Global Geometry and Analysis on Locally Symmetric Spaces with Indefinite-metric

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References

(Geometry)

[1] K– and T. Yoshino,
Compact Clifford–Klein forms of symmetric spaces—revisited, Pure and Appl. Math. Quarterly 1 (2005), 603-684, Special Issue: In Memory of Armand Borel,

(Analysis)

[2] K– ,
Hidden symmetries and spectrum of the Laplacian on an indefinite Riemannian manifold, Spectral Analysis in Geometry and Number Theory Contemp. Math., 484, Amer. Math. Soc., 2009. Special Issue: in honor of T. Sunada

[3] F. Kassel and K– ,
Poincaré series for non-Riemannian locally symmetric spaces, arXiv: 1209.4075.

[4] work in progress

Shorter strings produce a higher pitch than longer strings.

Thinner strings produce a higher pitch than thicker strings.

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}$$

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Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

Γ	$\subset$	G	$\supset$	H
discrete subgp		Lie group		subgroup

$SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R}) \supset SO(2)$

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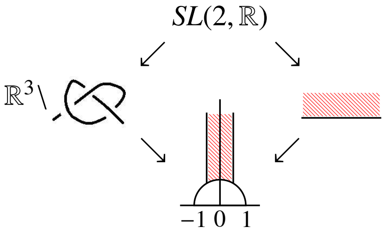

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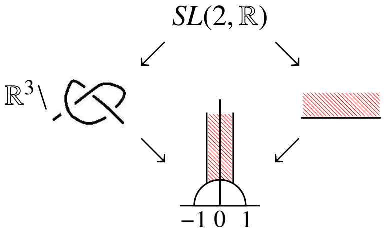
$$



Special cases are already deep and difficult.

- $\Gamma = \{e\}$
- H compact
- $G = \mathbb{R}^{p,q}$

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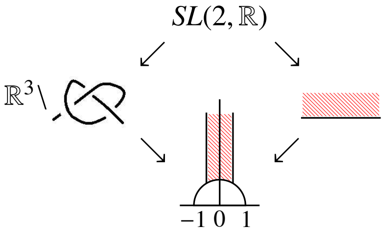

Special cases are already deep and difficult.

- $\Gamma = \{e\} \cdots$ non-commutative harmonic analysis on $L^2(G/H)$
  Gelfand, Harish-Chandra, T. Oshima, Delorme, . . .
- H compact
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Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$\Gamma \subset G \supset H$ — Γ: discrete subgp, G: Lie group, H: subgroup

$SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R}) \supset SO(2)$

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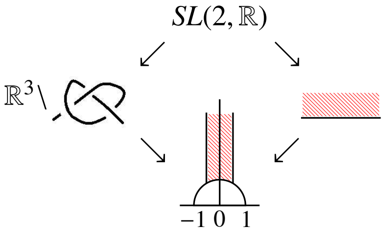

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  Siegel, Selberg, Piateski-Shapiro, Langlands, Arthur, Sarnak, Müller, . . .
- $G = \mathbb{R}^{p,q}$

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

Γ (discrete subgp) $\subset$ G (Lie group) $\supset$ H (subgroup) $\qquad SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R}) \supset SO(2)$

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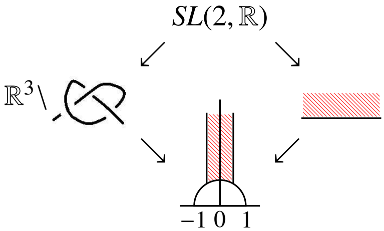

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- H compact $\cdots$ automorphic forms
  Siegel, Selberg, Piateski-Shapiro, Langlands, Arthur, Sarnak, Müller, . . .
- $G = \mathbb{R}^{p,q}$ (abelian, but non-Riemannian), $\Gamma = \mathbb{Z}^{p+q}$
  Oppenheim conjecture, Dani–Margulis, Ratner, . . .

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}$$

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New challenge: Initiate spectral analysis of Δ on $\Gamma\backslash G/H$
in a more general setting (non-abelian G and non-compact H).

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Further difficulties arise

- (geometry)

- (analysis)
- (representation theory)

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- (geometry) existence of good geometry $\Gamma\backslash G/H$?
  study of "local to global" beyond Riemannian setting
- (analysis)
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Further difficulties arise

- (geometry) existence of good geometry $\Gamma\backslash G/H$?
  study of "local to global" beyond Riemannian setting
- (analysis) Laplacian is no more elliptic.
- (representation theory) $\mathrm{vol}(\Gamma\backslash G) = \infty$ even when $\Gamma\backslash G/H$ is compact

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

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> New challenge: Initiate spectral analysis of Δ on $\Gamma\backslash G/H$
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Further difficulties arise

$\rightsquigarrow$ need to change methods for the study!

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1. Laplacian on $\mathbb{R}$
2. Quick course to Riemannian and Lorentzian geometry
3. Elementary definition of Laplacian Δ on pseudo-Riemannian manifold
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Eigenvalue of Laplacian (1-dim'l case)

$$\Delta = -\frac{d^2}{dx^2} \text{ on } \mathbb{R} \text{ (real line)}$$

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Definition $f(x)$ is an eigenfunction of Δ

$$\Longleftrightarrow \Delta \boxed{f(x)} = \lambda \boxed{f(x)}$$

λ: eigenvalue

Eigenvalue of Laplacian (3-dim'l case)

$$\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \text{ on } \mathbb{R}^3$$

Definition $f(x, y, z)$ is an eigenfunction of Δ

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λ : eigenvalue

This is the Helmholtz equation

$\sqrt{\lambda}$: wavenumber

$f(x, y, z)$: amplitude

Spectrum of Laplacian (1-dim'l case)

$$\Delta = -\frac{d^2}{dx^2} \text{ on } \mathbb{R} \text{ (real line)}$$

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$$\iff \Delta \boxed{f(x)} = \lambda \boxed{f(x)}$$

$$\iff -f''(x) = \lambda f(x)$$

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$$f''(x) = -m^2 \sin mx = -m^2 f(x)$$

$$\therefore \Delta f = m^2 f$$

Namely, $\begin{cases} f(x) = \sin mx : \text{eigenfunction of } \Delta \\ m^2 : \text{eigenvalue } (m : \text{wavenumber}) \end{cases}$

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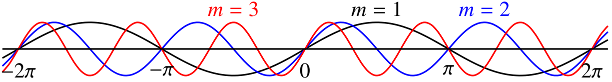

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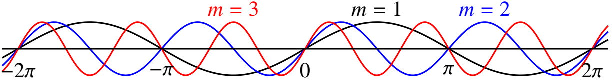

Periodic function For $m \in \mathbb{Z}$, $f(x) = \sin mx$ satisfies

$$\Delta f = m^2 f$$

$$f(x + 2\pi) = f(x) \quad \text{period } 2\pi$$

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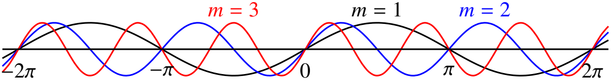


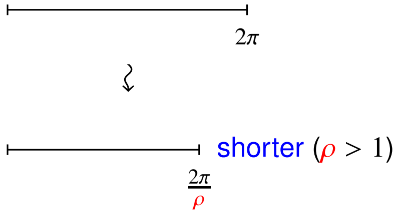
Periodic function For $m \in \mathbb{Z}$, $f(x) = \sin \rho m x$ satisfies

$$\Delta f = \rho^2 m^2 f$$

$$f\left(x + \frac{2\pi}{\rho}\right) = f(x) \qquad \text{period } \frac{2\pi}{\rho}$$

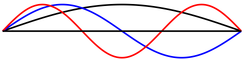
Shorter strings produce a higher pitch

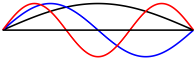
period



Shorter strings produce a higher pitch

frequency





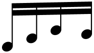
higher

period

shorter ($\rho > 1$)

Eigenvalues of Δ : $\{m^2\pi^2 : m = 1, 2, 3, \ldots\} \Rightarrow \{\rho^2 m^2 \pi^2 : m = 1, 2, 3, \ldots\}$

Shorter strings produce a higher pitch



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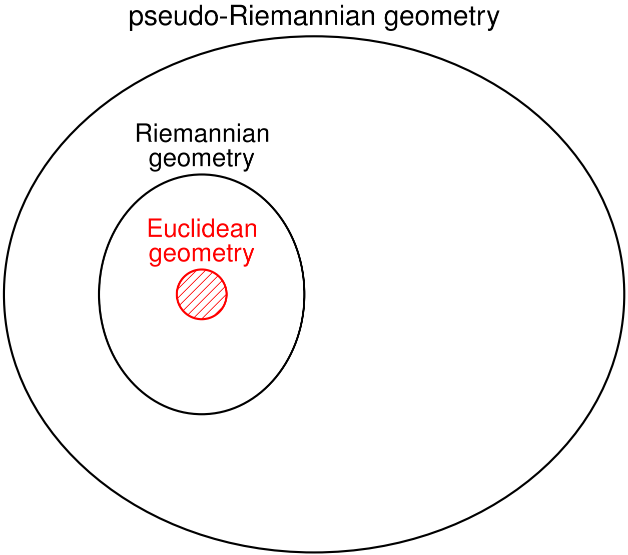
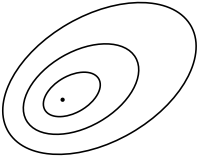
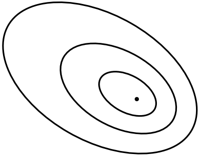
Introduction to pseudo-Riemannian geometry



Riemannian geometry

Loosely, balls with radius R (> 0) are defined at every point.



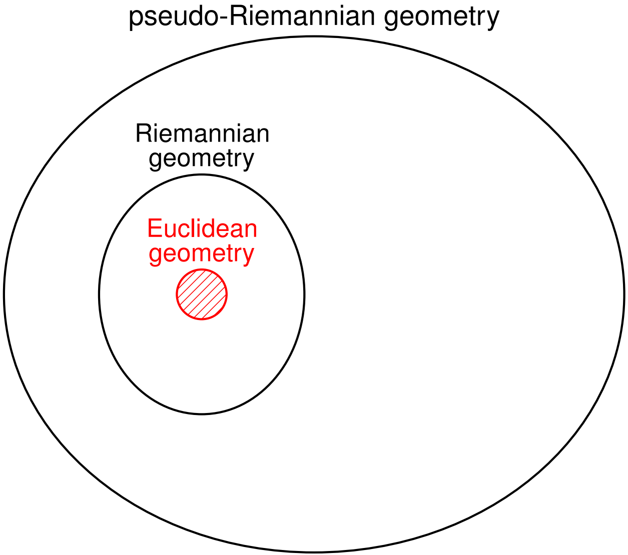


Riemannian geometry

Loosely, balls with radius R (> 0) are defined at every point.

A little more precisely, the ‘distance’ is integrated from the ‘infinitesimal distance.’

Quick course on pseudo-Riemannian geometry



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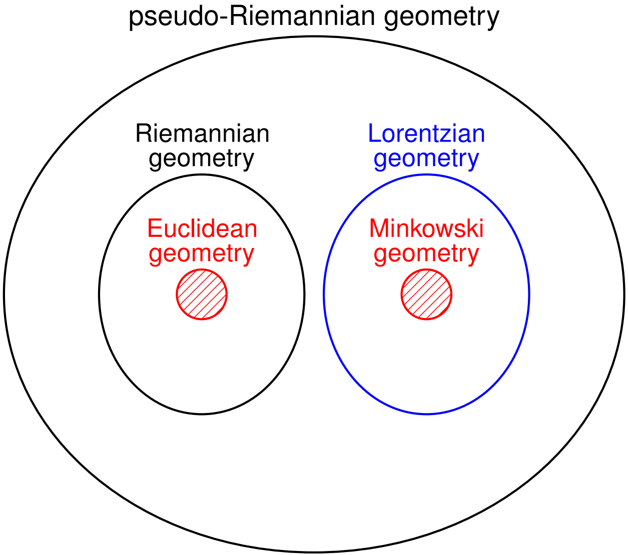
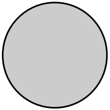


Riemannian ⇒ Lorentzian ⇒ pseudo-Riemannian

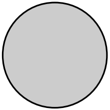
Euclidean space $\mathbb{R}^2$



$x^2 + y^2 \leq R^2$

Riemannian ⇒ Lorentzian ⇒ pseudo-Riemannian

Euclidean space $\mathbb{R}^2$ Minkowski space $\mathbb{R}^{1,1}$ $\mathbb{R}^{p,q}$



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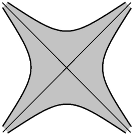


$\rightsquigarrow \cdots$

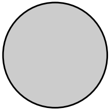
$x^2 + y^2 \leq R^2$

$|x^2 - y^2| \leq R^2$

$\left| \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 \right| \leq R^2$

Riemannian ⇒ Lorentzian ⇒ pseudo-Riemannian

Euclidean space $\mathbb{R}^2$ | Minkowski space $\mathbb{R}^{1,1}$ | $\mathbb{R}^{p,q}$



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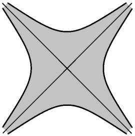

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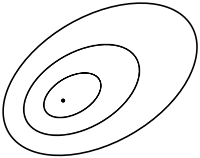
$x^2 + y^2 \leq R^2$ | $|x^2 - y^2| \leq R^2$ | $\left| \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 \right| \leq R^2$

$\wr$ | $\wr$ | $\wr$



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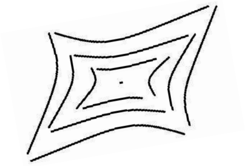
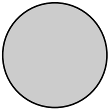


$\rightsquigarrow \cdots$

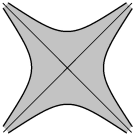
Riemannian manifold | Lorentzian manifold | pseudo-Riemannian

Riemannian ⇒ Lorentzian ⇒ pseudo-Riemannian

Euclidean space $\mathbb{R}^2$ | Minkowski space $\mathbb{R}^{1,1}$ | $\mathbb{R}^{p,q}$



$\rightsquigarrow$



$\rightsquigarrow \cdots$

$x^2 + y^2 \leq R^2$ | $|x^2 - y^2| \leq R^2$ | $\left| \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 \right| \leq R^2$

In higher dimensional case

generalize

$\mathbb{R}^n$ $\quad x_1{}^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 + x_n{}^2 \Longrightarrow$ Riemannian geometry

$\mathbb{R}^{n-1,1}$ $\quad x_1{}^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 - x_n{}^2 \Longrightarrow$ Lorentzian geometry

$\mathbb{R}^{p,q}$ $\quad x_1{}^2 + \cdots + x_p{}^2 - x_{p+1}{}^2 - \cdots - x_{p+q}{}^2 \Longrightarrow$ pseudo-Riemannian geometry

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Introduction to Laplacian Δ

Laplacian Δ

$\cdots$ 'Intrinsic' differential operator <u>in Riemannian geometry</u>

or more generally

<u>in pseudo-Riemannian geometry</u>

Goal: give an elementary definition of Laplacian Δ

$\cdots$ for undergraduate students

Laplacian in pseudo-Riemannian geometry

X Riemannian manifold

$f(x)$: function on X

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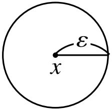

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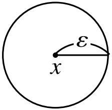

Obviously $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0$

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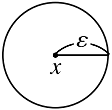

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Ex. $X = \mathbb{R}$ $\quad f_\varepsilon(x) = \dfrac{1}{2\varepsilon}\displaystyle\int_{x-\varepsilon}^{x+\varepsilon} f(t)dt$

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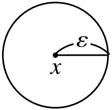

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An easy computation shows $\displaystyle\lim_{\varepsilon\to 0} \frac{f_\varepsilon(x) - f(x)}{\varepsilon^2} = \frac{1}{6}f''(x) = \frac{-1}{6}\Delta f(x)$

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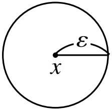

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Def (Laplacian) $\Delta f(x) := -6 \displaystyle\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}(f_\varepsilon(x) - f(x)) = -\operatorname{div}\circ\operatorname{grad} f$

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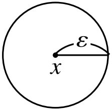

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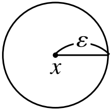
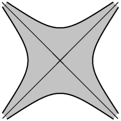


$\rightsquigarrow$



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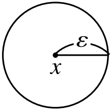

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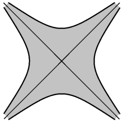

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$$\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right) \text{ for } X = \mathbb{R}^{2,2}$$

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Construction of eigenfunctions of Laplacian

How can we construct eigenfunctions f ?

$$\Delta f = \lambda f.$$

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$$\Delta f = \lambda f.$$

We shall explain its idea in the following case with $\lambda = 0$:

Example $\quad X = \mathbb{R}^{2,2}$

$= \mathbb{R}^4$ endowed with pseudo-Riemannian str.

$$dp^2 + dq^2 - dr^2 - ds^2$$

$$\Delta \equiv \Delta_{\mathbb{R}^{2,2}} = -\frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2}$$

Integral geometry

Idea of computer tomography (try to ‘see’ the inside)

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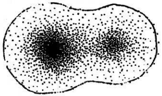

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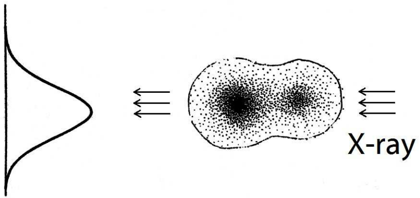

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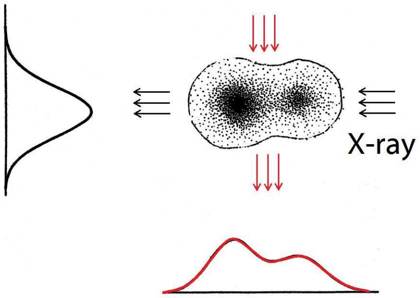

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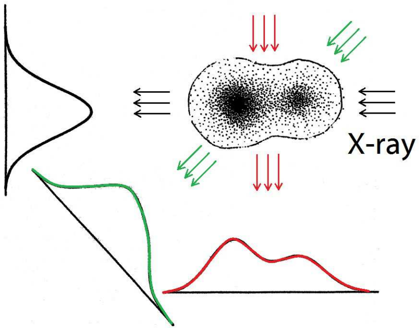

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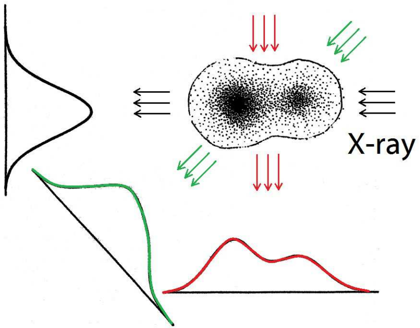

X = {points}, Y = {lines}

$$R : \Gamma(X) \to \Gamma(Y), \quad f \mapsto (Rf)(l) = \int_l f(x)$$

Integral geometry

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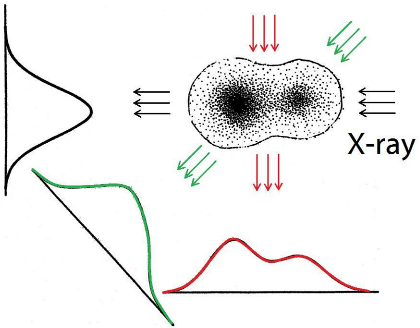

X = {points}, Y = {lines}

$$R : \Gamma(X) \to \Gamma(Y), \quad f \mapsto (Rf)(l) = \int_l f(x)$$

Find the inversion R^{-1} $\Longleftrightarrow$ 'see' the inside

Geometric construction of eigenfunctions

$X = \mathbb{R}^3$
$Y = \{\text{lines in } \mathbb{R}^3\} \doteqdot \mathbb{R}^4$

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Generic lines in $\mathbb{R}^3$ is given as

$$l_{a,b,c,d} := \{(t, at + b, ct + d) : t \in \mathbb{R}\}$$

with parameter $(a, b, c, d) \in \mathbb{R}^4$.

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$f(x, y, z)$ function on $X = \mathbb{R}^3$

$\mapsto$

$$(Rf)(a, b, c, d) := \int_{-\infty}^{\infty} f(t, at+b, ct+d)\,dt$$

function on $Y \doteqdot \mathbb{R}^4 \ni (a, b, c, d)$

Geometric construction of eigenfunctions

$$f(x, y, z) \mapsto Rf(a, b, c, d) = \int_{-\infty}^{\infty} f(t, at + b, ct + d)dt$$

$$\implies \left(\frac{\partial^2}{\partial a \, \partial d} - \frac{\partial^2}{\partial b \, \partial c} \right) Rf = 0 \text{ for any } f(x, y, z)$$

Geometric construction of eigenfunctions

$$f(x,y,z) \mapsto Rf(a,b,c,d) = \int_{-\infty}^{\infty} f(t, at+b, ct+d)dt$$
$$\Longrightarrow \left(\frac{\partial^2}{\partial a\,\partial d} - \frac{\partial^2}{\partial b\,\partial c}\right) Rf = 0 \text{ for any } f(x,y,z)$$

Proof.

$$\left(\frac{\partial^2}{\partial a\,\partial d} - \frac{\partial^2}{\partial b\,\partial c}\right) Rf(a,b,c,d)$$
$$= \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial a\,\partial d} - \frac{\partial^2}{\partial b\,\partial c}\right) f(t, at+b, ct+d)dt$$
$$= \int_{-\infty}^{\infty} t\left\{\frac{\partial^2 f}{\partial y\,\partial z}(t, at+b, ct+d) - \frac{\partial^2 f}{\partial y\,\partial z}(t, at+b, ct+d)\right\} dt$$
$$= 0$$

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$\Downarrow$ change of variables

$$a = p + s,\ b = r + q,\ c = r - q,\ d = p - s$$

$$\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2}\right) Rf = 0$$

$Rf(a,b,c,d)$ is an eigenfunction of the Laplacian $\Delta_{\mathbb{R}^{2,2}}$ with 0 eigenvalue for any function $f(x,y,z)$.

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F. John (1938): Use the Radon transform to construct a solution

$$\Delta_{\mathbb{R}^{2,2}} h = 0$$

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Periodic function and quotient space

Periodic function $f(x)$ on $\mathbb{R}$

$$f(x) = f(x+1)$$ period 1

Periodic function and quotient space

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$$f(x) = f(x+1) = f(x+2) = f(x+3) = \cdots \qquad \text{period } 1$$

Periodic function and quotient space

Periodic function $f(x)$ on $\mathbb{R}$

$$\begin{aligned} f(x) &= f(x+1) = f(x+2) = f(x+3) = \cdots \qquad \text{period } 1 \\ &= f(x-1) = f(x-2) = f(x-3) = \cdots \end{aligned}$$

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$\Longleftrightarrow$ $f(x)$ is a function of the decimal part of x

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$\rightsquigarrow$

visualize ?

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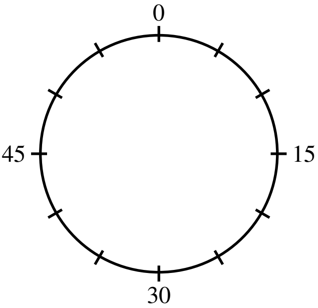
$\rightsquigarrow$

visualize ?

<u>Minutes</u>

$\mathbb{R}/60\mathbb{Z} \simeq$



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$\iff$ $f(x)$ is a function of $x \in \mathbb{R}/\mathbb{Z} \simeq S^1$

visualize ?

Minutes

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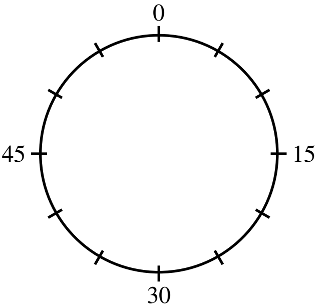

$\simeq S^1$ circle

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Periodic function $F(x, y)$ on $\mathbb{R}^2$

double period $\begin{cases} F(x,y) = F(x+a, y) \\ F(x,y) = F(x, y+b) \end{cases}$

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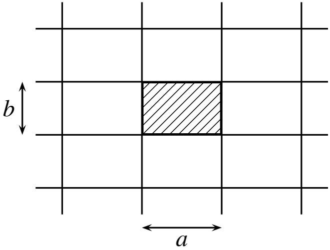

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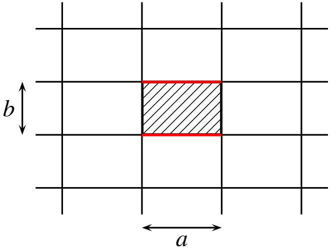

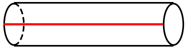

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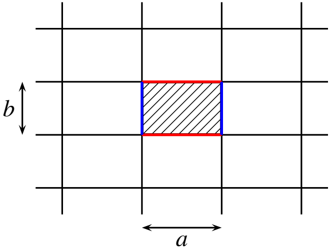

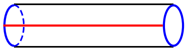

Periodic function and quotient space

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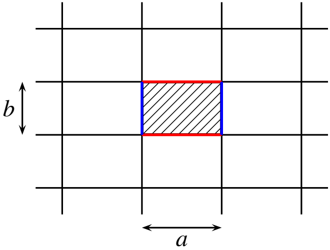
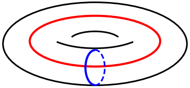
double period $\begin{cases} F(x,y) = F(x+a,y) \\ F(x,y) = F(x,y+b) \end{cases}$

$\Longleftrightarrow$ $F(x,y)$ is a function of $(x,y) \in \mathbb{R}^2/\mathbb{Z}\binom{a}{0} + \mathbb{Z}\binom{0}{b}$





torus

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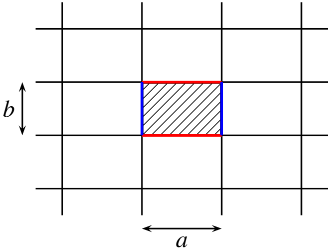

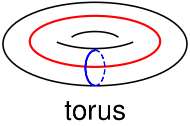

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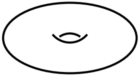

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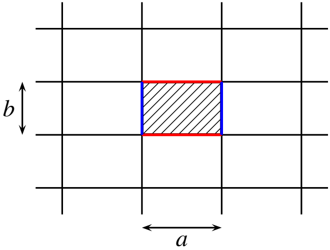
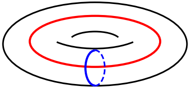
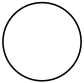
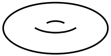




torus

Model space and quotients

	Model space		Quotients	
	$\mathbb{R}$	$\rightsquigarrow$	$\mathbb{R}/\mathbb{Z} \simeq$	
(Euclidean)	$\mathbb{R}^2$	$\rightsquigarrow$	$\mathbb{R}^2/\mathbb{Z}^2 \simeq$	

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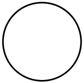
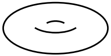
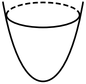
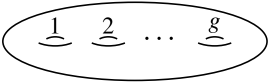
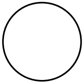
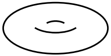
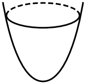
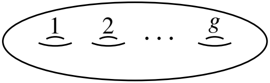
	$\mathbb{R}$	$\rightsquigarrow$	$\mathbb{R}/\mathbb{Z} \simeq$	
(Euclidean)	$\mathbb{R}^2$	$\rightsquigarrow$	$\mathbb{R}^2/\mathbb{Z}^2 \simeq$	
(Riemannian)	 hyperbolic space	$\rightsquigarrow$	 $(g \geq 2)$ hyperbolic manifold	

Model space and quotients

Model space | Quotients

$\mathbb{R} \rightsquigarrow \mathbb{R}/\mathbb{Z} \simeq$ 

(Euclidean) $\mathbb{R}^2 \rightsquigarrow \mathbb{R}^2/\mathbb{Z}^2 \simeq$ 

(Riemannian)  $\rightsquigarrow$ 

$(g \geq 2)$

hyperbolic space | hyperbolic manifold

Periodic functions on model space = functions on quotients

Model space and quotients

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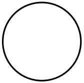
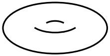
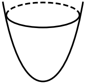
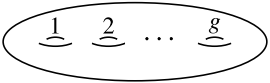
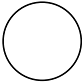
	$\mathbb{R}$	$\longrightarrow$	$\mathbb{R}/\mathbb{Z} \simeq$	
(Euclidean)	$\mathbb{R}^2$	$\longrightarrow$	$\mathbb{R}^2/\mathbb{Z}^2 \simeq$	
(Riemannian)	 hyperbolic space	$\longrightarrow$	 $(g \geq 2)$ hyperbolic manifold	

Δ Laplacian	$\rightsquigarrow$	Δ Laplacian

Model space and quotients

Model space — Quotients

$\mathbb{R} \rightsquigarrow \mathbb{R}/\mathbb{Z} \simeq$



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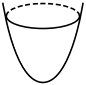
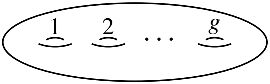


hyperbolic space

$\rightsquigarrow$



$(g \geq 2)$

hyperbolic manifold

Model space and quotients

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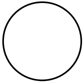
	$\mathbb{R}$	$\rightsquigarrow$	$\mathbb{R}/\mathbb{Z} \simeq$	
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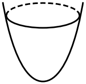
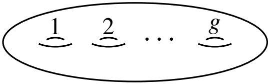
(Riemannian)	 hyperbolic space	$\rightsquigarrow$	 $(g \geq 2)$ hyperbolic manifold	
(Lorentzian)	AdS^n anti-de Sitter space	$\rightsquigarrow$	? anti-de Sitter manifold	

Most round shape (locally)

(M, g) : pseudo-Riemannian mfd,
geodesically complete

<u>Def.</u> (M, g) is a <u>space form</u>
$\iff$ sectional curvature κ is constant

Space forms (examples)

Space form $\cdots$

$$\begin{cases} \text{Signature } (p,q) \text{ of pseudo-Riemannian metric } g \\ \text{Curvature } \kappa \in \{+, 0, -\} \end{cases}$$

<u>E.g.</u> $q = 0$ (Riemannian mfd)

sphere S^n	$\mathbb{R}^n$	hyperbolic sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

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de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

Global nature of most round objects

Space form problem for pseudo-Riemannian mfds

Local Assumption

signature (p, q), curvature $\kappa \in \{+, 0, -\}$

$\Downarrow$

Global Results

- Do compact models exist?
- What groups can arise as their fundamental groups?

Global nature of most round objects

Space form problem for pseudo-Riemannian mfds

Local Assumption

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$\Downarrow$

Global Results

- Do compact models exist?

  Is the universe closed?

- What groups can arise as their fundamental groups?

Existence problem of compact space forms

Riemannian case

Compact space forms always exist:

- $\kappa > 0$ S^n
- $\kappa = 0$ $\mathbb{R}^n/\mathbb{Z}^n$
- $\kappa < 0$ hyperbolic space

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$\iff$ Cocompact discrete subgps of $O(n,1)$ (uniform lattice) exist

(Siegel, Borel, Makarov, Vinberg, Johnson–Millson, Gromov–Piateski-Shapiro $\cdots$)

Siegel, Borel: arithmetic

Makarov, Vinberg, Johnson–Millson, Gromov–Piateski-Shapiro: non-arithmetic

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Siegel, Borel: arithmetic; Makarov, Vinberg, Johnson–Millson, Gromov–Piateski-Shapiro: non-arithmetic

Lorentzian case

n dimensional compact space forms

- $\kappa > 0$ (de Sitter mfd) NOT exist (Calabi–Markus phenomenon)
- $\kappa = 0$ ALWAYS exist
- $\kappa < 0$ (anti-de Sitter mfd) exist $\Leftrightarrow$ n is odd

Space forms (examples)

Space form $\cdots$

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de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

More general case ?

Existence problem of compact space forms

- For pseudo-Riemannian manifold of signature (p, q)

<u>Theorem</u> Compact space forms of negative curvature κ exist if

Existence problem of compact space forms

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① q any, $p = 0$ $(\leftrightarrow \kappa > 0)$

② $q = 0$, p any (hyperbolic sp)

Proof (1950–)

①② (Riemmanian)

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④ $q = 3$, $p \equiv 0 \mod 4$

Proof (1950–)

①② (Riemmanian) ③ (Lorentzian) ④ (more general)

Existence problem of compact space forms

- For pseudo-Riemannian manifold of signature (p, q)

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⑤ $q = 7, p = 8$

(③④⑤: pseudo-Riemannian)

Proof (1950–)

(①② (Riemmanian); ③④⑤ (pseudo-Riemannian) Kulkarni '81, K– '94)

Existence problem of compact space forms

- For pseudo-Riemannian manifold of signature (p, q)

<u>Theorem</u> Compact space forms of negative curvature κ exist if

① q any, $p = 0$	($\leftrightarrow \kappa > 0$)
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⑤ $q = 7, p = 8$	

(③④⑤ are grouped by a brace.)

Proof (1950–)

(①② (Riemmanian); ③④⑤ (pseudo-Riemannian) Kulkarni '81, K– '94)

Obstruction:

Compact space forms of $\kappa < 0$ do not exist if

$p \leq q$ (Calabi–Markus, Wolf '62, K– '89),

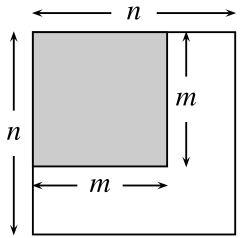
or pq is odd (generalized Hirzebruch's proportionality principle, K–Ono)

Compact manifolds modelled on $SL(n)/SL(m)$?

Problem: Does there exist compact Hausdorff quotients of

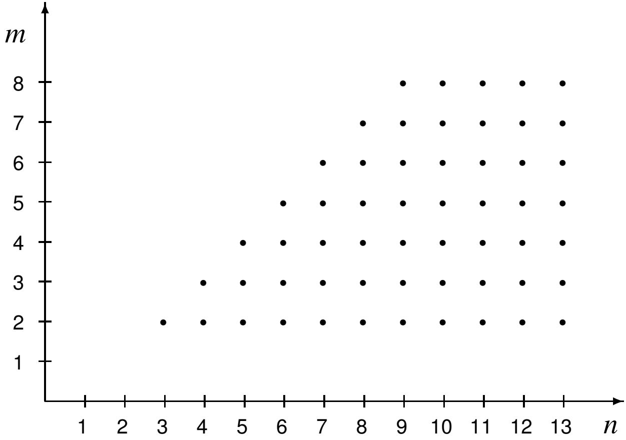
$$SL(n,\mathbb{F})/SL(m,\mathbb{F}) \quad (n > m,\ \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$$

by discrete subgps of $SL(n,\mathbb{F})$?



Compact quotients for $SL(n)/SL(m)$

Conjecture (non-existence of compact forms) has been proved for:



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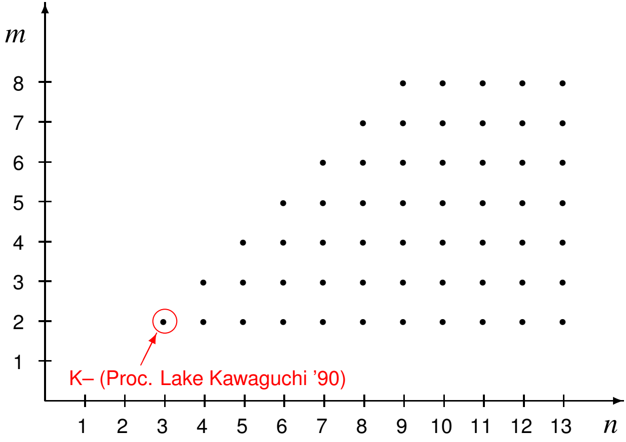

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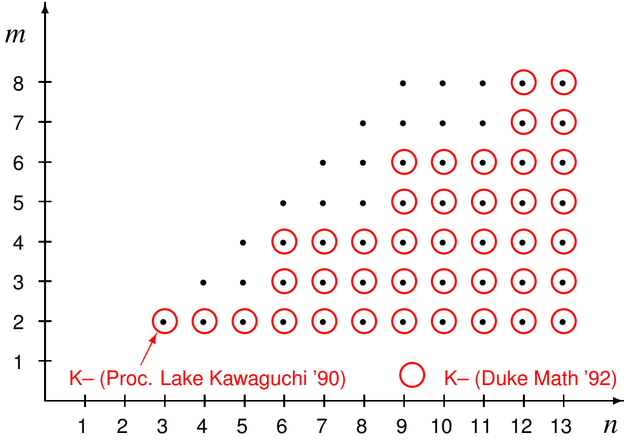

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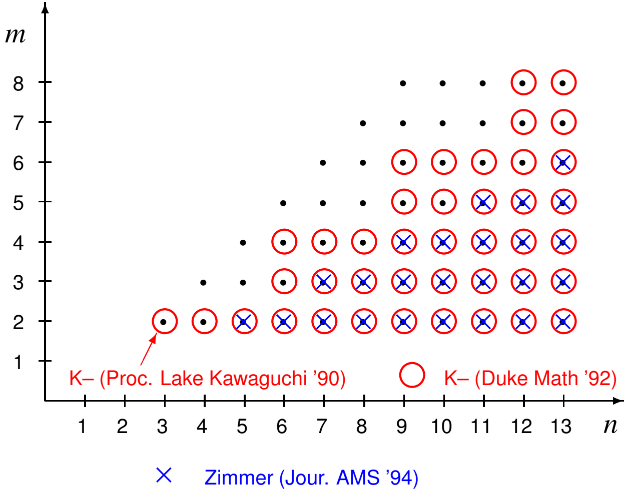

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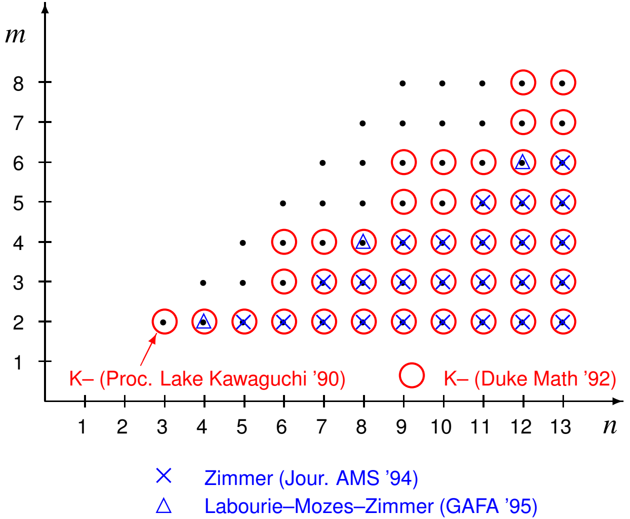

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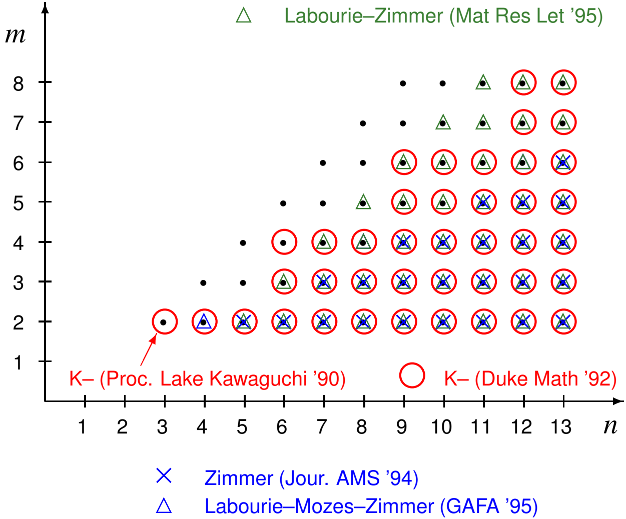

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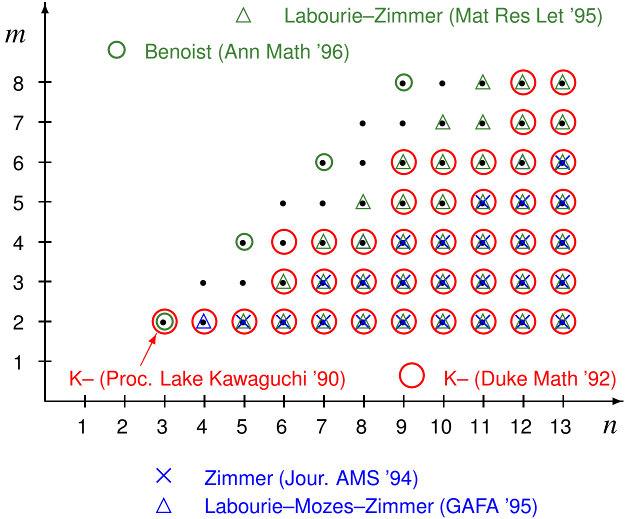

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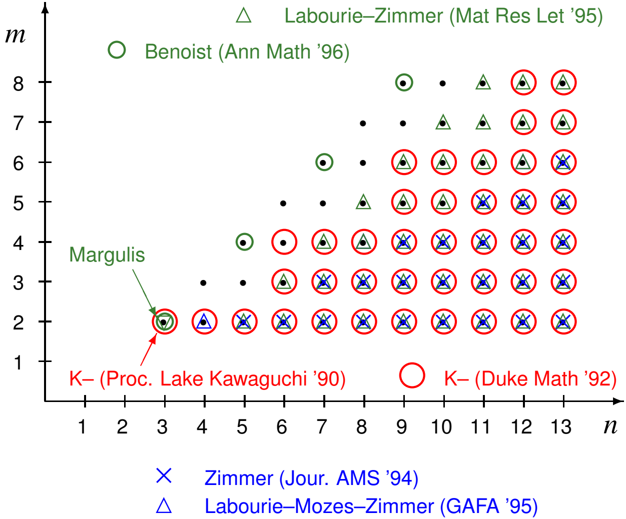

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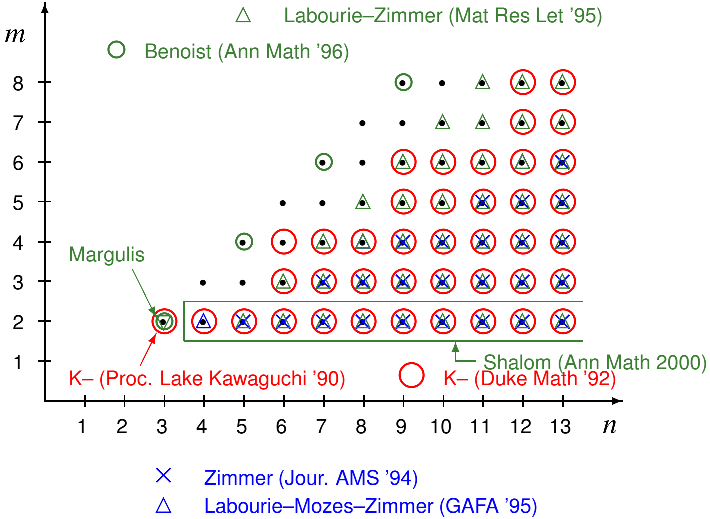

Methods for non-existence of compact forms

Conjecture $SL(n)/SL(m)$ $(n > m > 1)$
has no cocompact discontinuous group.

K–	criterion of proper actions	$\frac{n}{3} > [\frac{m+1}{2}]$
Zimmer	orbit closure thm (Ratner)	$n > 2m$
Labourier–Mozes–Zimmer		
	ergodic action	$n \geq 2m$
Benoist	criterion of proper actions	$n = m + 1,\ m$ even
Margulis	unitary representation	$(n \geq 5, m = 2)$
Shalom	unitary representation	$n \geq 4, m = 2$

Table of contents

1. Laplacian on $\mathbb{R}$
2. Quick course to Riemannian and Lorentzian geometry
3. Elementary definition of Laplacian Δ on pseudo-Riemannian
4. Construction of eigenfunctions of Δ and integral geometry
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Geometric and analytic questions — locally symmetric spaces

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Thinner strings produce a higher pitch than thicker strings.

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Geometric Question

Do compact locally symmetric spaces M exist?

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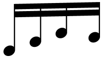
Analytic Question

Does point spectrum of the Laplacian Δ_M exist?
If so, construct L^2-eigenfunctions.



Geometric and analytic questions — locally symmetric spaces

Shorter strings produce a higher pitch than longer strings.

Thinner strings produce a higher pitch than thicker strings.

Geometric Question

Do compact locally symmetric spaces M exist?

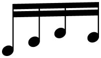
Rigidity / Deformation ?

Analytic Question

Does point spectrum of the Laplacian Δ_M exist?

If so, construct L^2-eigenfunctions.

Stability of eigenvalues ?



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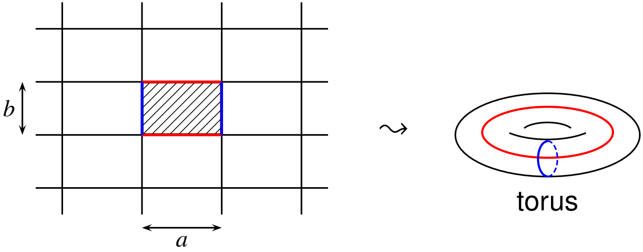

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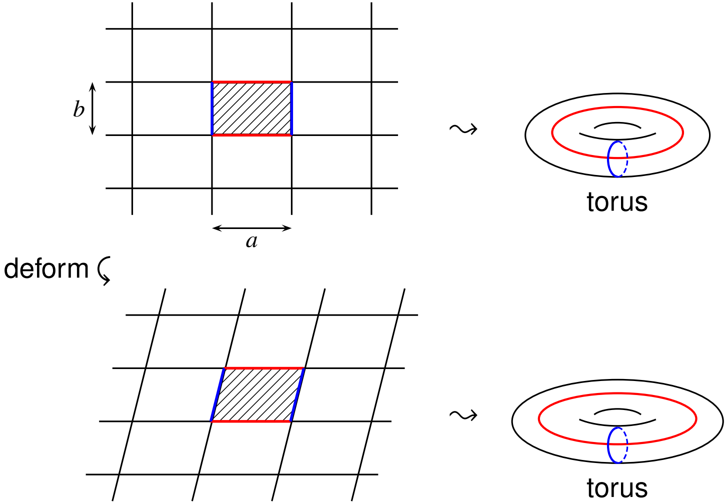

Deformation v.s. rigidity

$$\underset{\text{discrete}}{\Gamma} \subset \underset{\text{simple Lie gp}}{G} \supset H$$

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & & \searrow & \\ \Gamma\backslash G & & & & G/H \\ & \searrow & & \swarrow & \\ & & \Gamma\backslash G/H & & \end{array}$$

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- Riemannian case (H: compact) $\cdots$ Rigid in most cases

- pseudo-Riemannian case (H: non-compact) $\cdots$ "quite flexible"

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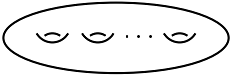
  $^{\exists}$non-trivial deformation $\Longleftrightarrow \Gamma\backslash G/H =$ 

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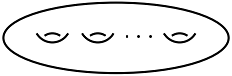
  $^{\exists}$non-trivial deformation $\Longleftrightarrow \Gamma\backslash G/H =$ 

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  Deformation (Goldman, Ghys, K– , Kassel, ...)

Eigenvalues and deformation of periods (1 dim'l case)

Fix $\rho > 0$ $\qquad X_\Gamma := \mathbb{R}/\frac{2\pi}{\rho}\mathbb{Z}$

$$\mathrm{Spec}(X_\Gamma, \Delta) := \left\{ \begin{matrix} \lambda : {}^\exists f \neq 0 \text{ of period } \dfrac{2\pi}{\rho} \text{ s.t.} \\ \Delta f = \lambda f \end{matrix} \right\}$$

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$$f(x) = \sin \rho m x, \ \ \cos \rho m x \quad (m = 0, \pm 1, \pm 2, \dots)$$

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$$\mathrm{Spec}(X_\Gamma, \Delta) = \{\rho^2 m^2 : m \in \mathbb{Z}\}$$

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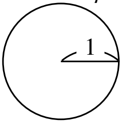
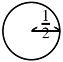
$\rho = 1$		$\{0, 1, 4, 9, 16, 25, \dots\}$
$\rho = 2$		$\{0, 4, 16, 36, 64, 100, \dots\}$

Eigenvalues and deformation of periods (1 dim'l case)

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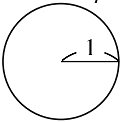

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↓stable ↓varies

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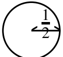

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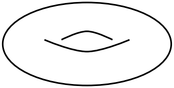
Deformation of torus (flat case)

$$\vec{e_1} := 2\pi(1,0), \quad \vec{e_2} := 2\pi(0,1)$$

$\mathbb{R}^2/\mathbb{Z}\vec{e_1} + \mathbb{Z}\vec{e_2}$



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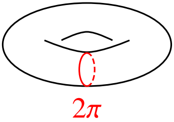

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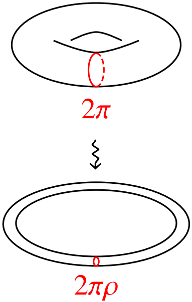

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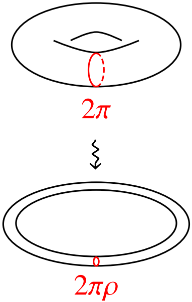

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$$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

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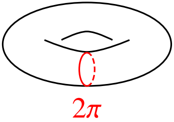
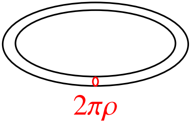


$\{m^2 + n^2 : m, n \in \mathbb{Z}\}$

$\rightsquigarrow$ $\rightsquigarrow$ Spectrum varies

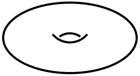
$\mathbb{R}^2/\mathbb{Z}\vec{e_1} + \mathbb{Z}\frac{1}{\rho}\vec{e_2}$



$\{m^2 + \rho^2 n^2 : m, n \in \mathbb{Z}\}$

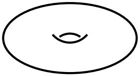
$$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$
$$ds^2 = dx^2 + dy^2$$

Eigenvalues vary on deformation space

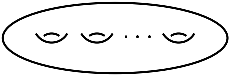
$\mathbb{T}^2 \simeq$  (flat)

Observation (flat case) Any non-zero eigenvalue of the Laplacian on $\mathbb{T}^2$ varies on the *deformation space*.

Eigenvalues vary on deformation space

$\mathbb{T}^2 \simeq$  (flat)

Observation (flat case) Any non-zero eigenvalue of the Laplacian on $\mathbb{T}^2$ varies on the *deformation space*.

$\Sigma_g \simeq$  (curvature $\equiv -1$)

Fact (Wolpert, 1994) Any non-zero eigenvalue of the Laplacian on the compact hyperbolic manifold Σ_g varies on the *Teichmüller space*.

$\cdots$ Spectrum is unstable

Shorter strings produce a higher pitch than longer strings.

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1. Laplacian on $\mathbb{R}$
2. Quick course to Riemannian and Lorentzian geometry
3. Elementary definition of Laplacian Δ on pseudo-Riemannian
4. Construction of eigenfunctions of Δ and integral geometry
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6. Deformation of geometry and stability of eigenvalues
7. Universal sound in anti-de Sitter manifolds

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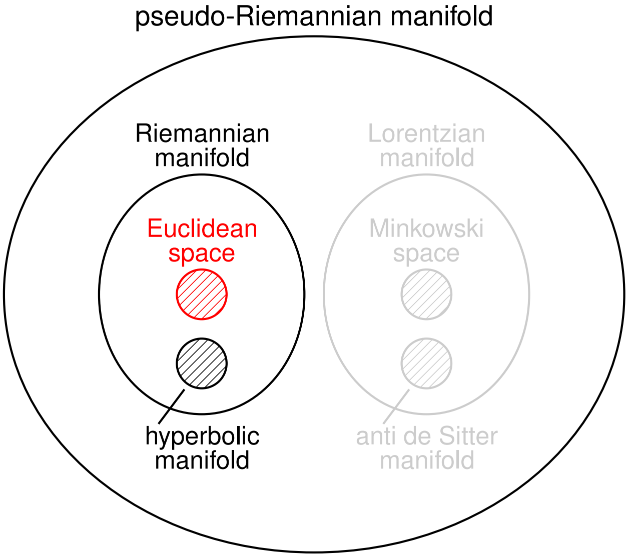
Quick course on pseudo-Riemannian manifold



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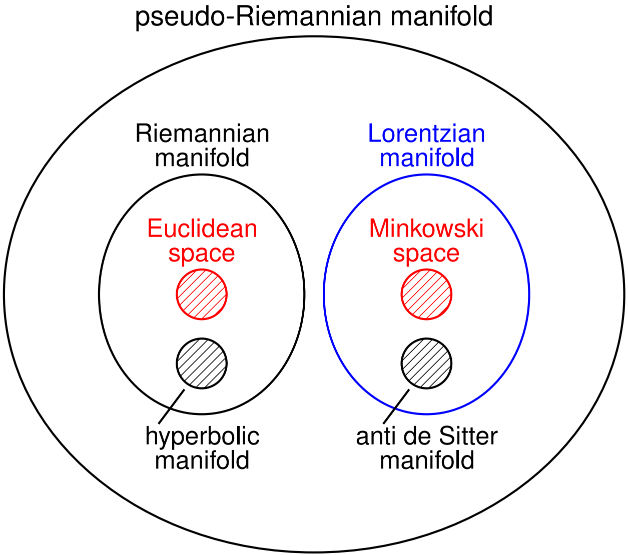

3-dim'l compact anti-de Sitter manifold

Def M: anti-de Sitter manifold

$\underset{\text{def}}{\Longleftrightarrow}$ Lorentz manifold with sectional curvature $\equiv -1$

Cf. Hyperbolic manifold

$\underset{\text{def}}{\Longleftrightarrow}$ Riemannian manifold with sectional curvature $\equiv -1$

3-dim'l compact anti-de Sitter manifold

Def M: anti-de Sitter manifold
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In the previous examples (flat case, hyperbolic manifold), any eigenvalue λ of the Laplacian varies except for $\lambda = 0$ (constant functions are obviously eigenfunctions). However,

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Theorem (joint with F. Kassel, arXiv: 1209.4075, 141 pages)
There exists infinitely many
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Remark	X	Riemannian	$\Longrightarrow$	Laplacian	Δ	is elliptic
		Lorentzian	$\Longrightarrow$		$\square$	hyperbolic

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Theorem (joint with F. Kassel, arXiv: 1209.4075, 141 pages)

There exists infinitely many 'stable' eigenvalues of the Laplacian $\square$ on 3-dimensional compact anti-de Sitter manifold.

eigenvalues $\cdots$ L^2-eigenfunctions

(Note: eigenfunctions are not always real analytic)

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'stable' $\underset{\text{def}}{=}$ 'does NOT vary under deformation'
of anti-de Sitter structure

The deformation space (modulo conjugation) has dimension $12g - 12$

‘Universal sound’ for anti-de Sitter manifolds

Usually,

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‘Universal sound’ for anti-de Sitter manifolds

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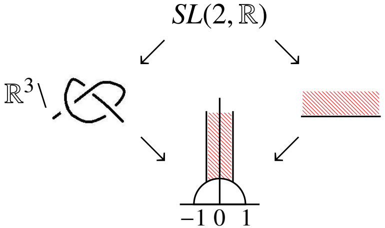
However,

for some locally symmetric spaces
such as 3-dim’l anti-de Sitter manifolds,
there exist countably many stable L^2-eigenvalues.

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

Γ (discrete subgp) $\subset$ G (Lie group) $\supset$ H (subgroup) $\qquad SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R}) \supset SO(2)$

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & & \searrow & \\ G/\Gamma & & & & G/H \\ & \searrow & & \swarrow & \\ & & \Gamma\backslash G/H & & \end{array}$$



New challenge: Initiate spectral analysis of Δ on $\Gamma\backslash G/H$ in a more general setting (non-abelian G and non-compact H).

Further difficulties arise

- (geometry) existence of good geometry $\Gamma\backslash G/H$?
  study of "local to global" beyond Riemannian setting
- (analysis) Laplacian is no more elliptic.
- (representation theory) volume $(\Gamma\backslash G) = \infty$ even when $\Gamma\backslash G/H$ is compact

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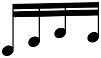


Construction of stable L^2-spectrum of Laplacian on $\Gamma\backslash G/H$

$$\Delta f = \lambda f \quad \text{on} \quad \Gamma\backslash G/H$$

Step 1 Construction of (non-periodic) eigenfunctions

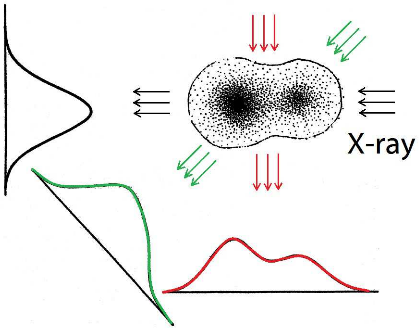
Step 2 Existence/Construction of Γ

Step 3 Construction of periodic eigenfunctions (Poincaré series)

- Geometric estimate
- Analytic estimate

Geometric construction of eigenfunctions

Idea of computer tomography (try to ‘see’ the inside)



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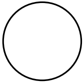
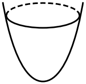
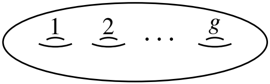
Model space and quotients

	Model space		Quotients	
	$\mathbb{R}$	$\rightsquigarrow$	$\mathbb{R}/\mathbb{Z} \simeq$	
(Euclidean)	$\mathbb{R}^2$	$\rightsquigarrow$	$\mathbb{R}^2/\mathbb{Z}^2 \simeq$	
(Riemannian)	 hyperbolic space	$\rightsquigarrow$	 $(g \geq 2)$ hyperbolic manifold	
(Lorentzian)	AdS^n anti-de Sitter space	$\rightsquigarrow$	? anti-de Sitter manifold	

Construction of stable L^2-spectrum of Laplacian on $\Gamma\backslash G/H$

$$\Delta f = \lambda f \quad \text{on} \quad \Gamma\backslash G/H$$

Step 1 Construction of (non-periodic) eigenfunctions
(integral geometry, Poisson transform)

Step 2 Existence/Construction of Γ (local to global)

Step 3 Construction of periodic eigenfunctions (Poincaré series)
non-periodic eigenfunction $\mapsto$ periodic eigenfunctions

$$f(x) \mapsto \sum_{\gamma\in\Gamma} f\,(\gamma \cdot x)$$

- Geometric estimate for proper actions $\Gamma \curvearrowright G/H$
(Kazhdan–Margulis, K– , Benoist, Kassel, . . .)
- Analytic estimate of eigenfunctions on G/H
(systems of PDEs, micro-local analysis)
(Sato–Kashiwara–Kawai, Oshima, . . .)

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Further approach to analysis on $\Gamma\backslash G/H$

Capture unstable eigenvalues $\cdots$ work in progress

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Idea: Use the theory of unitary representations

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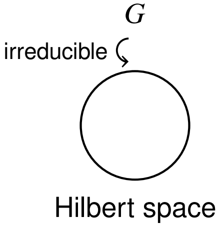
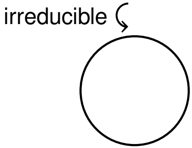


Further approach to analysis on $\Gamma\backslash G/H$

Capture unstable eigenvalues $\cdots$ work in progress

Idea: Use the theory of unitary representations

$$G \supset \overset{\text{subgroup}}{L} = \text{Zariski closure of } \Gamma$$



Hilbert space

Further approach to analysis on $\Gamma\backslash G/H$

Capture unstable eigenvalues $\cdots$ work in progress

Idea: Use the theory of unitary representations

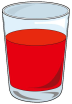
subgroup

$G \supset L$ = Zariski closure of Γ

irreducible ↻

Hilbert space

"decomposes" →



+

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"discretely decomposable branching law" (occasional)

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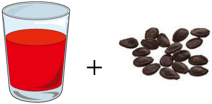

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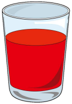

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$\updownarrow$

more differential equations!

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5. Period and geometry of discontinuous groups
6. Deformation of geometry and stability of eigenvalues
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Thank you very much!