A note on some theorem of Gelfand and Kirillov[a)]

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In this we show how Lie field $D(g)$[1,16] can be useful in investigations of universal enveloping algebra $U(g)$[4] of Lie algebra g. Due to validity of Ore condition in $D(g)$, any system of linear equations with coefficients and right hand sides from $U(g)$ can be solved in $D(g)$. Solving suitable system we could find new elements having "better" commutation relations by means of which, on the contrary, generators of g could be expressed.

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I. INTRODUCTION

The main idea of this article can be illustrated for the case $g = \mathrm{gl}(2,\mathbb{C})$ where $q := E_{12}^{-1}E_{11} \in D(g)$ and $p := E_{12}$ commute as Heisenberg pair and $C = E_{11} + E_{22}$ is an element from the center of $U(\mathrm{gl}(2,\mathbb{C}))$. These relations can be reversed and we obtain realization[2,3,5,8–15,19] of $\mathrm{gl}(2,\mathbb{C})$: $E_{11} = pq$, $E_{12} = p$, $E_{22} = -pq + C$.

Let us now consider $g = \mathrm{gl}(n,\mathbb{C})$. We will show that "small" extension $\widetilde{U}(g)$ of $U(g)$ is isomorphic to $U(H_{2(n-1)} \oplus \mathrm{gl}(n-1,\mathbb{C}) \oplus Z)$, where $\dim Z = 1$ and $H_{2(n-1)}$ is $2(n-1)$-dimensional Heisenberg algebra. Main technical tool for our proof are Capelli identities[18]. This result consists in generalization of Hamilton-Cayley equation for matrix $E = (E_{ij})_{i,j=1}^{n}$, namely

$$E^n + a_{n-1}E^{n-1} + ... + a_1 E + a_0 = 0, \tag{1}$$

where coefficients a_k belong to the center $Z(g)$ of g[17]. From this equation we obtain explicit form of "inverse" matrix E^{-1} of E. The alternative proof of (1) is given in the appendix.

II. PRELIMINARIES

Let E_{ij}, $i,j = 1,...,n$ be generators of the Lie algebra $\mathrm{gl}(n,\mathbb{C})$ satisfying usual commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}, \qquad i,j = 1,...,n.$$

Then the center $Z(U(g))$ of the enveloping algebra $U(g)$ is generated by Casimir elements (Gelfand generators)

$$C_s = \sum_{i1,i2,...,is=1}^{n} E_{i_1,i_2}E_{i_2,i_3}...E_{i_s,i_1}, \qquad s = 1,...,n.$$

Let us denote the $n \times n$ matrix having its (i,j)-entry defined as E_{ij} by E. Let us define the matrix $E(u)$, where $u \in \mathbb{C}$, by

$$E(u) = (E + (u - i + 1)\delta_{ij})_{i,j=1}^{n}$$

and take

$$C(u) = \mathrm{coldet}\, E(u) = \sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma\ (E + uI)_{\sigma(1),1}(E + (u-1)I)_{\sigma(2),2}...(E + (u-n+1)I)_{\sigma(n),n},$$

where I is $n \times n$ identity matrix. We often abbreviate $E + uI$ by $E + u$ in what follows. The column determinant $C(u)$ can be rewritten in the collected form

$$C(u) = \sum_{k=0}^{n} a_k u^k. \tag{2}$$

In the appendix we give an easy proof for the well known fact:[18]

Theorem 1. *a) We have*

$$[C(u), E_{ij}] = 0,$$

therefore the coefficients a_k in (2) are, in fact, functions of Casimir elements C_1, ..., C_{n-k}.
b) We have

$$C(-E + n - 1) = 0, \qquad C(-E^T) = 0,$$

where E^T is usual transposition, $(E^T)_{ij} = E_{ji}$.

Theorem 1 b) allows to write down the explicit formulas for "inverse" matrices $(E - n + 1)^{-1}$ and $(E^T)^{-1}$ which satisfy

$$(E-n+1)^{-1}(E-n+1) = (E-n+1)(E-n+1)^{-1} = C(0)I, \quad (E^T)^{-1}E^T = E^T(E^T)^{-1} = C(0)I.$$

Namely we have

$$(E - n + 1)^{-1} = \sum_{k=1}^{n} a_k(-E + n - 1)^{k-1}, \quad (E^T)^{-1} = \sum_{k=1}^{n} a_k(-E^T)^{k-1}.$$

Apparently, "inverse" matrices exist for any matrix of the form $E + u$ or $E^T + u$, $u \in \mathbb{C}$.

III. HEISENBERG PAIRS

Let us solve (in $D(g)$) the set of linear equations

$$E_{in} = \sum_{j=1}^{n} E_{ij} q_j, \quad i = 1, ..., n - 1. \tag{3}$$

for unknown variables q_1, ..., q_{n-1}. We have

$$q_j = C^{-1} E_{ij}^{-1} E_{jn},$$

where C is some central element from $Z(U(\mathrm{gl}(n - 1)))$.

Lemma 2. *We have*

$$[E_{kl}, q_j] = \delta_{jl} q_k. \tag{4}$$

Proof. Commuting (3) with E_{kl} we obtain the relation

$$\delta_{il} E_{kn} = \delta_{li} E_{kj} q_j - E_{il} q_k + E_{ij} [E_{kl}, q_j]$$

and then, using (3) again, we come to the equation

$$E_{ij} \left([E_{kl}, q_j] - \delta_{jl} q_k \right) = 0.$$

Applying inverse matrix E^{-1}, we get (4). □

Lemma 3. *We have*

$$[E_{in}, q_j] = q_j q_i.$$

Proof. Commuting eq. (3) with E_{kn} we have

$$0 = -\delta_{jk} E_{in} q_j + E_{ij} [E_{kn}, q_j].$$

Using again (3) we conclude, that

$$E_{ij} \left([E_{kn}, q_j] - q_j q_k \right) = 0.$$

□

Theorem 4. *We have*

$$[q_i, q_j] = 0.$$

Proof. We commute eq. (3) with q_k and use the results above. We obtain

$$q_k q_i = q_i q_k + E_{ij} [q_j, q_k],$$

which can be written in the form

$$(E_{ij} + \delta_{ij}) [q_j, q_k] = 0.$$

Commuting q_j with E_{ij} in (3) we get

$$E_{in} = q_j \left(E_{ij} + (n-1) \delta_{ij} \right) = q_j \left(E^T_{ji} + (n-1) \delta_{ji} \right). \tag{5}$$

This is linear equation for the unknown q_j with the following solution:

$$q_j = E_{in} (E^T + n - 1)^{-1}_{ij} \tilde{C}^{-1},$$

where $\tilde{C}$ is some central element from $Z(U(g))$. □

Theorem 5. *We have*

$$[q_i, E_{nj}] = \delta_{ij}, \quad E_{nn} = E_{ni} q_i.$$

Proof. Commuting eqs. (3) and (5) with E_{nk} we obtain

$$C([q_l, E_{nk}] - \delta_{lk}) = E^{-1}_{lk}(E_{nj} q_j - E_{nn}), \tag{6}$$
$$([q_l, E_{nk}] - \delta_{lk})\tilde{C} = (q_j E_{nj} - E_{nn} - n + 1)(E^T + n - 1)^{-1}_{kl}.$$

Multiplying the first equation by $\tilde{C}$ from the left hand side and the second by C from the right hand side and subtracting the equations we obtain

$$E^{-1}_{lk}(E_{nj} q_j - E_{nn})\tilde{C} = C(q_j E_{nj} - E_{nn} - n + 1)(E^T + n - 1)^{-1}_{kl}.$$

Multiplying by E_{il} from the left and by $(E^T + n - 1)_{li}$ from the right we get

$$C(E_{nj} q_j - E_{nn})\tilde{C} = C(q_j E_{nj} - E_{nn} - n + 1)\tilde{C}$$

and because the elements C, $\tilde{C}$, $E_{nj} q_j - E_{nn}$ and $E_{jn} q_j - E_{nn}$ commute with E_{ij} (and due to nonexistence of nonzero divisors of zero) we have

$$q_j E_{nj} - E_{nj} q_j = n - 1, \qquad E_{nn} = E_{nj} q_j.$$

From (6) we get the desired assertion. □

IV. SUBALGEBRA $\mathrm{gl}(n-1, \mathbb{C})$ OF $D(\mathrm{gl}(n, \mathbb{C}))$

Let us have any $c \in Z(U(g))$ and define

$$F_{ij} = E_{ij} - q_i p_j + c,$$

where

$$p_j = -E_{nj}.$$

Then we have

Theorem 6. *a)*

$$[F_{ij}, q_k] = [F_{ij}, p_k] = 0,$$
$$[F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj}.$$

Proof. These relations can be verified by direct computation. □

Let us define the following subalgebra of $D(\mathrm{gl}(n,\mathbb{C}))$:

$$\widetilde{U}(\mathrm{gl}(n,\mathbb{C})) = \mathrm{span}\left\{ q_1^{r_1}...q_{n-1}^{r_{n-1}} p_1^{s_1}...p_{n-1}^{s_{n-1}} F_{11}^{t_{11}} F_{12}^{t_{12}}...F_{n-1,n-1}^{t_{n-1,n-1}} c^u \middle| r_i, s_i, t_{ij}, u = 1, 2, ..., n-1 \right\}.$$

Because

$$\begin{aligned}
E_{ij} &= F_{ij} + q_i p_j - c, \\
E_{in} &= (F_{ij} + q_i p_j - c) q_j, \\
E_{ni} &= -p_i, \\
E_{nn} &= E_{ni} q_i,
\end{aligned}$$

$\widetilde{U}(\mathrm{gl}(n,\mathbb{C}))$ is extension of enveloping algebra $U(\mathrm{gl}(n,\mathbb{C}))$. On the other side subalgebra $\widetilde{U}(\mathrm{gl}(n,\mathbb{C}))$ is isomorphic to the enveloping algebra of direct sum of $(2n+1)$-dimensional Heisenberg algebra H_{2n+1}, Lie algebra $\mathrm{gl}(n-1,\mathbb{C})$ and 1-dimensional algebra $\mathrm{span}\{c\}$:

$$U\left(H_{2n+1} \oplus \mathrm{gl}(n-1,\mathbb{C}) \oplus \mathrm{span}\{c\}\right) \simeq \widetilde{U}(\mathrm{gl}(n,\mathbb{C})).$$

Appendix A: Inverse matrix to $E + u$

Let $\tau\colon \{1, ..., n\} \to \{1, ..., n\}$, not necessarily bijective. Let us denote

$$C_\tau(u) = \sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma\ (E+u)_{\sigma(1),\tau(1)} (E+u-1)_{\sigma(2),\tau(2)} ... (E+u-n+1)_{\sigma(n),\tau(n)}.$$

Lemma 7. *We have*

$$C_{\tilde{\tau}}(u) = -C_\tau(u),$$

where $\tilde{\tau}$ is equal to τ, except for two positions, which are interchanged, i. e. $\tilde{\tau}(s) = \tau(s+1)$, $\tilde{\tau}(s+1) = \tau(s)$, $\tilde{\tau}(j) = \tau(j)$ for $j = 1, ..., s-1, s+2, ..., n$ for some $s \in \{1, ..., n\}$.

Proof. We have

$$C_{\tilde{\tau}}(u)=\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\ (E+u)_{\sigma(1),\tau(1)}(E+u-1)_{\sigma(2),\tau(2)}$$
$$...(E+u-s+1)_{\sigma(s),\tau(s+1)}(E+u-s)_{\sigma(s+1),\tau(s)}...(E+u-n+1)_{\sigma(n),\tau(n)}=$$
$$-\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\ (E+u)_{\sigma(1),\tau(1)}(E+u-1)_{\sigma(2),\tau(2)}$$
$$...(E+u-s+1)_{\sigma(s+1),\tau(s+1)}(E+u-s)_{\sigma(s),\tau(s)}...(E+u-n+1)_{\sigma(n),\tau(n)}=$$
$$-\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\ (E+u)_{\sigma(1),\tau(1)}(E+u-1)_{\sigma(2),\tau(2)}$$
$$...\big((E+u-s)_{\sigma(s),\tau(s)}(E+u-s+1)_{\sigma(s+1),\tau(s+1)}+\delta_{\tau(s+1),\sigma(s)}E_{\sigma(s+1),\tau(s)}-$$
$$\delta_{\sigma(s+1),\tau(s)}E_{\sigma(s),\tau(s+1)}\big)...(E+u-n+1)_{\sigma(n),\tau(n)}=$$
$$-\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\ (E+u)_{\sigma(1),\tau(1)}(E+u-1)_{\sigma(2),\tau(2)}...(E+u-n+1)_{\sigma(n),\tau(n)}.$$

□

Because every permutation can be written as a product of transpositions of adjacent elements, we have

$$C_\tau(u)=\operatorname{sgn}\tau\ C(u), \tag{A1}$$

where $\operatorname{sgn}\tau=0$ in the case when τ is not bijective.

Lemma 8.

$$C(u)\in Z(U(\mathrm{gl}(n,\mathbb{C})).$$

Proof. Let us take, without loss of generality, $u=0$. Let us denote $F=(E_{ij}+(1-i)\delta_{ij})_{i,j=1}^n$. Because $\mathrm{gl}(n,\mathbb{C})$ is generated by the elements e_{ii}, $e_{j,j+1}$ and $e_{j+1,j}$, $i=1,...,n$, $j=1,...,n-1$, it is sufficient to show that $C(0)$ commutes with mentioned generators.

1. First we show that for each $\sigma\in S_n$ we have

$$[F_{\sigma(1)1}...F_{\sigma(n)n},e_{ii}]=0.$$

If $\sigma(i)=i$, then

$$[\underbrace{F_{\sigma(1)1}...}_{=A}F_{\sigma(i)i}\underbrace{...F_{\sigma(n)n}}_{=B},e_{ii}]=A[F_{ii},e_{ii}]B=A\cdot 0\cdot B=0.$$

If $\sigma(i) \neq i$, then

$$[\underbrace{F_{\sigma(1)1} \ldots}_{=A} F_{\sigma(i)i} \underbrace{\ldots}_{=B} F_{i\sigma^{-1}(i)} \underbrace{\ldots F_{\sigma(n)n}}_{=C}, e_{ii}] =$$

$$[Ae_{\sigma(i)i}Be_{i\sigma^{-1}(i)}C, e_{ii}] = A[e_{\sigma(i)i}, e_{ii}]Be_{i\sigma^{-1}(i)}C + Ae_{\sigma(i)i}B[e_{i\sigma^{-1}(i)}, e_{ii}]C,$$

and because of

$$[e_{\sigma(i)i}, e_{ii}] = e_{\sigma(i)i}, \quad [e_{i\sigma^{-1}(i)}, e_{ii}] = -e_{i\sigma^{-1}(i)},$$

we get

$$A[e_{\sigma(i)i}, e_{ii}]Be_{i\sigma^{-1}(i)}C + Ae_{\sigma(i)i}B[e_{i\sigma^{-1}(i)}, e_{ii}]C = Ae_{\sigma(i)i}Be_{i\sigma^{-1}(i)}C - Ae_{\sigma(i)i}Be_{i\sigma^{-1}(i)}C = 0.$$

2. Now we show that

$$[C(0), e_{i,i+1}] = 0.$$

Let us denote by $\widetilde{E}$ the matrix whose entries are commutators of $e_{i,i+1}$ and components of E, i. e.

$$\widetilde{E}_{jk} = [e_{i,i+1}, E_{jk}].$$

Then thanks to

$$[e_{i,i+1}, e_{jk}] = \delta_{i+1,j}e_{ik} - \delta_{ik}e_{j,i+1}$$

the matrix $\widetilde{E}$ takes the following form (it has nonzero elements only on the $i+1$-st row and in the i-th column):

$$\widetilde{E} = \begin{array}{cc} \begin{array}{c} i\text{-th column} \end{array} & \\ \begin{pmatrix} & & & -e_{1,i+1} & & & \\ & 0 & & -e_{1,i+2} & & 0 & \\ & & & \vdots & & & \\ e_{i1} & e_{i2} & \cdots & e_{ii} - e_{i+1,i+1} & \cdots & e_{i,n-1} & e_{in} \\ & & & \vdots & & & \\ & 0 & & -e_{n,i+1} & & 0 & \end{pmatrix} & \begin{array}{c} \\ \\ \\ \text{- i+1-th row} \\ \\ \\ \end{array} \end{array}.$$

Let us now enumerate $D = [e_{i,i+1}, C_n]$. From the linearity of determinant, it follows that

$$D = \sum_{j=1}^{n} D_j - D'_i,$$

where D_j is determinant of matrix, which we can get from matrix F by replacing its j-th column by j-th column of matrix

$$\begin{array}{cl} \begin{pmatrix} & 0 & & & & 0 & \\ & & & & & & \\ e_{i1} & e_{i2} & \cdots & e_{ii}+n-i & \cdots & e_{i,n-1} & e_{in} \\ & & & & & & \\ & 0 & & & & 0 & \end{pmatrix} & \text{- i+1-th row}, \end{array} \tag{A2}$$

and D'_i is determinant of matrix, which we can get from matrix F by replacing its i-th column by i-th column of matrix

$$\begin{array}{c} \text{i-th column} \\ \begin{pmatrix} & e_{1,i+1} & \\ 0 & e_{2,i+1} & 0 \\ & \vdots & \\ & (e_{i+1,i+1}+n-i) & \\ & \vdots & \\ 0 & e_{n,i+1} & 0 \end{pmatrix} \end{array}. \tag{A3}$$

(the constant $n-i$ is added on the position $(i+1, i)$ into (A2) and into (A3)).

The determinant D_j is not changed by the following transformation: in place of zeros in the j-th column we set to zero $i+1$-th row (except for the element e_{ij}). This is because of the following identity, valid for both commutative and non-commutative determinants:

$$\det \begin{pmatrix} & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \\ \star & \cdots & \star & w & \star & \cdots & \star \\ & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \end{pmatrix} = \det \begin{pmatrix} & & & \star & & & \\ & & & \vdots & & & \\ & & & \star & & & \\ 0 & \cdots & 0 & w & 0 & \cdots & 0 \\ & & & \star & & & \\ & & & \vdots & & & \\ & & & \star & & & \end{pmatrix},$$

where w and stars are arbitraty elements. Then we can see that the sum

$$\sum_{j=1}^{n} D_j = 0,$$

because suming up we get determimant with two equal rows, i-th and $i+1$-st. Finally, the determinant D'_i has the following form:

$$D'_i = \det \begin{array}{c} \begin{array}{cccc} \quad & \scriptstyle i\text{-th column} & \scriptstyle i+1\text{-st column} & \quad \end{array} \\ \begin{pmatrix} & e_{1,i+1} & e_{1,i+1} & \\ \star & e_{2,i+1} & e_{2,i+1} & \star \\ & \vdots & \vdots & \\ & (e_{i+1,i+1}+n-i) & (e_{i+1,i+1}+n-i-1) & \\ & \vdots & \vdots & \\ \star & e_{n,i+1} & e_{n,i+1} & \star \end{pmatrix} \end{array}. \tag{A4}$$

Enumerating D'_i we encounter the 2×2 determimnants of the following two types:

$$\text{a)} \ \det \begin{pmatrix} e_{j,i+1} & e_{j,i+1} \\ e_{k,i+1} & e_{k,i+1} \end{pmatrix}, \quad \text{where } j \neq i+1, \ k \neq i+1,$$

this determinant is equal to zero, because $e_{j,i+1}$ and $e_{k,i+1}$ commute (if $j \neq i+1$ and $k \neq i+1$), and

$$\text{b)} \ \det \begin{pmatrix} e_{j,i+1} & e_{j,i+1} \\ e_{i+1,i+1}+n-i & e_{i+1,i+1}+n-i-1 \end{pmatrix}, \quad \text{where } j \neq i+1,$$

this determinant is equal to zero, because we get

$$\det \begin{pmatrix} e_{j,i+1} & e_{j,i+1} \\ e_{i+1,i+1}+n-i & e_{i+1,i+1}+n-i-1 \end{pmatrix} = (n-i-1-(n-i))e_{j,i+1} + [e_{j,i+1}, e_{i+1,i+1}] =$$

$\square$

Let us now define τ_k by

$$\tau_k(1) = 1, \ \ldots, \ \tau_k(k-1) = k-1, \ \tau_k(k) = k+1, \ \ldots, \tau_k(n-1) = n, \ \tau_k(n) = k.$$

The relation (A1) can be rewritten as

$$\sum_{\sigma \in S_n} \operatorname{sgn} \sigma \ (E+u)_{\sigma(1),\tau_k(1)}(E+u-1)_{\sigma(2),\tau_k(2)} \ldots (E+u-n+1)_{\sigma(n),\tau_k(n)} = \operatorname{sgn} \tau_k \ C(u),$$

which implies

$$\sum_{j=1}^{n}\left(\sum_{\sigma\in S_n,\ \sigma(n)=j}\operatorname{sgn}\sigma\ (E+u)_{\sigma(1),\tau_i(1)}(E+u-1)_{\sigma(2),\tau_i(2)}\right.$$
$$\left.\dots\ (E+u-n+2)_{\sigma(n-1),\tau_i(n-1)}\right)(E+u-n+1)_{j,k}=(-1)^{n-k}\delta_{ik}C(u),$$

which can be interpreted as a matrix equation

$$\sum_{j=1}^{n}B_{ij}(E+u-n+1)_{jk}=\delta_{ik}C(u),$$

which implies that matrix B is left inverse to $E+u-n+1$. Similarly, we can find right inverse using row instead of column determinants.

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