Symmetry breaking for infinite dimensional representation

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If G and V have a topology, then we also assume that $\pi(g)$ and π are continuous.

Example: H a subgroup of G ,
$V = \{$ functions on G/H $\}$.
G acts on V by

$$\pi(g)F(x) = F(gx)$$

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- continuous functions
- square integrable functions on G/H
- Schwartz functions on G/H

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- $L^2(G/H)$ square integrable functions on G/H

- Schwartz space $\mathcal{S}(G/H)$.

- Distributions $\mathcal{D}(G/H)$

If G is a Lie group we can consider the **continuous** action of G on

- $L^2(G/H)$ square integrable functions on G/H
  Hilbert space

- Schwartz space $\mathcal{S}(G/H)$.
  Frechet space

- Distributions $\mathcal{D}(G/H)$
  dual space to $\mathcal{S}(G/H)$

Problem:

Understand the restriction of a representation to the subgroup $G' \subset G$

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Understand the breaking of the symmetry

Given a representation $\pi : G \to End(V_\pi)$ of G and a representation $\rho : G' \to End(W_\rho)$ of G'. What does symmetry breaking mean?

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Determine

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of G' equivariant continuous operators.

or

or

Determine

$$\mathrm{Hom}_{G'}(V_\pi \otimes W_\rho, \mathbb{C})$$

of G' equivariant continuous linear functionals on $V_\pi \otimes W_\rho$.

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of G' equivariant continuous linear functionals on $V_\pi \otimes W_\rho$.

or

Determine

$$H^0_{U(\mathfrak{g}')}(\mathfrak{g}, \text{Hom}(V_\pi \otimes W_{\rho,\mathbb{C}}\ )$$

Outline of lecture

I. Introduction

II. Examples:
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A.) $G = SL(2,\mathbb{R})$, $G' = N$ nilpotent subgroup
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C.) $G = O(n+1,1)$, $G' = O(n,1)$, special family of unitary irreducible representations

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We consider spherical principal series representations π_ν , induced from a character χ_ν of the upper triangular matrices B, trivial on the diagonal matrix $-I$. These representations can be realized on the space V_ν of differentiable sections of a line bundle over the sphere $P(1) = G/B$.

$N \subset B$ subgroup with diagonal entries 1 and Lie algebra $\mathfrak{n}$.

The diagonal matrices M act on $V_\nu^{\mathfrak{n}}$. A character χ_λ of B defines an operator in $\mathrm{Hom}_G(V_\nu, V_\lambda)$. For the realization of these representations on a Hilbert space Knapp and Stein constructed nontrivial operators in $\mathrm{Hom}_G(V_\nu, V_{-\nu})$ using analytic continuation. The identity operator is in $\mathrm{Hom}_G(V_\nu, V_\nu)$.

There are 2 orbits of N on P^1, namely $\mathcal{O}_o = \mathbb{R}$ and $\mathcal{O}_c =$ point. We have the exact sequence

$$0 \to \mathcal{S}(\mathcal{O}_o) \to V_\nu \to \tilde{M}_\nu \to 0$$

Here $\mathcal{S}(\mathcal{O}_o)$ is the Schwartz space on the large orbit $\mathcal{O}_o$. Integration defines the unique N invariant linear functional on the Schwartz space $\mathcal{S}(\mathcal{O}_o)$ and the dual $M_{-\nu}$ of $\tilde{M}_\nu$ is a Verma module.

Note that $H^1(\mathfrak{n}, M_{-\nu})$ is trivial unless ν is a negative integer. If ν the connection homomorphism is non zero if ν is even and zero if ν is odd.

So we obtain:

Theorem 1. • *If ν is negative odd then there are 2 nontrivial intertwining operator in Hom*$_G(V_\nu, V_{-\nu})$*. One them is a differential operator.*

- *If ν is negative even then there are one nontrivial intertwining operator in Hom*$_G(V_\nu, V_{-\nu})$*. It is a differential operator.*

- *Otherwise there is only one nontrivial intertwining operator in Hom*$_G(V_\nu, V_{-\nu})$*. It is a integral operator.*

B.) $G = PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$, G' diagonal copy of $PGL(2,\mathbb{R})$ (joint work in progress with Raul Gomez)

We consider G' invariant trilinear forms on $V_{\nu_1,\epsilon_1} \otimes V_{\nu_2,\epsilon_2} \otimes V_{\nu_3,\epsilon_3}$ realized on section of a line bundles over $X = P^1 \times P^1 \times P^1$. Here ϵ is a character of $PGL(2,\mathbb{R})/PGL(2,\mathbb{R})^0$

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These forms have been considered by Bernstein and Reznikov, for discrete series representations also by Rankin Cohen, for more general groups and representations such forms were constructed by Kobayashi, Orsted, Pevzner.

Our problem:

How many forms are there.

There are 5 orbits of G' on X.

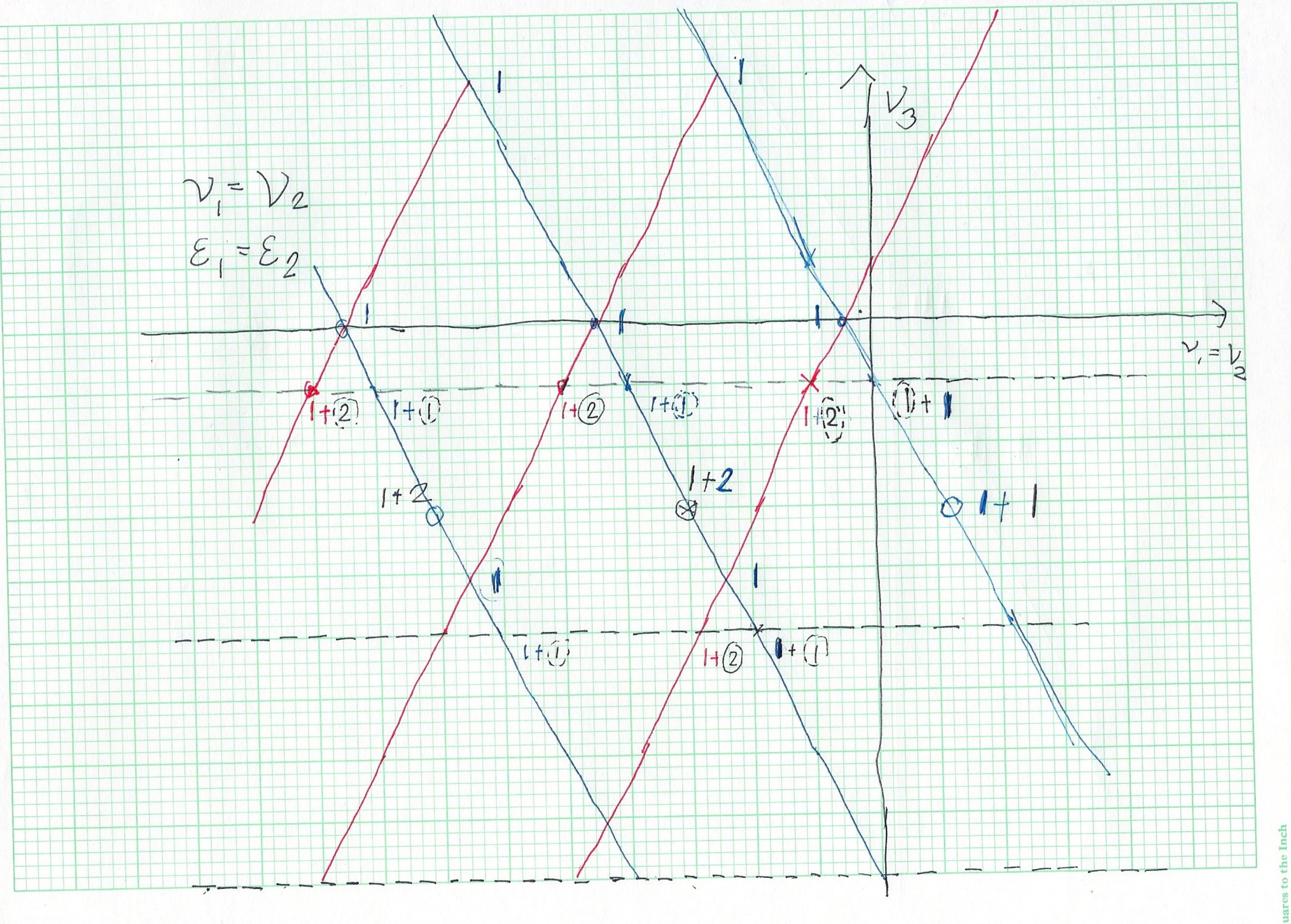
$\mathcal{O}_3$ open and isomorphic to G'.

$\mathcal{O}_1, O_2, O_3$ of dimension

$\mathcal{O}_5$ closed orbit

Our invariant forms are distributions with support in a G' invariant set.

The following is slide is represents the results in a special case: We assume that $\nu_1 = \nu_2$.



$\nu_1 = \nu_2$
$\varepsilon_1 = \varepsilon_2$
ν_3
$\nu_1 = \nu_2$
1+2
1+2
1+1

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We consider **irreducible unitary representations** A_i of O(n+1,1) with the same infinitesimal character as the trivial representation and representations B_j of G'=O(n,1) with the same properties.

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We consider **irreducible unitary representations** A_i of O(n+1,1) with the same infinitesimal character as the trivial representation and representations B_j of G'=O(n,1) with the same properties.

The results are represented graphically in the following tables. The first row are representations of G, the second row are representations of G'' . Symmetry breaking operators between these representations are represented by black arrows .

Results:

Symmetry breaking for O(2m+1,1), O(2m.1)

$$
\begin{array}{ccccccccc}
A_0 & & A_1 & & \cdots & \cdots & \cdots & A_{m-1} & & A_m \\
\downarrow & \swarrow & \downarrow & \swarrow & \cdots & & & \downarrow & \swarrow & \downarrow \\
B_0 & & B_1 & & \cdots & \cdots & \cdots & B_{m-1} & & B_m
\end{array}
$$

Symmetry breaking for O(2m+2,1), O(2m+1.1)

$$
\begin{array}{ccccccccccc}
A_0 & & A_1 & & \cdots & \cdots & \cdots & A_{m-1} & & A_m & A_{m+1} \\
\downarrow & \swarrow & \downarrow & \swarrow & \cdots & & & \downarrow & \swarrow & \downarrow & \swarrow \\
B_0 & & B_1 & & \cdots & \cdots & \cdots & B_{m-1} & & B_m &
\end{array}
$$