On the reducible points for scalar generalized Verma modules

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Problem

- $\mathfrak{g}$: complex simple Lie algebra
- $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$: maximal parabolic subalgebra of $\mathfrak{g}$
- $M_{\mathfrak{q}}[t]$: scalar generalized Verma module induced from $\mathfrak{q}$

Problem:

Classify all the complex paramters $t \in \mathbb{C}$, for which $M_{\mathfrak{q}}[t]$ are reducible.

Today:

Give answers to the problem for certain maximal parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ of type E_6, E_7, E_8.

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Outline

1. Setting
2. Reducible criteria
3. Results

Setting

- $\mathfrak{g}$: complex simple Lie algebra
- $\mathfrak{h}$: Cartan subalgebra
- Δ : the root system with respect to $\mathfrak{h}$
- $\mathfrak{b}$: Borel subalgebra
- Δ^+ : the positive system with respect to $\mathfrak{b}$
- Π : the simple system for Δ^+
- $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$
- $\alpha^\vee$: the coroot of $\alpha \in \Delta$
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- $\mathfrak{q}$: standard parabolic subalgebra
- $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$: Levi decomposition ($\mathfrak{l}$: Levi part, $\mathfrak{n}$: nilpotent radical)
- $\Delta(\mathfrak{l}) := \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{l}\}$ ($\mathfrak{g}_\alpha$: the root space for $\alpha \in \Delta$)
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Now we set

$$\mathrm{P}_{\mathfrak{l}}^{+} := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in 1 + \mathbb{Z}^{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\}.$$

For $\lambda \in \mathrm{P}_{\mathfrak{l}}^{+}$, we write

$$V_{\mathfrak{l}}(\lambda) := \text{the simple } \mathfrak{l}\text{-module with highest weight } \lambda.$$

Definition:

For $\lambda \in \mathrm{P}_{\mathfrak{l}}^{+}$, the *generalized Verma module* $M_{\mathfrak{q}}(\lambda)$ with highest weight $\lambda - \rho$ is defined as

$$M_{\mathfrak{q}}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} V_{\mathfrak{l}}(\lambda - \rho).$$

Remark:

When $\dim V_{\mathfrak{l}}(\lambda - \rho) = 1$, we say that $M_{\mathfrak{q}}(\lambda)$ is of *scalar type*.

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Hereafter, we assume that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra.

- $\alpha_\mathfrak{q}$: the simple root corresponding to $\mathfrak{q}$
- $\lambda_\mathfrak{q}$: the fundamental weight for $\alpha_\mathfrak{q}$

Facts:

1. $\dim V_\mathfrak{l}(\lambda - \rho) = 1 \iff \lambda = t\lambda_\mathfrak{q} + \rho(\mathfrak{l})$ for $t \in \mathbb{C}$
2. $\rho(\mathfrak{n}) = c_\mathfrak{q}\lambda_\mathfrak{q}$ for some $c_\mathfrak{q} \in \frac{1}{2}\mathbb{Z}^{\geq 0}$

Consequenlty, scalar generalized Verma modules $M_\mathfrak{q}(\lambda) = M_\mathfrak{q}(t\lambda_\mathfrak{q} + \rho(\mathfrak{l}))$ may be parametrized by $t \in \mathbb{C}$. We then write

$$M_\mathfrak{q}[t] \equiv M_\mathfrak{q}(t\lambda_\mathfrak{q} + \rho(\mathfrak{l})) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{(t-c_\mathfrak{q})\lambda_\mathfrak{q}}$$

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We say that $t_0 \in \mathbb{C}$ is a *reducible point* for $M_\mathfrak{q}[t]$ if $M_\mathfrak{q}[t_0]$ is reducible.

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Previous classification results: $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$

- $\mathfrak{n}$ abelian (i.e. one-step nilpotent, $[\mathfrak{n}, \mathfrak{n}] = 0$):
  - Enright–Howe–Wallach and Jakobsen: Indivisually classified all the reducible points so that the irreducible quotient of $M_{\mathfrak{q}}[t]$ is unitarizable. ('83)
  - Haian He: Recently classified all the reducible points for $M_{\mathfrak{q}}[t]$. (arXiv:1501.01884)
- $\mathfrak{n}$ k-step nilpotent with $k \geq 2$: not known

Today:

Mainly observe maximal parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ with $\mathfrak{n}$ non-Heisenberg two-step nilpotent, namely,

- $\mathfrak{n}$ is two-step nilpotent ($[\mathfrak{n}, \mathfrak{n}] \neq 0, [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$),
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Reducible criteria

Jantzen's criterion (a specialization for maximal parabolic $\mathfrak{q}$):

If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra, then

$$M_{\mathfrak{q}}[t] \text{ is irreducible} \iff \sum_{\beta \in S_t} Y\big(s_\beta(t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}))\big) = 0,$$

where

- $S_t := \{\beta \in \Delta(\mathfrak{n}) \mid \langle t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \beta^\vee \rangle \in 1 + \mathbb{Z}^{\geq 0}\}$
- $Y(\lambda) := D^{-1} \sum_{w \in W(\mathfrak{l})} (-1)^{\ell(w)} e^{w\lambda}$

with

- D : the Weyl denominator,
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Point: Jantzen's criterion is very powerful. However, it is in general not easy to evaluate $\sum_{\beta \in S_t} Y\big(s_\beta(t\lambda_\mathfrak{q} + \rho(\mathfrak{l}))\big)$.

Facts: (See, for instance, [Humpreys, GSM 94, Theorem 9.12].)

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be a parabolic subalgebra (not necessarily maximal) and let $\lambda \in \mathrm{P}^+_\mathfrak{l}$ (not necessarily $\dim V_\mathfrak{l}(\lambda - \rho) = 1$).

1. If $\langle \lambda, \beta^\vee \rangle \notin 1 + \mathbb{Z}^{\geq 0}$ for all $\beta \in \Delta(\mathfrak{n})$, then $M_\mathfrak{q}(\lambda)$ is irreducible.
2. The converse also holds if λ is regular.

Point: By "Humphreys' fact," we only need to apply Jantzen's criterion only for t so that $t\lambda_\mathfrak{q} + \rho(\mathfrak{l})$ is singuler.

Difficulty: It is a bit of work to evaluate $\langle t\lambda_\mathfrak{q} + \rho(\mathfrak{l}), \beta^\vee \rangle$ for all $\beta \in \Delta(\mathfrak{n})$ when $\mathfrak{g}$ is of type E_6, E_7, and E_8.

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Two further reducibility criteria

Let's go back to $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, maximal parabolic with $\mathfrak{n}$ non-Heisenberg two-step nilpotent.

- Criterion 1:

Proposition:

If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is as above, then $M_{\mathfrak{q}}[t]$ is reducible only if $t \in \frac{1}{2}\mathbb{Z}^{>0}$.

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- Criterion 2: First observe the following:

Observation:

If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is as above, then one can give a two-grading $\mathfrak{g} = \bigoplus_{k=-2}^{2} \mathfrak{g}(k)$ on $\mathfrak{g}$ so that

$$\mathfrak{q} = \bigoplus_{k \geq 0} \mathfrak{g}(k) \text{ with } \mathfrak{l} = \mathfrak{g}(0) \text{ and } \mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

Definition:

We set $\mathrm{ht}(\mathfrak{g}(k)) :=$ the height of a root $\beta \in \Delta(\mathfrak{g}(k))$ so that

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Theorem A:

Let $\mathfrak{g}$ be a complex simple simply laced Lie algebra, and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be as above. Suppose that $\mathrm{ht}(\mathfrak{g}(2)) - \mathrm{ht}(\mathfrak{g}(1)) + \mathrm{ht}(\mathfrak{g}(0)) > c_{\mathfrak{q}}$. If there exists $m \in \frac{1}{2}\mathbb{Z}^{>0}$ so that

- $\gamma \in S_m$ and
- $\langle m\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$,

then $M_{\mathfrak{q}}[t]$ is reducible for all $t \in m + \mathbb{Z}^{\geq 0}$,

where

- $S_m = \{\beta \in \Delta(\mathfrak{n}) \mid \langle m\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \beta^\vee \rangle \in 1 + \mathbb{Z}^{\geq 0}\}$
- γ : the highest root of $\mathfrak{g}$
- $c_{\mathfrak{q}}$: the number in $\frac{1}{2}\mathbb{Z}^{\geq 0}$ so that $\rho(\mathfrak{n}) = c_{\mathfrak{q}}\lambda_{\mathfrak{q}}$

Results

- $\mathfrak{g}$ = complex simple Lie algebra of type E_6, E_7, or E_8.
- $\mathfrak{q}$ = maximal parabolic subalgebra of non-Heisenberg two-step nilpotent type

Write such parabolic subalgebra by a pair of the type of $\mathfrak{g}$ and the simple root $\alpha_{\mathfrak{q}}$:

$$(E_6, \alpha_3),\ (E_6, \alpha_5),\ (E_7, \alpha_2),\ (E_7, \alpha_6),\ (E_8, \alpha_1).$$

(Here the Bourbaki convention is used for the numbering of simple roots.)

Theorem B:

If $\mathfrak{g}$ and $\mathfrak{q}$ are as above, then $M_{\mathfrak{q}}[t]$ is reducible exactly at the following t:

(E_6, α_3) :	$t \in (2 + \mathbb{Z}_{\geq 0}) \cup (\frac{3}{2} + \mathbb{Z}_{\geq 0})$
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(E_7, α_6) :	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0})$
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- $\mathfrak{g}$ = complex simple Lie algebra of type E_6, E_7, or E_8.

- $\mathfrak{q}$ = maximal parabolic subalgebra of non-Heisenberg two-step nilpotent type

Write such parabolic subalgebra by a pair of the type of $\mathfrak{g}$ and the simple root $\alpha_{\mathfrak{q}}$:

$$(E_6, \alpha_3),\ (E_6, \alpha_5),\ (E_7, \alpha_2),\ (E_7, \alpha_6),\ (E_8, \alpha_1).$$

(Here the Bourbaki convention is used for the numbering of simple roots.)

Theorem B:

If $\mathfrak{g}$ and $\mathfrak{q}$ are as above, then $M_{\mathfrak{q}}[t]$ is reducible exactly at the following t:

(E_6, α_3) :	$t \in (2 + \mathbb{Z}_{\geq 0}) \cup (\frac{3}{2} + \mathbb{Z}_{\geq 0})$
(E_6, α_5) :	$t \in (2 + \mathbb{Z}_{\geq 0}) \cup (\frac{3}{2} + \mathbb{Z}_{\geq 0})$
(E_7, α_2) :	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{5}{2} + \mathbb{Z}_{\geq 0})$
(E_7, α_6) :	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{1}{2} + \mathbb{Z}_{\geq 0})$
(E_8, α_1) :	$t \in (1 + \mathbb{Z}_{\geq 0}) \cup (\frac{5}{2} + \mathbb{Z}_{\geq 0})$

Proof:

- The proof is a case-by-case analysis.
- The summary of $c_{\mathfrak{q}}$, $\mathrm{ht}(\mathfrak{g}(0))$, $\mathrm{ht}(\mathfrak{g}(1))$, and $\mathrm{ht}(\mathfrak{g}(2))$:

Parabolic $\mathfrak{q}$	$c_{\mathfrak{q}}$	$\mathrm{ht}(\mathfrak{g}(0))$	$\mathrm{ht}(\mathfrak{g}(1))$	$\mathrm{ht}(\mathfrak{g}(2))$
(E_6, α_3) :	$\frac{9}{2}$	4	8	11
(E_6, α_5) :	$\frac{9}{2}$	4	8	11
(E_7, α_2) :	7	6	13	17
(E_7, α_6) :	$\frac{13}{2}$	7	12	17
(E_7, α_1) :	$\frac{23}{2}$	11	22	29

- Observe that we have $\mathrm{ht}(\mathfrak{g}(2)) - \mathrm{ht}(\mathfrak{g}(1)) + \mathrm{ht}(\mathfrak{g}(0)) > c_{\mathfrak{q}}$ for each case.

(Ex: (E_6, α_3):

$$\mathrm{ht}(\mathfrak{g}(2)) - \mathrm{ht}(\mathfrak{g}(1)) + \mathrm{ht}(\mathfrak{g}(0)) = 11 - 8 + 4 = 7 > \frac{9}{2} = c_{\mathfrak{q}})$$

Case of (E_6, α_3)

- It suffices to consider $t \in \frac{1}{2}\mathbb{Z}^{>0} = (1 + \mathbb{Z}^{\geq 0}) \cup (\frac{1}{2} + \mathbb{Z}^{\geq 0})$.

- Handle the cases that $t \in 1 + \mathbb{Z}^{\geq 0}$ and $t \in \frac{1}{2} + \mathbb{Z}^{\geq 0}$, separately, since roots in S_t are different ($S_t = \{\beta \in \Delta(\mathfrak{n}) \mid \langle t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \beta^\vee \rangle \in 1 + \mathbb{Z}^{\geq 0}\}$):

  - If $t \in 1 + \mathbb{Z}^{\geq 0}$, then
  $$S_t = \{\beta \in \Delta(\mathfrak{g}(2)) \mid \mathrm{ht}(\beta) > 9 - 2t\}.$$
  - If $t \in \frac{1}{2} + \mathbb{Z}^{\geq 0}$, then
  $$S_t = \{\beta \in \Delta(\mathfrak{g}(1)) \mid \mathrm{ht}(\beta) > (9/2) - t\} \cup \{\beta \in \Delta(\mathfrak{g}(2)) \mid \mathrm{ht}(\beta) > 9 - 2t\}.$$

Theorem A:

Let $\mathfrak{g}$ be a complex simple simply laced Lie algebra, and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be as above. Suppose that $\mathrm{ht}(\mathfrak{g}(2)) - \mathrm{ht}(\mathfrak{g}(1)) + \mathrm{ht}(\mathfrak{g}(0)) > c_{\mathfrak{q}}$. If there exists $m \in \frac{1}{2}\mathbb{Z}^{>0}$ so that

- $\gamma \in S_m$ and
- $\langle m\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$,

then $M_{\mathfrak{q}}[t]$ is reducible for all $t \in m + \mathbb{Z}^{\geq 0}$,

where

- $S_m = \{\beta \in \Delta(\mathfrak{n}) \mid \langle m\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \beta^\vee \rangle \in 1 + \mathbb{Z}^{\geq 0}\}$
- γ : the highest root of $\mathfrak{g}$
- $c_{\mathfrak{q}}$: the number in $\frac{1}{2}\mathbb{Z}^{\geq 0}$ so that $\rho(\mathfrak{n}) = c_{\mathfrak{q}}\lambda_{\mathfrak{q}}$

By inspection, for γ the highest root of $\mathfrak{g}$, we have

$$\gamma \in S_2 \text{ and } \langle 2\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$$

and

$$\gamma \in S_{\frac{3}{2}} \text{ and } \langle \tfrac{3}{2}\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l}).$$

Note: For $\alpha \in \Delta$, we have

$$\langle t\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \alpha^\vee \rangle = \begin{cases} \mathrm{ht}(\alpha) & \text{if } \alpha \in \Delta(\mathfrak{l}) \\ (t - c_{\mathfrak{q}}) + \mathrm{ht}(\alpha) & \text{if } \alpha \in \Delta(\mathfrak{g}(1)) \\ 2(t - c_{\mathfrak{q}}) + \mathrm{ht}(\alpha) & \text{if } \alpha \in \Delta(\mathfrak{g}(2)). \end{cases}$$

By Theorem A, $M_{\mathfrak{q}}[t]$ is reducible for all $t \in (2 + \mathbb{Z}^{\geq 0}) \cup (\frac{3}{2} + \mathbb{Z}^{\geq 0})$.

- To determine the cases that $t = 1, \frac{1}{2}$, we use more involved observation on Jantzen's criterion and heights of roots.

By inspection, for γ the highest root of $\mathfrak{g}$, we have

$$\gamma \in S_2 \text{ and } \langle 2\lambda_{\mathfrak{q}} + \rho(\mathfrak{l}), \gamma^\vee \rangle > \mathrm{ht}(\mathfrak{l})$$

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By Theorem A, $M_{\mathfrak{q}}[t]$ is reducible for all $t \in (2 + \mathbb{Z}^{\geq 0}) \cup (\frac{3}{2} + \mathbb{Z}^{\geq 0})$.

• To determine the cases that $t = 1, \frac{1}{2}$, we use more involved observation on Jantzen's criterion and heights of roots.

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