Toeplitz operators with discontinuous symbols on the sphere

Tatyana Barron

University of Western Ontario, Canada

Plan of the talk:

- What are Toeplitz operators ? (including Toeplitz operators on the circle, Berezin-Toeplitz operators on Kähler manifolds)

- Quantization (Kähler/Berezin-Toeplitz)

- My recent work with Ma, Marinescu, Pinsonnault on Berezin-Toeplitz operators with C^m symbols

- Kähler quantization on $\mathbb{P}^1$ and my recent work with D. Itkin

Toeplitz operators on S^1

- $S^1 = \{z \in \mathbb{C} | \, |z| = 1\}$, μ normalized Lebesgue measure
- $e_n = e_n(z) = z^n$, $z \in S^1$, $n \in \mathbb{Z}$ (bounded, measurable, form an orthonormal basis in $L^2 = L^2(S^1, d\mu)$)
- $f \in L^2$ is *analytic* if $\int_{S^1} f \bar{e}_n d\mu = 0$ for all $n < 0$.
- H^2 = the space of all functions in L^2 which are analytic
- $P : L^2 \to H^2$ the orthogonal projector

Definition. [Brown, Halmos, *J. Reine Angew. Math.*'64]

Let φ be a bounded measurable function on S^1.

Toeplitz operator: $T_\varphi = PM_\varphi : H^2 \to H^2$

I.e. $T_\varphi g = P(\varphi g)$ for g in H^2.

Some immediate observations:

- if $\varphi(z) = 1$ then T_φ is the identity operator

- $\alpha, \beta \in \mathbb{C}$, f, g bounded measurable functions on S^1

  $\Longrightarrow T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$

- if $\varphi(z) = z$ then $T_\varphi = M_z$

- if $\varphi(z) = z^{-1}$ then $T_\varphi(c_0 + c_1 z + c_2 z^2 + ...) = c_1 + c_2 z + c_3 z^2 + ...$

- two T.o. don't necessarily commute

- product ot T.o. is not necessarily a T.o.

It is possible to generalize this from S^1 to disk, spheres, balls, $\mathbb{C}^n$, domains in $\mathbb{C}^n$.

- measure
- $P : L^2 \to H^2$
- *Toeplitz operator*: $T_\varphi = PM_\varphi : H^2 \to H^2$

(Berezin-)Toeplitz operators

[Berezin, Boutet de Monvel and Guillemin]

(X, ω) compact Kähler manifold

assume: ω is integral

$\Longrightarrow \exists\ L \to X$ holomorphic Hermitian line bundle s. t. the curvature of the Hermitian connection is equal to $-2\pi i\omega$ ($\Longrightarrow c_1(L) = [\omega]$, L is ample)

$V^{(k)} = H^0(X, L^{\otimes k})$ space of holomorphic sections of $L^{\otimes k}$

(finite-dimensional complex vector space,

dim $V^{(k)} \sim$ const $k^n +$ l.o.t. as $k \to \infty$).

For $f \in C^\infty(X)$ *(Berezin)-Toeplitz operators*

$$T_f = \oplus_{k=0}^{\infty} T_f^{(k)},\ T_f^{(k)} = \Pi^{(k)} \circ M_f^{(k)} \in End(V^{(k)})$$

where

$$M_f^{(k)} : V^{(k)} \to L^2(X, L^{\otimes k})$$

$$s \mapsto fs,$$

and $\Pi^{(k)} : L^2(X, L^{\otimes k}) \to V^{(k)}$ is the orthogonal projector.

Remark. For $\alpha, \beta \in \mathbb{C}$, $f, g \in C^\infty(X)$

$$T_{\alpha f + \beta g}^{(k)} = \alpha T_f^{(k)} + \beta T_g^{(k)}$$

I.e. the map $C^\infty(X) \to End(V^{(k)})$, $f \mapsto T_f^{(k)}$, is linear.

Quantization

quantization: classical mechanics $\longrightarrow$ quantum mechanics

function $f\ \mapsto$ operator $\hat{f}$ on a Hilbert space $\mathcal{H}$

"Quantization" is a way to associate a linear operator $T_f^{(k)}$ to a function $f \in C^\infty(X)$ so that a version of Dirac's quantization conditions holds.

X classical phase space

$\mathcal{H} = V^{(k)}$ space of quantum-mechanical wave functions

The positive integer k is formally interpreted as $1/\hbar$, where $\hbar$ is the Planck constant. The limit $k \to \infty$ is called *the semi-classical limit*.

The quantum observables $T_f^{(k)}$ are (Berezin-)Toeplitz operators.

Dirac's conditions:

for a classical phase space X we want a linear map

$C^\infty(X) \to \{$ operators on $\mathcal{H}\}$ such that

$1 \to const(\hbar)I$

$\{f, g\} \to const(\hbar)[\hat{f}, \hat{g}]$

Related mathematics: since 1950-1960s (Kostant, Kirillov, Guillemin,...)

<u>Second quantization:</u> language of quantum field theory

- $\mathcal{H} = H^0(X, L^{\otimes k})$ (choose and fix sufficiently large k)

- Fock space

$$F(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$$

- bosonic (symmetric) Fock space, fermionic (antisymmetric) Fock space, creation and annihilation operators

- operators on $\mathcal{H} \rightsquigarrow$ operators on $F(\mathcal{H})$

Theorem.[Bordemann, Meinrenken, Schlichenmeier, *Comm. Math. Phys.*'94; based on techniques developed by Boutet de Monvel and Guillemin in the '80s]

For $f, f_1, ..., f_p \in C^\infty(X)$, as $k \to \infty$

$$||T^{(k)}_{f_1}...T^{(k)}_{f_p} - T^{(k)}_{f_1 \cdots f_p}|| = O(\frac{1}{k})$$

$$tr(T^{(k)}_{f_1}...T^{(k)}_{f_p}) = k^n(\int_X f_1...f_p \frac{\omega^n}{n!} + O(\frac{1}{k}))$$

$$||ik[T^{(k)}_{f_1}, T^{(k)}_{f_2}] - T^{(k)}_{\{f_1,f_2\}}|| = O(\frac{1}{k})$$

$$||[T^{(k)}_{f_1}, T^{(k)}_{f_2}]|| = O(\frac{1}{k})$$

There is $C > 0$ s.t.

$$|f|_\infty - \frac{C}{k} \leq ||T^{(k)}_f|| \leq |f|_\infty$$

$f \in C^\infty(X)$ has been a quite standard assumption in the subject.

In [T. Barron, X. Ma, G. Marinescu, M. Pinsonnault. Semi-classical properties of Berezin-Toeplitz operators with C^k-symbol. *J. Math. Phys.* 55, 042108 (2014) 25 pages.] we address the case when f is C^m (not necessarily C^∞). Work done in this paper implies, in particular, the following:

Theorem 1. Let $f_1, ..., f_p \in C^m(X)$. As $k \to \infty$

$$||T^{(k)}_{f_1} ... T^{(k)}_{f_p} - T^{(k)}_{f_1 ... f_p}|| = \begin{cases} O(k^{-1}) & \text{if } m = 2 \\ o(k^{-1/2}) & \text{if } m = 1 \\ o(1) & \text{if } m = 0 \end{cases}$$

[BMMP '14]:

Theorem 2. Suppose $f \in L^\infty(X)$ and there is $x_0 \in X$ such that $f(x_0) = |f|_\infty$ and f is continuous at x_0. Then

$$\lim_{k\to\infty} ||T_f^{(k)}|| = |f|_\infty.$$

If $f \in C^1(X)$ then $\exists\, C > 0$ s.t.

$$|f|_\infty - \frac{C}{\sqrt{k}} \leq ||T_f^{(k)}|| \leq |f|_\infty.$$

If $f \in C^2(X)$ then $\exists\, C > 0$ s.t.

$$|f|_\infty - \frac{C}{k} \leq ||T_f^{(k)}|| \leq |f|_\infty.$$

Also

Theorem 3.[BMMP '14] Let $f, f_1, ..., f_p \in C^m(X)$. As $k \to \infty$

$$\frac{1}{k^n} tr(T^{(k)}_{f_1} ... T^{(k)}_{f_p}) = \int_X f_1 ... f_p \frac{\omega^n}{n!} + \begin{cases} O(k^{-1}) & \text{if } m = 2 \\ O(k^{-1/2}) & \text{if } m = 1 \\ o(1) & \text{if } m = 0 \end{cases}$$

Theorem 4.[BMMP '14] Let $f, g \in C^m(X)$. As $k \to \infty$

$$||ik[T^{(k)}_{f_1}, T^{(k)}_{f_2}] - T^{(k)}_{\{f_1, f_2\}}|| = \begin{cases} o(1) & \text{if } m = 2 \\ o(k^{-1/2}) & \text{if } m = 3 \\ O(k^{-1}) & \text{if } m = 4 \end{cases}$$

Comments:

- Paper [BMMP '14] does not contain examples.

- A standard example is $X = S^2 \cong \mathbb{P}^1$, with the Fubini-Study form. Explicit constructions of Toeplitz operators on sphere (and torus), *with C^∞ symbol*, were worked out in

  [BGPU] A. Bloch, F. Golse, T. Paul, A. Uribe. Dispersionless Toda and Toeplitz operators. *Duke Math. J.* 117 (2003), no. 1, 157-196.

  See also [BU] D. Borthwick, A. Uribe. On the pseudospectra of Berezin-Toeplitz operators. *Methods Appl. Anal.* 10 (2003), no. 1, 31-65.

My recent work with David Itkin:

examples related to the context of Theorems 1, 2 above, on the sphere, for $T_f^{(k)}$ with discontinuous f.

$X = S^2 \cong \mathbb{P}^1$, with the Fubini-Study form

L the hyperplane bundle

$V^{(k-1)}$ the space of polynomials in z of degree $\leq k-1$

$$\langle \phi, \psi \rangle = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\phi(z)\overline{\psi(z)}}{(1+|z|^2)^{k+1}} dz d\bar{z}$$

Orthonormal basis in $V^{(k-1)}$

$$\varphi_j^{(k)} = \sqrt{\frac{k!}{j!(k-j-1)!}} z^j, \ j = 0, ..., k-1$$

Realize X as

$$\{(\xi, \eta, \zeta) \in \mathbb{R}^3 \mid \xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4}\}$$

Stereographic projection $S^2 - \{(0, 0, \frac{1}{2})\} \to \mathbb{C}$, $(\xi, \eta, \zeta) \mapsto z = x + iy$, where

$$x = \frac{\xi}{1 - \zeta}, \; y = \frac{\eta}{1 - \zeta}$$

Also we have:

$$\xi = \frac{x}{x^2 + y^2 + 1}, \; \eta = \frac{y}{x^2 + y^2 + 1}, \; \xi = \frac{x^2 + y^2}{x^2 + y^2 + 1}$$

Define $f, g, h : S^2 \to \mathbb{R}$ by

$$f(\xi,\eta,\zeta) = \begin{cases} 1 & \text{if } \zeta < 1/2 \\ 0 & \text{if } \zeta \geq 1/2 \end{cases}$$

$$g(\xi,\eta,\zeta) = \begin{cases} 1 & \text{if } \zeta < 1/5 \\ 0 & \text{if } \zeta \geq 1/5 \end{cases}$$

$$h(\xi,\eta,\zeta) = \begin{cases} 0 & \text{if } \zeta < 1/2 \\ \zeta & \text{if } \zeta \geq 1/2 \end{cases}$$

Stereographic projection gives the following functions on the xy-plane:

$$\hat{f}(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 < 1 \\ 0 & \text{if } x^2+y^2 \geq 1 \end{cases}$$

$$\hat{g}(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 < 1/4 \\ 0 & \text{if } x^2+y^2 \geq 1/4 \end{cases}$$

$$\hat{h}(x,y) = \begin{cases} 0 & \text{if } x^2+y^2 < 1 \\ \frac{x^2+y^2}{x^2+y^2+1} & \text{if } x^2+y^2 \geq 1 \end{cases}$$

Write $T_\rho^{(k)}$ as a matrix, in the basis $(\varphi_j^{(k)})$. The jl-th matrix element of $T_\rho^{(k)}$ is

$$(T_\rho^{(k)})_{jl} = \langle T_\rho^{(k)} \varphi_j^{(k)}, \varphi_j^{(k)} \rangle =$$

$$\langle \Pi^{(k)} M_\rho^{(k)} \varphi_j^{(k)}, \varphi_j^{(k)} \rangle = \langle \rho \varphi_j^{(k)}, \varphi_j^{(k)} \rangle =$$

$$\frac{i}{2\pi} \frac{k!}{\sqrt{j!l!(k-j-1)!(k-l-1)!}} \int_{\mathbb{C}} \hat{\rho}(z,\bar{z}) z^j \bar{z}^l \frac{dz d\bar{z}}{(1+|z|^2)^{k+1}}$$

Proposition. *(T.B., D. Itkin)* As $k \to \infty$

$$\text{(i) } ||T_f^{(k)}|| = 1 - \frac{1}{2^k}$$

$$\text{(ii) } ||T_g^{(k)}|| = 1 - \frac{1}{(5/4)^k}$$

$$\text{(iii) } ||T_f^{(k)} T_g^{(k)} - T_{fg}^{(k)}|| = \frac{1}{2^k}\Big(1 - \frac{1}{(5/4)^k}\Big)$$

$$\text{(iv) } 1 - \frac{2}{k} \leq ||T_h^{(k)}|| \leq 1$$

Sketch of proof: for $\rho = f, g, h$

$$(T_\rho^{(k)})_{jl} = 0 \text{ for } j \neq l$$

$$||T_\rho^{(k)}|| = \max_{0 \le j \le k-1} (T_\rho^{(k)})_{jj}$$

$$||T_f^{(k)}|| = 2k! \max_{0 \le j \le k-1} \frac{1}{j!(k-j-1)!} \int_0^1 \frac{r^{2j+1}}{(1+r^2)^{k+1}} dr$$

$$||T_g^{(k)}|| = 2k! \max_{0 \le j \le k-1} \frac{1}{j!(k-j-1)!} \int_0^{1/2} \frac{r^{2j+1}}{(1+r^2)^{k+1}} dr$$

$$||T_h^{(k)}|| = 2k! \max_{0 \le j \le k-1} \frac{1}{j!(k-j-1)!} \int_1^{\infty} \frac{r^{2j+3}}{(1+r^2)^{k+2}} dr$$

Proof of (i), (ii), (iv) is finished by long tedious calculations,

(iii) is an immediate consequence of (i), (ii). $\square$