\eta}\left(1 - v_0(3v_0 - 2v_0^3 - 4v_1 + v_0(1-v_0^2-4v_0v_1)\tilde{\eta}^2)\right)\right)\Bigg]\epsilon$$
$$+\frac{\tilde{\eta}(1+\tilde{\eta}^2)(1-v_0^2-4v_0v_2)}{4\sqrt{(1+\tilde{\eta}^2)(1-v_0^2)}(1+\tilde{\eta}^2v_0^2)}\,\epsilon\log\epsilon\Bigg\}. \tag{3.4}$$

To fix v_0, v_1, and v_2, one can use the small ϵ expansion of the angular difference

$$\Delta\phi_1 = \phi_1(\tau, L) - \phi_1(\tau, -L) \equiv p,$$

where we identified the angular difference $\Delta\phi_1$ with the magnon momentum p on the worldsheet. The result is [7]

$$v_0 = \frac{\cot\frac{p}{2}}{\sqrt{\tilde{\eta}^2 + \csc^2\frac{p}{2}}}, \tag{3.5}$$

and

$$v_1 = \frac{v_0(1-v_0^2)\left[1 - \log 16 + \tilde{\eta}^2\left(2 - v_0^2(1+\log 16)\right)\right]}{4(1+\tilde{\eta}^2 v_0^2)}, \tag{3.6}$$
$$v_2 = \frac{1}{4}v_0(1-v_0^2).$$

By using (3.5), (3.6) in (3.4), one finds

$$C_3^{\tilde\eta} \approx \frac{16 c_\Delta^d}{3\tilde\eta}\Bigg\{\mathrm{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right) + \frac{(1+\tilde\eta^2)\sin^2\frac{p}{2}}{4\sqrt{\tilde\eta^2+\csc^2\frac{p}{2}}} \times \tag{3.7}$$

$$\left[\left(2\sqrt{\tilde\eta^2+\csc^2\frac{p}{2}}\,\mathrm{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right) - \tilde\eta(1+\log 16)\right)\epsilon + \tilde\eta\epsilon\log\epsilon\right]\Bigg\}.$$

The expansion parameter ϵ in the leading order is given by [7]

$$\epsilon = 16\ \exp\left[-\left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde\eta^2}}{\tilde\eta}\mathrm{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde\eta^2\sin^2\frac{p}{2}}{(1+\tilde\eta^2)\sin^2\frac{p}{2}}}\right].$$

Here we used Eq.(2.5) for the string tension T.

The final expression for the normalized structure costant is given by

$$C_3^{\tilde\eta} \approx \frac{16 c_\Delta^d}{3\tilde\eta}\Bigg\{\mathrm{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right) \tag{3.8}$$

$$-4\frac{\tilde\eta(1+\tilde\eta^2)\sin^3\frac{p}{2}}{\sqrt{1+\tilde\eta^2\sin^2\frac{p}{2}}}\left[1+\frac{J_1}{g}\sqrt{\frac{\tilde\eta^2+\csc^2\frac{p}{2}}{1+\tilde\eta^2}}\right]$$

$$\times\exp\left[-\left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde\eta^2}}{\tilde\eta}\mathrm{arcsinh}\left(\tilde\eta\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde\eta^2\sin^2\frac{p}{2}}{(1+\tilde\eta^2)\sin^2\frac{p}{2}}}\right]\Bigg\}.$$

Let us point out that in the limit $\tilde\eta\to 0$, (3.8) reduces to

$$C_3 \approx \frac{16}{3}c_\Delta^d\sin\frac{p}{2}\left[1-4\sin\frac{p}{2}\left(\sin\frac{p}{2}+\frac{J_1}{g}\right)\exp\left(-\frac{J_1}{g\sin\frac{p}{2}}-2\right)\right],$$

which reproduces the result for the undeformed case found in [11].

Another check is that this satisfies Eq.(2.19) with Δ computed in [7]

$$\Delta \equiv E - J_1 \approx 2g\sqrt{1+\tilde{\eta}^2}\Bigg\{\frac{1}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right) \tag{3.9}$$

$$-4\frac{(1+\tilde{\eta}^2)\sin^3\frac{p}{2}}{\sqrt{1+\tilde{\eta}^2\sin^2\frac{p}{2}}}\times$$

$$\exp\left[-\left(\frac{J_1}{g}+\frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}}\text{arcsinh}\left(\tilde{\eta}\sin\frac{p}{2}\right)\right)\sqrt{\frac{1+\tilde{\eta}^2\sin^2\frac{p}{2}}{(1+\tilde{\eta}^2)\sin^2\frac{p}{2}}}\right]\Bigg\}.$$

4 Concluding Remarks

Here we obtained the *exact* semiclassical the 3-point correlation function between two finite-size giant magnons "heavy" string states and the "light" dilaton operator with zero momentum in the η-deformed $AdS_5\times S^5$. It is given in terms of the complete elliptic integrals of first and third kind. We proved the consistency of our result by taking a derivative of the conformal dimension w.r.t. the coupling constant. We also provided the leading finite-size effect expansion of the structure constant.

It will be interesting to compute other three-point correlation functions of the η-deformed background to which our results may be useful.

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